

Recent results and open problems on parabolic equations with gradient nonlinearities *

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Abstract

We survey recent results and present a number of open problems concerning the large-time behavior of solutions of semilinear parabolic equations with gradient nonlinearities. We focus on the model equation with a dissipative gradient term

$$u_t - \Delta u = u^p - b|\nabla u|^q,$$

where $p, q > 1, b > 0$, with homogeneous Dirichlet boundary conditions. Numerous papers were devoted to this equation in the last ten years, and we compare the results with those known for the case of the pure power reaction-diffusion equation ($b = 0$). In presence of the dissipative gradient term a number of new phenomena appear which do not occur when $b = 0$. The questions treated concern: sufficient conditions for blowup, behavior of blowing up solutions, global existence and stability, unbounded global solutions, critical exponents, and stationary states.

1 Introduction

The large-time behavior of solutions of nonlinear reaction-diffusion equations has received considerable interest since the 60's. A model case of such equation is

$$u_t - \Delta u = |u|^{p-1}u. \tag{1.1}$$

Various sufficient conditions for blowup and global existence were provided and qualitative properties were investigated, such as: nature of the blowup set, rate and profile of blowup, maximum existence time and continuation after blowup, boundedness of global solutions and convergence to a stationary state. We refer for these to the books and survey articles [6, 41, 33, 57, 54, 14].

More recently, a number of works have addressed the same type of questions for semilinear parabolic equations where the nonlinearity also depends on the spatial derivatives of u . A rough and partial classification of such equation

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can be made according to two criteria. The first one is the nature the gradient dependence of the nonlinearity, namely, through a convective term, like $a \cdot \nabla(u^q)$, or through a term of Hamilton-Jacobi type $b|\nabla u|^q$. The second criterion is the presence (or not) of a reaction term, like u^p . Typical equations resulting from the combination of these criteria are

$$u_t - \Delta u = a \cdot \nabla(u^q), \quad (1.2)$$

$$u_t - \Delta u = u^p + a \cdot \nabla(u^q), \quad (1.3)$$

$$u_t - \Delta u = b|\nabla u|^q, \quad (1.4)$$

$$u_t - \Delta u = u^p - b|\nabla u|^q. \quad (1.5)$$

(Here $u^p \equiv |u|^{p-1}u$, $a \in \mathbb{R}^N$, $b \in \mathbb{R}$.) Each of these equations has been rather well studied in the past ten years. However, reviewing all of them would be somehow too dispersive, and we prefer to focus on one particular equation, which already provides a rich variety of aspects. The purpose of this article is thus to survey the existing literature on the equation (CW). We refer the interested reader to [15] for (1.2), [2] for (1.3), [8] for (1.4), and to the references in these papers. Outside of this classification, let us also mention the equation

$$u_t - u_{xx} = f(u)|u_x|^{q-1}u_x,$$

which exhibits interesting phenomena (related to derivative blowup – see e.g. [4, 49]).

We will consider the associated initial-boundary value problem of Dirichlet type:

$$\begin{aligned} u_t - \Delta u &= |u|^{p-1}u - b|\nabla u|^q, & t > 0, \quad x \in \Omega, \\ u(t, x) &= 0, & t > 0, \quad x \in \partial\Omega, \\ u(0, x) &= \phi(x) \geq 0, & x \in \Omega. \end{aligned} \quad (1.6)$$

In what follows, we assume that $p > 1$, $q \geq 1$, and Ω is a domain of \mathbb{R}^N , bounded or unbounded, sufficiently regular (say, uniformly regular of class C^2). Also, *unless otherwise stated, we assume $b > 0$* . (A few results will however concern the case $b < 0$.)

It is known that (1.6) admits a unique, maximal in time, classical solution $u \geq 0$, for all $\phi \geq 0$ sufficiently regular, e.g., $\phi \in C^1(\overline{\Omega})$ with $\phi|_{\partial\Omega} = 0$ if Ω is bounded, or $\phi \in W_0^{1,s}(\Omega)$ with $s > N \max(p, q)$ if Ω is unbounded. This regularity of ϕ will be assumed throughout the paper, unless otherwise stated. We denote by $T^* = T^*(\phi)$ the maximum existence time of u , and we say that u blows up in finite time if $T^*(\phi) < \infty$. When $\phi \geq 0$ and $b > 0$, it is known [37, 53] that gradient blowup cannot occur for (1.6), that is: $T^*(\phi) < \infty$ implies $\limsup_{t \rightarrow T^*} \|u(t)\|_\infty = \infty$.

Since we only consider nonnegative solutions of (1.6), it is clear that the gradient term here represents a dissipation when $b > 0$. In fact, the dynamics of this equation can be partially understood as a competition between the reaction term u^p , which may cause blowup as in the equation (1.1), and the gradient

term, which fights against blowup. The solutions will exhibit different large-time behaviors, according to the issue of this competition. Similar mechanisms of competition have been studied in the case of nonlinear wave equations of the type

$$u_{tt} - \Delta u = |u|^{p-1}u - |u_t|^{q-1}u_t,$$

where $p > 1$, $q \geq 1$ (see [23]).

Equation (1.6) was first introduced in [10] in order to investigate the possible effect of a damping gradient term on global existence or nonexistence. On the other hand, a model in population dynamics was proposed in [46], where (1.6) describes the evolution of the population density of a biological species, under the effect of certain natural mechanisms. In particular, the dissipative gradient term represents the action of a predator which destroys the individuals during their displacements (it is assumed that the preys are not vulnerable at rest). A further discussion of this model can be found in [1], where the related degenerate equation

$$u_t - \Delta(u^m) = u^p - |\nabla(u^\alpha)|^q$$

with $m > 1$, $\alpha > 0$ was studied.

As it will turn out, the large-time behavior of the solution of problem (1.6) will generally depend on all the values of the parameters, on the initial data, and on the domain Ω . However, of particular importance will be the fact that $p > q$ or $q \geq p$. These cases are respectively reviewed in § 2 and 3. Finally, § 4 is devoted to stationary solutions of (1.6). Throughout the paper, we will indicate a number of open problems related to the results we will review.

2 The case $p > q$

2.1 Existence of blowup: the general result

The following result [52] states that finite-time blowup occurs for large data whenever $p > q$.

Theorem 2.1 *Assume $p > q$, $\Omega \subset \mathbb{R}^N$ (bounded or unbounded) and $\psi \not\equiv 0$ ($\psi \geq 0$). Then there exists $\lambda_0 = \lambda_0(\psi) > 0$ such that for all $\lambda > \lambda_0$, the solution of (1.6) with initial data $\phi = \lambda\psi$ blows-up in finite time.*

We will see in § 3 that this result is optimal, in the sense that blowup *never* occurs if $q \geq p$, at least in bounded domains.

The basic idea of the proof is to compare u with a subsolution that blows up in finite time. In fact, one constructs a *self-similar* subsolution, whose profile is compactly supported. Interestingly, it is possible to find blowing-up self-similar subsolutions, whether or not (1.6) has the invariance properties normally associated with self-similar solutions. The similarity exponents depend on p and q , and can be chosen within a certain range of values.

The result of Theorem 2.1 actually extends to more general nonlinearities $F(u, \nabla u)$ and also to some degenerate problems.

We mention that the conclusion of Theorem 2.1 was obtained earlier, by completely different methods, in [29] in the special case $q = 2$, and in [36] in the special case $N = 1$, b small.

2.2 Other conditions for blowup

Besides the preceding general blowup result, various blowup conditions of more specific type are known, often under the restriction $q \leq 2p/(p+1)$. Some of them concern non-decreasing solutions. A sufficient condition on the initial data for having $u_t \geq 0$ is $\Delta\phi + \phi^p - b|\nabla\phi|^q \geq 0$ (see [10, 50]). The following theorem [10, 3] establishes blowup under an additional assumption of negative initial energy, in the spirit of the results of [32] and [5] for equation (1.1).

Theorem 2.2 *Assume $q \leq 2p/(p+1)$ and $\Omega \subset \mathbb{R}^N$ (bounded or unbounded). Assume that ϕ (sufficiently regular) satisfies*

$$E(\phi) = \frac{1}{2}\|\nabla\phi\|_2^2 - \frac{1}{p+1}\|\phi\|_{p+1}^{p+1} < 0$$

and is such that $u_t \geq 0$. Moreover, suppose that $-E(\phi)/\|\phi\|_2^2$ is large enough if $q < 2p/(p+1)$, or that b is sufficiently small if $q = 2p/(p+1)$. Then $T^ < \infty$.*

In some situations, the energy assumption can be relaxed, leading to blowup of all nontrivial non-decreasing solutions [45, 46].

Theorem 2.3 *Assume $q = 2p/(p+1)$, $\Omega = \mathbb{R}^N$, $(N-2)p < N+2$, and b small enough. Suppose also that ϕ is such that $u_t \geq 0$. Then $T^* < \infty$.*

We note that initial data ϕ satisfying the requirements of Theorems 2.2 and 2.3 are shown to exist. Moreover, in case of Theorem 2.3, it is possible to find suitable ϕ such that $E(\phi) > 0$ (so that the result is not covered by Theorem 2.2).

For equation (1.1) in $\Omega = \mathbb{R}^N$ a classical result, essentially due to Fujita (see [20, 33]), asserts that no nonnegative nontrivial global solutions exist for $p \leq 1 + 2/N$, whereas both blowing-up and global positive solutions do exist if $p > 1 + 2/N$. The value $p_c = 1 + 2/N$ is thus said to be the Fujita critical exponent of the problem.

Open problem 1. Is there a Fujita critical exponent for equation (1.6) in \mathbb{R}^N when $q = 2p/(p+1)$ and b is small?

Partial facts are known about this problem. First, if $p > 1 + 2/N$, for any $b > 0$ (and any q actually), there always exist positive global solutions. This follows from a straightforward comparison argument with the global solutions of the case $b = 0$. When $q = 2p/(p+1)$ and b is large, both blowing-up and stationary positive solutions do exist. Therefore no Fujita-like result can hold in this case.

On the contrary, when $q = 2p/(p+1)$, $p \leq 1 + 2/N$ and b is small, the existence of positive global solutions is unknown (at least it is known that no positive stationary solutions exist). On account of the similarity of scaling properties between equations (1.1) and (1.6) when $q = 2p/(p+1)$, the authors of [3] conjectured the nonexistence of positive global solutions.

In one space dimension on a bounded interval, when $q \leq 2p/(p+1)$, with b small if $q = 2p/(p+1)$, it is known [10] that (1.6) admits a unique positive stationary solution v . In this case, a very simple blowup condition, which does not require the monotonicity of u , was obtained in [16].

Theorem 2.4 *Assume $\Omega = (a, b)$, $-\infty < a < b < \infty$, $q \leq 2p/(p+1)$ with b small if $q = 2p/(p+1)$. Suppose that $\phi \geq v$, $\phi \not\equiv v$, where v is the unique positive stationary solution. Then $T^* < \infty$.*

For equation (1.1) in \mathbb{R}^N , a criterion for blowup in terms of the growth of ϕ as $|x| \rightarrow \infty$ was found in [31]. The following theorem [53] improves the result of [31] by allowing any domain containing a cone, and imposing the growth condition on ϕ only in that cone. The result holds for (1.1) and for (1.6) as well.

Theorem 2.5 *Assume that $2p/(p+1) \leq q < p$ and that Ω contains a cone Ω' . There exists a constant $C = C(\Omega') > 0$ such that if ϕ satisfies*

$$\liminf_{|x| \rightarrow \infty, x \in \Omega'} |x|^{2/(p-1)} \phi(x) > C, \quad (2.1)$$

then $T^* < \infty$.

It can be proved that the decay condition (2.1) is optimal: there exist global solutions for initial data which decay like $\varepsilon|x|^{-2/(p-1)}$ when $\varepsilon > 0$ is small. Recently, a similar optimal result was obtained in [40] for a very general class of “smaller” unbounded domains, of paraboloid type. The corresponding decay condition on the initial data is related in a precise way to the growth of the domain at infinity.

Open problem 2. Does the result of Theorem 2.5 remain valid when $1 \leq q < 2p/(p+1)$?

Let us remark that all the results in §2.2 involve the limiting value $q = 2p/(p+1)$. The origin of this number can be easily understood from scaling considerations. Indeed, for $q = 2p/(p+1)$, the equation (1.6) exhibits the same scale invariance as the equation (1.1). Namely, if u solves (1.6), say, in \mathbb{R}^N , then so does $u_\alpha(t, x) \equiv \alpha^{2/(p-1)} u(\alpha^2 t, \alpha x)$. This property will play an important role in § 2.3 (self-similar solutions), and in §2.4 and §3.

2.3 Description of blowup

Several results on the blowup behavior of non-global solutions of (1.6) have been recently obtained, although still relatively little is known in comparison with the most studied case of (1.1).

The estimates of the blowup rates were proved in [11, 50, 12, 18] in the case $q < 2p/(p+1)$. We summarize the results in the following theorem.

Theorem 2.6 *Assume $q < 2p/(p+1)$ and let $u \geq 0$ be a solution of (1.6), such that $T < \infty$. The estimate*

$$C_1(T-t)^{1/(p-1)} \leq \|u(t)\|_\infty \leq C_2(T-t)^{1/(p-1)}, \quad \text{as } t \rightarrow T \quad (2.2)$$

holds in each of the following cases:

- (i) [11] $\Omega = \mathbb{R}^N$, $p \leq 1 + 2/N$;
- (ii) [50] $\Omega = \mathbb{R}^N$ or $\Omega = B_R$, u radially symmetric, $u_r \leq 0$, $u_t \geq 0$, $p < (N+2)/(N-2)_+$. Moreover this remains valid for $q = 2p/(p+1)$ and b small;
- (iii) [12] Ω convex bounded and ($u_t \geq 0$ or $p \leq 1 + 2/N$);
- (iv) [18] Ω arbitrary, $p \leq 1 + 2/(N+1)$.

This theorem shows that for $q < 2p/(p+1)$ (or =), the blowup rate is the same as for (1.1). Recall that for (1.1), the upper bound in (upper) holds for all subcritical p , i.e. $p < (N+2)/(N-2)_+$, (see [58, 19, 25], and also [34] for further recent results), whereas it may fail for large supercritical p (see [26]). Also, the lower bound in (2.2) holds for (1.1) for all $p > 1$ (see, e.g., [19]).

There are basically four different techniques to prove the upper blowup estimate in (2.2) for (1.1) (the lower bound is much easier). Three of them use some re-scaling arguments, either of elliptic or parabolic type, which means that one re-scales, respectively, only space or both space and time variables, so that the limiting equation obtained is either elliptic or parabolic. The technique of [58], which relies on elliptic re-scaling (for monotone symmetric solutions) was used (and improved) in [50]. That of [25], relying on elliptic re-scaling and energy methods, does not seem applicable here, because the equation (1.6) has no variational structure. The technique in [19], relying on maximum principle arguments, was successfully adapted in [12]. The method of [27], which relies on parabolic re-scaling and Fujita-type theorems (and was designed for problems with nonlinear boundary conditions), was used in [11, 18].

Concerning the blowup set and profile of solutions of (1.6), the following very interesting result was proved in [12].

Theorem 2.7 *Assume that Ω is a ball, u is radially symmetric and $u_r \leq 0$, $r = |x|$. Then 0 is the only blowup point and*

$$u(t, r) \leq C_\alpha r^{-\alpha} \quad \text{for all } \alpha > \alpha_0, \quad (2.3)$$

where

$$\alpha_0 = \begin{cases} 2/(p-1), & \text{if } q < 2p/(p+1), \\ q/(p-q), & \text{if } q \geq 2p/(p+1). \end{cases}$$

Furthermore, this estimate is optimal in the sense that, if in addition $N = 1$ and $u_t \geq 0$, then (2.3) holds for no $\alpha < \alpha_0$.

The proof relies in particular on nontrivial modifications of the maximum principle arguments of [19]. Recall that for (1.1), under the assumptions of Theorem 2.7, (2.3) holds for all $\alpha > 2/(p-1)$ (see [19]). Actually, the final profile is given by

$$u(T, r) \sim C(\log r)^{1/(p-1)} r^{-2/(p-1)}, \quad \text{as } r \rightarrow 0 \quad (2.4)$$

(for radially symmetric decreasing solutions, this is known in \mathbb{R}^N or on a bounded interval – see [56]). Also, observe that $q/(p-q) > 2/(p-1)$ for $q > 2p/(p+1)$. Theorem 2.7 thus indicates that the blowup profile of solutions of (1.6) is basically similar to that in (1.1) as long as $q < 2p/(p+1)$, whereas for q greater than this critical value, the gradient term induces an important effect on the profile, which becomes more singular.

Under the assumptions of case (ii) of Theorem 2.6, the following information on the blowup profile is also obtained in [50]: there exists a constant $C > 0$ (independent of u) such that

$$\frac{u(t, |y|\sqrt{T-t})}{u(t, 0)} \geq 1 - C|y|$$

for t close to T . However, this estimate is only of interest for $|y|$ small.

As for the blowup set of non-global solutions, it is proved in [12] that when $q < 2p/(p+1)$ and Ω is convex and bounded, the blowup set of any solution of (1.6) is a compact subset of Ω .

In some special cases, a further insight into the description of blowup can be gained by studying the existence of backward self-similar solutions, that is, solutions of the form

$$u(t, x) = (T-t)^{-1/(p-1)} W(x/(T-t)^m), \quad -\infty < t < T, \quad x \in \mathbb{R}^N, \quad (2.5)$$

with $m = 1/2$. From the scaling considerations of § 2.2, it is easily seen that such solutions can exist only if $q = 2p/(p+1)$. The following result is proved in [51].

Theorem 2.8 *Assume $\Omega = \mathbb{R}^N$, $q = \frac{2p}{p+1}$, and $0 < b < 2$. There exists $p_0 = p_0(b, n) > 1$, such that for all p with $1 < p < p_0$, the equation (1.6) has a solution of the form (2.5) with $m = 1/2$, where W is positive, C^2 , radially symmetric and radially decreasing in \mathbb{R}^N .*

Moreover, for all such solution, there exists a constant $C > 0$ such that the corresponding function W satisfies $\lim_{|x| \rightarrow \infty} |x|^{2/(p-1)} W(x) = C$.

In particular, u blows up at the single point $x = 0$, and it holds

$$u(T, x) = C|x|^{-2/(p-1)}, \quad \text{for all } x \neq 0.$$

It is to be noted that no nontrivial, backward, self-similar solutions exist for $b = 0$ and p subcritical. Also the blowup profile above is different from all the profiles known for (1.1). Namely, it is slightly less singular, by a logarithmic factor, than the corresponding profile for (1.1) (see formula (2.4) above).

Comparison of Theorems 2.7 and 2.8 yields the interesting and a bit surprising observation that the gradient term can have different effects on the blowup profile: when the perturbation is mild ($q = 2p/(p+1)$ in Theorem 2.8), slightly less singular profile; when the perturbation is strong ($2p/(p+1) < q < p$ in Theorem 2.7), more singular profile.

Different kinds of self-similar blowup behaviors, and a description of the blowup set as well, were obtained in the case $b < 0$, $q = 2$. Note that the gradient term now has a positive sign, enhancing blowup. Also, the transformation $v = e^u - 1$ changes the first equation in (1.6) into the equation $v_t - \Delta v = (1+v) \log^p(1+v)$. One has single-point blowup if $1 < p < 2$, regional blowup if $p = 2$, and global blowup if $p > 2$ (see [30, 29, 21, 22]).

The authors of [29] interpret the above result in the following way. While the term u^p alone would force the solution to develop a spike at the maximum point, hence causing single point blowup, the gradient term tends to push up the steeper parts of the profile $u(t, \cdot)$. This enhances regional or even global blowup, the influence of the gradient term becoming more important as the value of p decreases.

Concerning self-similar profiles, in the case $b < 0$, $q = 2$, for radial solutions in \mathbb{R}^N it is proved in [21, 22] that blowup solutions behave asymptotically like a self-similar solution w of the following Hamilton-Jacobi equation without diffusion:

$$w_t = |\nabla w|^2 + w^p,$$

with w having the form (2.5), for $m = (2-p)/2(p-1)$. Note that this kind of self-similar behavior is quite different from that in Theorem 2.8 above (or from those known for $b = 0$ and p super-critical); indeed, m describes the range $(-\infty, 1/2)$ for $p \in (1, \infty)$.

Let us mention that for the related equation with exponential source

$$u_t - \Delta u = e^u - |\nabla u|^2, \tag{2.6}$$

some results on blowup sets and profiles were obtained in [7]. The analysis therein is strongly based on the observation that the transformation $v = 1 - e^{-u}$ changes (2.6) into the linear equation $v_t - \Delta v = 1$.

Open problem 3. The value of p_0 in Theorem 2.8 is not explicitly known (because the proof involves a limiting argument). Can one specify the allowable values of p , or even extend the result to all $p > 1$, and also to all $b > 0$? On the other hand, is the self-similar solution unique for each value of the parameters? Is the self-similar profile of Theorem 2.8 representative of all blowup behaviors when $q = 2p/(p+1)$, or do there exist different profiles?

Open problem 4. What is the blowup rate when $2p/(p+1) < q < p$? On the basis of the blowup profiles found in [12] in that range of parameters, and of the parabolicity of the problem, one could conjecture a rate of the order $(T-t)^{-q/2(p-q)}$, but there is no evidence that this guess is true.

2.4 Behavior of global solutions

An obvious property of equation (1.6) in bounded domains is the stability of the solution $u \equiv 0$: for all (nonnegative) data of sufficiently small L^∞ norm, the solution is global, bounded, and decays exponentially to 0. This follows, via the comparison principle, from the same well-known property for equation (1.1) (see, e.g., [28]).

Even for $\Omega = \mathbb{R}^N$, some kind of stability was found in [44] in the case $q = 2p/(p+1)$, regardless of the sign and of the size of b . It is shown there that the solution of (1.6) is global, decays to 0, and is asymptotically self-similar, whenever the initial data is small with respect to a special norm related to the heat semigroup. On the other hand, exact self-similar global solutions, of the form

$$u(t, x) = (t+1)^{-1/(p-1)} U(|x|(t+1)^{-1/2})$$

are constructed in [55] by different methods (shooting arguments for the corresponding ODE).

The next natural question concerning global solutions is whether they are bounded or not and, if they are, whether they satisfy a priori estimates for all $t \geq 0$. This question has received much attention in the case of (1.1): roughly speaking, the answer is yes for sub-critical p ($(N-2)p < N+2$), and no otherwise. For problem (1.6), the following result was recently obtained in [39].

Theorem 2.9 *Assume $q < 2p/(p+1)$ and either*

$$1 < p \leq 1 + \frac{2}{n+1}, \quad \text{or} \quad \Omega = \mathbb{R}^n \quad \text{and} \quad 1 < p \leq 1 + \frac{2}{n}.$$

Suppose that $\phi \in C_b^1(\bar{\Omega})$, $\phi \geq 0$, $\phi|_{\partial\Omega} = 0$ and that $T^ = \infty$. Then u is uniformly bounded for $t \geq 0$ and satisfies the a priori estimate*

$$\sup_{t \geq 0} \|u(t)\|_{C^1} \leq C(\|\phi\|_{C^1}),$$

where $C(\|\phi\|_{C^1})$ remains bounded for $\|\phi\|_{C^1}$ bounded.

In the case of (1.1), the known techniques for proving boundedness and a priori estimates of global solutions make essential use of the existence of a Liapunov functional, namely the energy

$$E(t) = \frac{1}{2} \|\nabla u(t)\|_2^2 - \frac{1}{p+1} \|u(t)\|_{p+1}^{p+1},$$

and no Liapunov functional is known for problem (1.6) in general. The proof of Theorem 2.9 thus relies on a different method based on re-scaling and Fujita-type theorems, in the spirit of [27] and [18]. We refer to [38] and [39] for related questions for other gradient-dependent nonlinearities. Due to the method of proof, the result of Theorem 2.9 is restricted to $p \leq 1 + (2/N)$. In the special case of time-increasing solutions however, the energy functional decreases along the trajectories, which enables one to obtain the following result [16, 45, 46].

Theorem 2.10 *Assume $(N - 2)p < N + 2$, and either $q < 2p/(p + 1)$ and Ω bounded, or $q = 2p/(p + 1)$ and b small. Suppose that ϕ is such that $u_t \geq 0$ and $T^* = \infty$. Then u is uniformly bounded for $t \geq 0$ and converges in L^∞ to a stationary solution.*

The scaling properties of the equation (1.6) (see §2.2) suggest that both re-scaling and energy arguments require $q \leq 2p/(p + 1)$. It turns out that this is a genuine restriction. Indeed, the following result (see [13], Theorem 3.3 (iv) and its proof) shows that, even in 1 dimension on a bounded interval, there exist *unbounded* non-decreasing global solutions for certain values of b , whenever $p > q = 2$. (Note that $2p/(p + 1) \rightarrow 2$ as $p \rightarrow \infty$.)

Theorem 2.11 *Assume $\Omega = (0, L)$, $0 < L < \infty$, $p > q = 2$. For some $b = b_0(L) > 0$, there exist (infinitely many) ϕ such that $u_t \geq 0$, $T^* = \infty$, and $\lim_{t \rightarrow \infty} \|u(t)\|_\infty = \infty$.*

More precisely, it is proved in [13] that $u(t)$ approaches the (unique) *singular* stationary solution v_s as $t \rightarrow \infty$, whenever ϕ lies between the maximal regular stationary solution and v_s . Further sharp stability/instability results for equilibria of (1.6) are given in [13] for $q = 2$ and $N = 1$.

Open problem 5. What can be said about boundedness of global solutions for $2p/(p + 1) < q < p$, $q \neq 2$?

The results in the next section for $q \geq p$ will confirm that, unlike the situation for (1.1), the existence of unbounded global solutions is a quite general phenomenon in presence of a dissipative gradient term.

3 The case $q \geq p$

3.1 Geometry of Ω and existence of unbounded solutions

When $q \geq p$, it was proved in [16, 37] that for *bounded* domains, blowup cannot occur, neither in finite nor in infinite time. Starting from this result, the study of the case $q \geq p$ in arbitrary unbounded domains was undertaken in [53]. It turns out that the geometry of Ω at infinity plays a determinant role in the problem. The relevant concept is the *inradius* of Ω :

$$\rho(\Omega) = \sup\{r > 0; \Omega \text{ contains a ball of radius } r\} = \sup_{x \in \Omega} \text{dist}(x, \partial\Omega).$$

The following result [53, 47] gives a characterization in terms of $\rho(\Omega)$ of the domains Ω in which all solutions of (1.6) are global and bounded for $q \geq p$.

Theorem 3.1 *Assume $q \geq p$.*

(i) *If $\rho(\Omega) < \infty$, then for all ϕ , the solution u of (1.6) is global and bounded.*

(ii) If $\rho(\Omega) = \infty$, then there exists ϕ such that the solution u of (1.6) is unbounded (with either $T^* < \infty$ and $\limsup_{t \rightarrow T^*} \|u(t)\|_\infty = \infty$, or $T^* = \infty$ and $\lim_{t \rightarrow \infty} \|u(t)\|_\infty = \infty$).

(See paragraph after Theorem 3.6 below for some ideas on the proof.) One important property of the inradius, is that its finiteness is also equivalent to the validity of the *Poincaré inequality* in $W_0^{1,k}(\Omega)$, $1 \leq k < \infty$:

$$\|v\|_k \leq C_k(\Omega) \|\nabla v\|_k, \quad \forall v \in W_0^{1,k}(\Omega). \quad (3.1)$$

(The equivalence is true under mild regularity assumptions on Ω , for instance if Ω satisfies a uniform exterior cone condition – see [47] and the references therein for details.)

As an illustration, we have $\rho(\Omega) < \infty$ if Ω is contained in a strip, and $\rho(\Omega) = \infty$ if Ω contains a cone. A typical example of "largest" possible domains satisfying $\rho(\Omega) < \infty$ is the complement of a periodic net of balls

$$\Omega = \mathbb{R}^N \setminus \bigcup_{z \in \mathbf{Z}^N} \overline{B}(Rz, \epsilon), \quad 0 < \epsilon < R/2.$$

In the opposite direction, the "smallest" possible kind of unbounded domain for which $\rho(\Omega) = \infty$ is the reunion of a sequence of disjoint balls of growing up radii, connected by thin bridges.

Using the above relation between $\rho(\Omega)$ and the Poincaré inequality, it is proved in [53] that in case (i) of Theorem 3.1, $u(t, \cdot)$ decays exponentially to 0 in $L^k(\Omega)$, for large $k \leq \infty$, as $t \rightarrow \infty$. This happens in each of the following situations:

- (a) $b > b_0(\Omega) > 0$ large enough and ϕ is any initial data;
- (b) $b > 0$ and $\|\phi\|_k$ is sufficiently small (independent of b).

By the way, let us mention that the stability of the 0 solution for equation (1.1) in unbounded domains is also strongly related to $\rho(\Omega)$ (see [47, 48]).

Theorem 3.1 (ii) does not conclude whether blowup occurs in finite or infinite time. Some cases of global unbounded solutions – i.e. $\|u(t)\|_\infty \rightarrow \infty$ as $t \rightarrow \infty$ – will be described in §3.3. One of the more interesting questions on equation (1.6) then remains the following:

Open problem 6. Can *finite time* blowup occur when $q \geq p$? This is unknown even for $\Omega = \mathbb{R}^N$ (note that the existence of a blowing-up solution in some domain Ω would imply the same conclusion in \mathbb{R}^N by comparison).

However, the following result [53] shows that in any domain, finite time blowup cannot occur if $q \geq p$ and ϕ is compactly supported.

Theorem 3.2 Assume $q \geq p$ and $\Omega \subset \mathbb{R}^N$ (bounded or unbounded). If ϕ is compactly supported in \mathbb{R}^N , then $T^* = \infty$.

Actually, the conclusion of Theorem 3.2 remains valid whenever ϕ decays exponentially in at least one direction [53].

3.2 Critical blowup exponents

As a consequence of Theorems 2.1 and 3.1, it follows that the critical blowup exponent for problem (1.6) is given by $q = p$, whenever $\rho(\Omega) < \infty$.

For bounded domains, this was conjectured in [36], where the conjecture was verified in the case when Ω is a bounded interval and b is small.

Corollary 3.3 *Assume $\rho(\Omega) < \infty$.*

(i) *If $p > q$, then there exists ϕ such that u blows up in finite time.*

(ii) *If $q \geq p$, then for all ϕ , u is global and bounded.*

If one restricts to *compactly supported* initial data, it follows from Theorems 2.1 and 3.2 that the critical blowup exponent is still given by $q = p$ for *any* domain, including \mathbb{R}^N .

Corollary 3.4 *Assume $\Omega \subset \mathbb{R}^N$ (bounded or unbounded).*

(i) *If $p > q$, then there exists ϕ , compactly supported, such that u blows up in finite time.*

(ii) *If $q \geq p$, then for all ϕ compactly supported, u is global (possibly unbounded).*

3.3 Unbounded global solutions

Under additional assumptions on Ω , one can prove that some unbounded **global** solutions do actually exist [53].

Theorem 3.5 *Assume that $q \geq p$ and that Ω contains a cone. Then there exists ϕ , compactly supported, such that the solution u of (1.6) satisfies $T^* = \infty$ and*

$$\lim_{t \rightarrow \infty} \|u(t)\|_{\infty} = \infty.$$

If $\Omega = \mathbb{R}^N$, one further obtains solutions which blow up *everywhere* in infinite time [53].

Theorem 3.6 *Assume $q \geq p$ and $\Omega = \mathbb{R}^N$. Then there exists ϕ , compactly supported, such that the solution u of (1.6) satisfies $T^* = \infty$ and*

$$\forall x \in \mathbb{R}^N, \lim_{t \rightarrow \infty} u(t, x) = \infty.$$

Note that the conclusions of Theorems 3.5 and 3.6 remain true for large sets of initial data, namely for any compactly supported initial data lying above ϕ (this follows from Theorem 3.2 and the comparison principle).

The proofs of Theorems 3.5 and 3.6 rely on the construction of ordered, global, unbounded sub- and supersolutions. The main difficulty in constructing the subsolution comes from the gradient term, whose power is larger than that of

the source term. The idea is to build a radial expanding wave, whose maximum at the origin grows up to ∞ as $t \rightarrow \infty$, while its gradient remains uniformly bounded. As for supersolutions, a pair of them is constructed under the form of traveling waves, propagating in two opposite directions. These supersolutions prevent u from blowing up in finite time.

The subsolutions above are also an essential ingredient for proving the existence of unbounded global solutions when $\rho(\Omega) = \infty$ (see Theorem 3.1 (ii)). More precisely, one superposes a sequence of expanding wave subsolutions, whose supports eventually fill a collection of balls of arbitrary large radii, included in Ω .

Open problem 7. Does there exist unbounded global solutions whenever $\rho(\Omega) = \infty$ and $q \geq p$?

Open problem 8. What is the precise grow-up rate of $\|u(t)\|_\infty$ for unbounded global solutions of (1.6) ? For the solutions constructed in the proof of Theorem 3.6, we only have the rough estimate $C_1 t \leq \|u(t)\|_\infty \leq C_2 e^{C_3 t}$, as $t \rightarrow \infty$.

Global blowup, as described in Theorem 3.6, can occur only for $\Omega = \mathbb{R}^N$. Indeed, define the blowup set of u as

$$E = \{x_0 \in \overline{\Omega} \cup \{\infty\}; \exists x_n \rightarrow x_0, \exists t_n \rightarrow T^*, u(t_n, x_n) \rightarrow \infty\}.$$

The blowup set then satisfies the following alternative [53].

Theorem 3.7 *Assume $q \geq p$ and $\Omega \subset \mathbb{R}^N$ (unbounded). Assume that ϕ is such that u is unbounded, with either $T^* < \infty$ or $T^* = \infty$.*

(i) *If $\Omega \neq \mathbb{R}^N$, then $E = \{\infty\}$.*

(ii) *If $\Omega = \mathbb{R}^N$, then either $E = \mathbb{R}^N \cup \{\infty\}$ or $E = \{\infty\}$.*

Open problem 9. Does there exist ϕ such that $E = \{\infty\}$ when $q \geq p$ and $\Omega = \mathbb{R}^N$? Theorem 3.6 provides some ϕ such that $\Omega = \mathbb{R}^N$ and $E = \mathbb{R}^N \cup \{\infty\}$.

Finally, we have the analogue of Theorem 2.5 when $q \geq p$, except that it is not known whether $T^* = \infty$ or $T^* < \infty$ [53].

Proposition 3.8 *Assume that $q \geq p$ and that Ω contains a cone Ω' . There exists a constant $C = C(\Omega') > 0$ such that if ϕ satisfies*

$$\liminf_{|x| \rightarrow \infty, x \in \Omega'} |x|^{2/(p-1)} \phi(x) > C,$$

then the solution u of (1.6) is unbounded (with $T^ \leq \infty$).*

4 Stationary states

The stationary states of (1.6) were thoroughly investigated in [10, 3, 9, 17, 13, 57, 43, 35]. We conclude this survey by a brief account of results on (positive classical) stationary solutions of (1.6), i.e. solutions of the elliptic problem

$$\begin{aligned}\Delta u + u^p - b|\nabla u|^q &= 0, & x \in \Omega \\ u(x) &= 0, & x \in \partial\Omega.\end{aligned}\tag{4.1}$$

The best results available concern the case when $\Omega = \mathbb{R}^N$ or Ω is a ball B_R . By the results of [24], any positive solution to (4.1) on \mathbb{R}^N or on a ball must be radial. Searching solutions of (4.1) thus leads to an ODE. Let $p_S = (N+2)/(N-2)$, with $p_S = \infty$ if $N \leq 2$. For the elliptic problem associated with (1.1) ((4.1) with $b = 0$), which is classically known as Lane-Emden's equation, it is well-known that positive solutions exist on a ball (resp. on \mathbb{R}^N) if and only if $p < p_S$ (resp. $p \geq p_S$).

The existence and non-existence properties of solutions to (4.1) in a given domain Ω exhibit an interesting and sharp dependence on the parameters p , q , b . This dependence is even more crucial than that of the blowup properties for the evolution equation. As a consequence, the picture is already somehow complicated, even though some ranges of the parameters are not yet completely explored and several questions remain open.

Without getting into too much detail, we here attempt to summarize the situation. In what follows, by "existence" (or "nonexistence"), we understand the existence of at least one classical positive solution of (4.1) on Ω .

First consider the case $\Omega = \mathbb{R}^N$.

- (i) If $p > p_S$: existence (for all $q > 1$) [43];
- (ii) If $p = p_S$: existence if and only if $q < p$ [43];
- (iii) If $p < p_S$:
 - (iii1) existence if $q < 2p/(p+1)$ or $q = 2p/(p+1)$ and b is large enough [10];
 - (iii2) nonexistence if $p \leq N/(N-2)_+$ and $q > 2p/(p+1)$ [43];
 - (iii3) nonexistence if $p < N/(N-2)_+$ and $q = 2p/(p+1)$ with b small [10, 17, 57];
 - (iii4) nonexistence if $N \geq 3$, $N/(N-2) < p < p_S$ and $q > \bar{q}$, for some (explicitly determined) $\bar{q} \in (2p/(p+1), p)$ [43].

Moreover, there is numerical evidence that solutions exist for some values of q between $2p/(p+1)$ and \bar{q} [42].

Next we turn to the case when Ω is a ball B_R in \mathbb{R}^N . Contrary to the case $\Omega = \mathbb{R}^N$, the super-critical range $p > p_S$ is hardly explored. We thus classify the results in terms of the value of q as a function of p .

- (i) If $1 < q < 2p/(p+1)$ and $p < p_S$: existence [10];

- (ii) If $q = 2p/(p+1)$;
 - (ii1) if $p \geq p_S$ [43] or if $p < p_S$ and b is large [10]: nonexistence;
 - (ii2) if $p \leq N/(N-2)_+$ and b is small: nonexistence [10, 17, 57];
- (iii) If $2p/(p+1) < q < p$ and $p < p_S$: existence for b small [10] and nonexistence for b large [9];
- (iv) If $q \geq p > 1$: existence if and only if $b \leq b_0$, for some $b_0 = b_0(p, N) > 0$ [37, 57];

Some partial results are known when Ω is an arbitrary bounded domain with smooth boundary (these results are obtained via topological degree theory).

- (i) If $p < p_S$: existence for b small enough [57];
- (ii) If $q \geq p > 1$: existence if and only if $b \leq b_0$, for some $b_0 = b_0(p, N)$ [37, 57];

Last, we mention that some results on the number of stationary states can be found in [10, 9, 13, 57, 35, 43].

If we analyze the results above, we find several “critical” values of the parameters with respect to the existence of positive stationary solutions. The value $p = p_S$ is critical in the case of the whole space, as it is for the equation without gradient term. Concerning q , there are at least two critical values $q = 2p/(p+1)$ and $q = p$. There might possibly exist a third critical value $\bar{q} \in (2p/(p+1), p)$, in which case $N/(N-2)$ would also be critical for p when $N \geq 3$. (Incidentally, when $q = 2p/(p+1)$, it happens that $p \geq N/(N-2)_+$ is a necessary and sufficient condition for the existence of singular stationary solutions of the form $C|x|^{-r}$ for all $b > 0$.) Moreover, the size of b can also be determinant when $q \geq 2p/(p+1)$.

In comparison with these properties, it is interesting to recall from § 3.2 that $q = p$ is the only critical blowup exponent for the evolution problem (at least in bounded domains), and that the values of $p > 1$ and $b (> 0)$ do not play much role in global existence or nonexistence.

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