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A NOTE ON NONLINEAR ELLIPTIC SYSTEMS INVOLVING MEASURES

Shulin Zhou

Abstract. We study the existence and regularity of solutions to nonlinear elliptic systems whose right-hand side is a measure-valued function. Using a sign condition instead of structure conditions, we obtain the same as those presented by Dolzmann, Hungerbühler and Müller in [1].

1. Introduction

In [1], Dolzmann, Hungerbühler and Müller use Young measures and the approximate derivatives to study the existence and regularity of solutions to nonlinear elliptic systems of the form

$$-\operatorname{div}\sigma\big(x,u(x),Du(x)\big)=\mu \quad \text{in } \Omega\,, \tag{1.1}$$

$$u=0 \quad \text{on } \partial\Omega\,, \tag{1.2}$$

with measure-valued right hand side on an open bounded domain Ω in $\mathbb{R}^n$. They obtained a series of complete and optimal results which are very important when the exponent $p\in(1,2-1/n)$. They assume that the matrix σ satisfies the following hypotheses:

(H0) (continuity) $\sigma:\Omega\times\mathbb{R}^m\times\mathbb{M}^{m\times n}\to\mathbb{M}^{m\times n}$ is a Carathéodory function.

(H1) (monotonicity) For all $x\in\Omega$, $u\in\mathbb{R}^m$ and F, $G\in\mathbb{M}^{m\times n}$,

$$\big(\sigma(x,u,F)-\sigma(x,u,G)\big):(F-G)\ge 0.$$

(H2) (coercivity and growth) There exist constants $c_1,c_3>0$, $c_2\ge 0$ and p,q with $1<p<n$ and $q-1<(p-1)n/(n-1)$ such that for all $x\in\Omega$, $u\in\mathbb{R}^m$ and $F\in\mathbb{M}^{m\times n}$,

$$\sigma(x,u,F):F\ge c_1|F|^p-c_2,$$
$$|\sigma(x,u,F)|\le c_3|F|^{q-1}+c_3.$$

(H3) (structure condition) For all $x\in\Omega$, $u\in\mathbb{R}^m$, $F\in\mathbb{M}^{m\times n}$ and $M\in\mathbb{M}^{m\times m}$ of the form $M=\mathrm{Id}-a\otimes a$ with all $a\in\mathbb{R}^m$, $|a|\le 1$,

$$\sigma(x,u,F):MF\ge 0.$$

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Here $\mathbb{M}^{m\times n}$ denotes the space of real $m\times n$ matrices equipped with the inner product $M:N=\sum_{i=1}^m\sum_{j=1}^n M_{ij}N_{ij}$ and the tensor product $a\otimes b$ of two vectors $a,b\in\mathbb{R}^m$ is defined to be the $m\times m$ matrix of entries $(a_ib_j)_{i,j}$, with $i,j=1,\dots,m$.

For equation (1.1) ($m=1$), the hypotheses (H0)-(H2) are sufficient to prove the existence and regularity results. However, for system (1.1) ($m>1$) it seems hopeless to obtain such results without further assumptions. Therefore, the structure condition (H3), often called the angle condition, is assumed in [1] and [2]. In this note we assume that the matrix σ satisfies another type of assumption

(H3)' (sign condition) For all $i=1,\dots,m$,

$$\sigma_i(x,u,F)\cdot F_i\geq 0\,.$$

Here σ_i and F_i are respectively the i-th row vectors of σ, F, and the dot denotes the inner product in $\mathbb{R}^n$.

This indicates that system (1.1) is weakly-coupled. Using (H3)' to replace (H3) and assuming (H0)-(H2), we can prove all the results in [1] and [2]. To make this note short we are only concerned with system (1.1).

In [1], a notion of solution where the weak derivative Du is replaced by the approximate derivative $\operatorname{ap}Du$ was introduced. This kind of notion is useful when one considers the case $1<p\leq 2-1/n$ since solutions of system (1.1) in general do not belong to the Sobolev space $W^{1,1}$. We introduce the definition of the solution in [1] for the sake of convenience.

Definition 1. *A measurable function $u:\Omega\to\mathbb{R}^m$ is called a solution of system (1.1) if*

(i) u is almost everywhere approximately differentiable,
(ii) $\eta\circ u\in W^{1,1}(\Omega,\mathbb{R}^m)$ for all $\eta\in C_0^1(\mathbb{R}^m,\mathbb{R}^m)$,
(iii) $\sigma\big(x,u(x),\operatorname{ap}Du(x)\big)\in L^1(\Omega;\mathbb{M}^{m\times n})$,
(iv) the system

$$-\operatorname{div}\sigma\big(x,u(x),\operatorname{ap}Du(x)\big)=\mu$$

holds in the sense of distributions.

Moreover, u is said to satisfy the boundary condition (1.2) if $\eta\circ u\in W_0^{1,1}(\Omega;\mathbb{R}^m)$ for all $\eta\in C_0^1(\mathbb{R}^m,\mathbb{R}^m)\cap L^\infty(\mathbb{R}^m,\mathbb{R}^m)$ that satisfy $\eta\equiv\mathrm{Id}$ on $B(0,\rho)$ for some $\rho>0$ and $|D\eta(y)|\leq C(1+|y|)^{-1}$ for some constant $C<\infty$.

We will prove the following conclusion.

Theorem 2. *Suppose (H0), (H1), (H2) and (H3)' hold, and that one of the following conditions is satisfied.*

(i) $F\mapsto\sigma(x,u,F)$ is a C^1 function.
(ii) There exists a function $W:\Omega\times\mathbb{R}^m\times\mathbb{M}^{m\times n}\to\mathbb{R}$ such that $\sigma(x,u,F)=\frac{\partial W}{\partial F}(x,u,F)$ and $F\mapsto W(x,u,F)$ is convex and C^1.
(iii) σ is strictly monotone with respect to F, i.e., σ is monotone and $\big(\sigma(x,u,F)-\sigma(x,u,G)\big):(F-G)=0$ implies $F=G$.

Also assume that μ is an $\mathbb{R}^m$-valued Radon measure on Ω with finite mass. Then Problem (1.1)-(1.2) has a solution in the sense of Definition 1, which satisfies the weak Lebesgue space estimate

$$\|u\|^*_{L^{s^*,\infty}(\Omega)}+\|\operatorname{ap}Du\|^*_{L^{s,\infty}(\Omega)}\leq C(c_1,c_2,\|\mu\|_{\mathcal{M}},\operatorname{meas}\Omega)$$

where

$$s = \frac{n}{n-1}(p-1), \quad s^* = \frac{ns}{n-s} = \frac{n}{n-p}(p-1)\,.$$

Theorem 2 remains true if we assume that (H0), (H2), (H3)' and the matrix σ is strictly p-quasi-monotone, see Corollary 4 in [1]. We will follow the same procedure as in [1] to prove Theorem 2. Instead of writing all details of the proof, we will only prove Lemmas 10 and 11 in [1] where the structure condition (H3) has been used. Actually we will only show how to prove the essential parts of Lemmas 10 and 11 when we use (H3)' to replace (H3). Our main idea is to choose slightly different test functions.

The relationship between (H3) and (H3)' will also be established in this section. In the scalar case $m = 1$, (H3) or (H3)' is redundant. In the vector case $m = 2$, (H3) implies (H3)' by choosing the vector $a = (0, 1) \in \mathbb{R}^2$ or $a = (1, 0) \in \mathbb{R}^2$ in (H3). Nevertheless there seems no direct implication between (H3) and (H3)' when $m \geq 3$. To our knowledge, there are two known examples satisfying (H0)-(H3).

Example 3.

$$\sigma(x, u, F) = a(u)|F|^{p-2}F$$

where $a : \mathbb{R}^m \to \mathbb{R}$ is a continuous function, which is bounded from above and below by positive constants.

Example 4.

$$\sigma(x, u, F) = \big((FA) : F\big)^{(p-2)/2} FA$$

where $A = A(x, u) : \Omega \times \mathbb{R}^m \to \mathbb{M}^{n\times n}$ is a symmetric matrix satisfying the ellipticity condition $\nu_1|\zeta|^2 \leq \sum_{i,j=1}^n A_{ij}(x, u)\zeta_i\zeta_j \leq \nu_2|\zeta|^2$ with constants $\nu_1, \nu_2 > 0$.

It is not difficult to verify that Examples 3 and 4 satisfy the sign condition (H3)'. Next, we present a linear elliptic system which satisfies (H0)-(H2) and (H3)', but does not satisfy (H3). To some extent, this example shows the restriction of the structure condition (H3). Since the sign condition (H3)' requires that the components of the matrix σ can be only weakly-coupled, this shows the limitation of (H3)'. Therefore, it would be interesting to prove the same results in [1] and [2] under a weaker condition than (H3) and (H3)'.

Example 5. *Let*

$$F = \begin{pmatrix} F_{11} & F_{12} \\ F_{21} & F_{22} \end{pmatrix} \in \mathbb{M}^{2\times 2}, \quad \sigma(F) = \begin{pmatrix} \varepsilon F_{11} & \varepsilon^\alpha F_{12} \\ F_{21} & F_{22} \end{pmatrix} \in \mathbb{M}^{2\times 2}$$

with $1 \leq \alpha$, $0 < \varepsilon \leq 1/5$.

It is obvious that $\sigma(F)$ satisfy (H0), (H1), (H2) and (H3)'. We will show that $\sigma(F)$ does not satisfy (H3).

Choosing $a = (\varepsilon^{1/2}, (1-\varepsilon)^{1/2}) \in \mathbb{R}^2$, we have $|a| = 1$ and

$$M = \mathrm{Id} - a \otimes a = \begin{pmatrix} 1-\varepsilon & -\varepsilon^{1/2}(1-\varepsilon)^{1/2} \\ -\varepsilon^{1/2}(1-\varepsilon)^{1/2} & \varepsilon \end{pmatrix} \in \mathbb{M}^{2\times 2}.$$

Now choosing $F = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$, we have $\sigma(F) = \begin{pmatrix} \varepsilon & \varepsilon^\alpha \\ 1 & 1 \end{pmatrix}$ and

$$MF = \begin{pmatrix} 1-\varepsilon-\varepsilon^{1/2}(1-\varepsilon)^{1/2} & 1-\varepsilon-\varepsilon^{1/2}(1-\varepsilon)^{1/2} \\ \varepsilon-\varepsilon^{1/2}(1-\varepsilon)^{1/2} & \varepsilon-\varepsilon^{1/2}(1-\varepsilon)^{1/2} \end{pmatrix}.$$

Therefore,

$$\begin{aligned}\sigma(F):MF &= (\varepsilon+\varepsilon^{\alpha})\big(1-\varepsilon-\varepsilon^{1/2}(1-\varepsilon)^{1/2}\big)+2\big(\varepsilon-\varepsilon^{1/2}(1-\varepsilon)^{1/2}\big)\\ &< \varepsilon+\varepsilon^{\alpha}+2\varepsilon-2\varepsilon^{1/2}(1-\varepsilon)^{1/2}\\ &< 4\varepsilon-2\varepsilon^{1/2}(1-\varepsilon)^{1/2}\\ &< 2\varepsilon^{1/2}\big(2\varepsilon^{1/2}-(1-\varepsilon)^{1/2}\big)\le 0\,,\end{aligned}$$

which implies that (H3) does not hold for this example.

2. Key Lemmas

In [1], the following *spherical truncation function*

$$T_\alpha(y)=\min\{1,\frac{\alpha}{|y|}\}y \tag{2.1}$$

is used with $\alpha>0$ and $y=(y_1,\dots,y_m)\in\mathbb{R}^m$. Roughly speaking, $T_\alpha(u)$ is used as a test function for system (1.1) in [1]. Direct calculation shows that

$$DT_\alpha(y)=\begin{cases}\mathrm{Id}, & \text{for } |y|\le\alpha,\\ \frac{\alpha}{|y|}(\mathrm{Id}-\frac{y}{|y|}\otimes\frac{y}{|y|}), & \text{for } |y|>\alpha.\end{cases}$$

From this viewpoint, it is easy to understand why the structure condition (H3) is assumed in [1] and [2]. In this note we introduce the following *cubic truncation function*

$$\begin{aligned}\mathcal{T}_\alpha(y)&=\big(T_\alpha(y_1),\dots,T_\alpha(y_m)\big)\\ &=\big(\max\{-\alpha,\min\{\alpha,y_1\}\},\dots,\max\{-\alpha,\min\{\alpha,y_m\}\}\big)\end{aligned} \tag{2.2}$$

to build up some simpler test functions when the sign condition (H3)' is satisfied.

To prove Theorem 2, we only need to prove Lemmas 10 and 11 in [1] as we explained in the introduction. First we recall

Lemma 10. *Let $\Omega\subset\mathbb{R}^n$ be an open set and $f\in L^1(\Omega;\mathbb{R}^m)$. Assume that σ satisfies (H2) and (H3)' with $p=q$ and that $u\in W_0^{1,p}(\Omega;\mathbb{R}^m)$ is a solution of*

$$-\operatorname{div}\sigma\big(x,u(x),Du(x)\big)=f(x) \tag{2.3}$$

in the sense of distributions. Then

$$u\in L^{s^*,\infty}(\Omega;\mathbb{R}^m),\quad Du\in L^{s,\infty}(\Omega;\mathbb{M}^{m\times n})$$

where

$$s=\frac{n}{n-1}(p-1),\quad s^*=\frac{ns}{n-s}=\frac{n}{n-p}(p-1).$$

Moreover,

$$\|u\|^*_{L^{s^*,\infty}(\Omega)}+\|Du\|^*_{L^{s,\infty}(\Omega)}\le C(c_1,c_2,\|f\|_{L^1},\operatorname{meas}\Omega). \tag{2.4}$$

Proof. In the weak formulation of (2.3) we use the test function $\mathcal{T}_\alpha(u)$ to replace $T_\alpha(u)$ in [1]. Then we have

$$\sum_{i=1}^{m} \int_{|u_i|\le\alpha} \sigma_i(x,u,Du)\cdot Du_i\,dx = \int_\Omega f\cdot\mathcal{T}_\alpha(u)\,dx \le m\alpha\|f\|_{L^1}.$$

By (H2) and (H3)', we obtain

$$\begin{aligned}\int_{\max\{|u_1|,\dots,|u_m|\}\le\alpha}(c_1|Du|^p-c_2)\,dx &\le \int_{\max\{|u_1|,\dots,|u_m|\}\le\alpha}\sigma(x,u,Du):Du\,dx\\ &\le \sum_{i=1}^{m}\int_{|u_i|\le\alpha}\sigma_i(x,u,Du)\cdot Du_i\,dx,\end{aligned}$$

which implies

$$\int_{|u|\le\alpha}|Du|^p\,dx \le \int_{\max\{|u_1|,\dots,|u_m|\}\le\alpha}|Du|^p\,dx \le C(\alpha\|f\|_{L^1}+\operatorname{meas}\Omega). \tag{2.5}$$

Actually (2.5) is exactly (4.5) in [1], which is the starting point for proving (2.4). Now following the same procedure as in [1], we obtain (2.4).

And now we prove Lemma 11 in [1]. It contains a div-curl inequality, which is the crucial ingredient for the argument in [1].

Let $f^k(x):\Omega\to\mathbb{R}^m$ denote a bounded sequence in $L^1(\Omega)$. Suppose that $u^k\in W^{1,1}(\Omega;\mathbb{R}^m)$ is the weak solution of the system

$$-\operatorname{div}\sigma\big(x,u^k(x),\,Du^k(x)\big)=f^k(x) \tag{2.6}$$

with $Du^k\in L^r_{\rm loc}(\Omega;\mathbb{M}^{m\times n})$ and $\sigma^k(x)=\sigma\big(x,u^k(x),Du^k(x)\big)\in L^{r'}_{\rm loc}(\Omega;\mathbb{M}^{m\times n})$ for some $r\in(1,\infty)$. Suppose further that (1) there exist an $s>0$ such that $\int_\Omega|Du^k|^s\,dx\le C$ uniformly in k, (2) σ^k is equi-integrable, (3) $u^k\to u$ in measure and u is almost everywhere approximately differentiable.

And we may assume that $\{Du^k\}$ generates a Young measure ν (see [3]) such that ν_x is a probability measure for almost every $x\in\Omega$, see Theorem 5 (iii) in [1]. Now we recall

Lemma 11. *Suppose the sequence $\{u^k\}$ is constructed as above. Then (after passage to a subsequence) the sequence $\sigma^k(x)$ converges weakly to $\bar\sigma(x)$ in $L^1(\Omega)$ where*

$$\bar\sigma(x)=\langle\nu_x,\sigma(x,u(x),\cdot)\rangle=\int_{\mathbb{M}^{m\times n}}\sigma(x,u(x),\lambda)\,d\nu_x(\lambda)\,.$$

Moreover, the following inequality holds,

$$\int_{\mathbb{M}^{m\times n}}\sigma(x,u(x),\lambda):\lambda\,d\nu_x(\lambda)\le\bar\sigma(x):\operatorname{ap}Du(x)\quad\text{for a.e. } x\in\Omega\,. \tag{2.7}$$

Proof. First we choose $\varphi_1\in C_0^\infty(\Omega;\mathbb{R})$ with $\varphi_1\ge0$ and $\int_\Omega\varphi_1\,dx=1$. Choosing the test function $\mathcal{T}_1(u^k-v)\varphi_1$ in (2.6) where $v\in C^1(\Omega;\mathbb{R}^m)$ is a suitable comparison function as in [1], we have

$$\int_\Omega\sigma^k:D\big(\mathcal{T}_1(u^k-v)\varphi_1\big)\,dx=\int_\Omega f^k\cdot\mathcal{T}_1(u^k-v)\varphi_1\,dx\,. \tag{2.8}$$

Let $h^k = \sigma^k : D\big(\mathcal{T}_1(u^k - v)\varphi_1\big)$. Then

$$\begin{aligned} h^k &= \sigma^k : D\mathcal{T}_1(u^k - v)D(u^k - v)\varphi_1 + \sigma^k : \mathcal{T}_1(u^k - v) \otimes D\varphi_1 \\ &= \sum_{i=1}^m \sigma_i^k \cdot Du_i^k \chi_{\{|u_i^k - v_i| \le 1\}}\varphi_1 - \sum_{i=1}^m \sigma_i^k \cdot Dv_i \chi_{\{|u_i^k - v_i| \le 1\}}\varphi_1 \\ &\quad + \sigma^k : \mathcal{T}_1(u^k - v) \otimes D\varphi_1 . \end{aligned}$$

In view of (H3)', the first term on the right hand side of the last equality is nonnegative. Recalling our assumptions, we conclude that $(h^k)^-$ is equi-integrable. By Theorem 5, Lemma 6 in [1] and the equi-integrability of $\sigma^k(x)$ together with the convergence of $u^k(x)$ in measure we obtain

$$\begin{aligned} \liminf_{k\to\infty} \int_\Omega h^k \, dx \ge \int_\Omega \varphi_1 \int_{\mathbb{M}^{m\times n}} \sigma\big(x, u(x), \lambda\big) : D\mathcal{T}_1(u - v)\big(\lambda - Dv(x)\big)\, d\nu_x(\lambda)\, dx \\ + \int_\Omega \bar\sigma : \mathcal{T}_1(u - v) \otimes D\varphi_1 \, dx \, . \end{aligned}$$

The right hand side of (2.8) may be estimated as in [1]. And then we use the blow-up method in [1] to prove (2.7) with some minor changes.

Now the lemmas involving (H3) in [1] have been proved if (H3) is replaced by (H3)'. Therefore, Theorem 2 and other results in [1] hold.

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References

[1] G. Dolzmann, N. Hungerbühler & S. Müller, *Non-linear elliptic systems with measure-valued right hand side*, Math. Z. **226(4)** (1997), 545–574.

[2] R. Landes, *On the existence of weak solutions of perturbated systems with critical growth*, J. reine angew. Math. **393** (1989), 21–38.

[3] D. Kinderlehrer & P. Pedregal, *Gradient Young measures generated by sequences in Sobolev spaces*, J. Geom. Anal. **4** (1994), 59–90.

Shulin Zhou
Department of Mathematics, Peking University,
Beijing 100871, P. R. China
E-mail address: `zsl@sxx0.math.pku.edu.cn`