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One-sided Mullins-Sekerka Flow Does Not Preserve Convexity *

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Abstract

The Mullins-Sekerka model is a nonlocal evolution model for hypersurfaces, which arises as a singular limit for the Cahn-Hilliard equation. Assuming the existence of sufficiently smooth solutions we will show that the one-sided Mullins-Sekerka flow does not preserve convexity.

Introduction

The Mullins-Sekerka flow is a nonlocal generalization of the mean curvature flow arising from physics [10, 11]. Similar to Stefan-type problems there is a one-sided and a two-sided version. Recently it has been shown rigorously that the two-sided model arises as a singular limit of the Cahn-Hilliard equation [1]. This has been known formally since the work of Pego [11]. In the literature the Mullins-Sekerka model has been often called Hele-Shaw model. However, there are two different problems which are called Hele-Shaw problems, compare for example [1, 2] with [4]. The problem studied in this paper is the same as the one-sided version of the Hele-Shaw problem as formulated in [1, 2]. To avoid this confusion one should probably call the Hele-Shaw flow of [1, 2] the Mullins-Sekerka flow.

One can ask whether the properties of the mean curvature flow can be generalized to the Mullins-Sekerka flow. Not all results can be expected to generalize, due to the nonlocal character of the Mullins-Sekerka problem, in particular not those that rest on a local argument for the mean curvature flow. There has been some progress made towards the question of existence, see [5] for the one-sided version and [2] for the two-sided version. Recently Luckhaus has announced further results concerning existence, however, no details are know by the author. It is known that the mean curvature flow preserves convexity [6, 9]. It is therefore a natural question to ask whether this is also true for the Mullins-Sekerka flow.

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Under the assumption of short-term existence of sufficiently smooth solutions this question is answered negatively in this paper for the one-sided case.

The 2-dimensional case

We look at a curve Γ_0 and the free boundary problem governed by the evolution law given by

$$\begin{cases} \Delta u = 0 & \text{inside of } \Gamma_t\,, \\ u = \kappa & \text{on } \Gamma_t\,, \\ v = -\frac{\partial u}{\partial n}\,. \end{cases} \tag{1}$$

Here n is the *outer* unit normal to Γ_t, v and κ are the normal velocity and the curvature of Γ_t, respectively. The signs are chosen in such a way that a circle has positive curvature, and a shrinking curve has negative velocity.

The principal idea is to look at a shape given by a straight tube with two circular end caps. By the strong maximum principle $\frac{\partial u}{\partial n} > 0$ on the circular parts of Γ_0. Let us restrict our attention to the right part of the figure. $\frac{\partial u}{\partial n} > 0$ implies $u_x > 0$ on the circular path. On the straight part we have $u_x \equiv 0$, as u is identically zero there. We also have $u_x \equiv 0$ on the y-axis by symmetry for u. Ignoring for the moment the discontinuity of u_x we conclude $u_x > 0$ in the interior of the right half by the maximum principle.

As $u_x \equiv 0$ on the (upper) straight line, we must have $\frac{\partial u_x}{\partial n} = u_{xy} < 0$ on the right half of it by another application of the maximum principle. Even another application of the maximum principle for the function u tells us that $\frac{\partial u}{\partial n} = u_y < 0$ on the upper straight line. Therefore on the right half of the upper line $|u_y|$ decreases towards the center. By symmetry we get the distribution of the initial speed sketched in Figure 1.

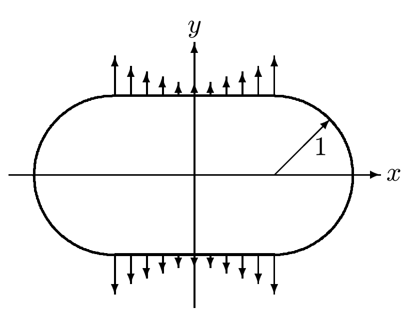

Figure 1: Distribution of initial velocity

Therefore the center will move out slower than the rest of the straight line, the figure will evolve into a nonconvex shape.

This example has one fatal flaw. While the straight lines want to move out, the circular parts want to move in. This of course will break up the curve instantaneously, and (1) cannot possibly be satisfied with an initial configuration like this. The solution to the dilemma is obviously to smoothen out the corners.

For the sequel we will assume that the one-sided Mullins-Sekerka flow allows a smooth solution provided the initial configuration Γ_0 is C^∞.

The difficulty for a smooth domain lies now in showing that u_x still has the sign we want. On the parts of Γ_0 where $u \equiv 0$ or $u \equiv 1$ we get the correct sign by the maximum principle as before. However, the maximum principle does not help on the transition parts.

The following discussion is restricted to the right lower quarter of Γ_0. Let γ be the transition path from the straight line to the circular part, and $\kappa : [0, L] \mapsto [0, 1]$ be the curvature on γ, parametrized by arc length. We choose κ to be a monotonous function.

Then γ is given by

$$\begin{cases} x(s) &= \int_0^s \cos\left(\int_0^\sigma \kappa(t)\,dt\right)\,d\sigma + x_0\,, \\ y(s) &= \int_0^s \sin\left(\int_0^\sigma \kappa(t)\,dt\right)\,d\sigma + y_0\,. \end{cases}$$

Let $\overline{\gamma}$ be the curve in 3-space over γ parametrized by $(x(s), y(s), \kappa(s))$, and let β be the projection of $\overline{\gamma}$ onto the y-u-plane.

Proposition 1 *For a suitable choice of κ the curve β is concave down.*

One can choose, for example,

$$\kappa(s) = \frac{1}{C}\int_0^{s/L} e^{\frac{1}{t(t-1)}}\,dt\,, \quad s \in [0, L]\,,$$

where C is chosen to have $\kappa(L) = 1$. The curve β is described by

$$y(u) = \int_0^{\kappa^{-1}(u)} \sin\left(\int_0^\sigma \kappa(t)\,dt\right)\,d\sigma + y_0\,.$$

Concavity can be checked with methods from elementary calculus, the rather technical details will be omitted here.

We show now how the proposition can be used to construct upper barriers for u. Pick any point Q on $\overline{\gamma}$, and let B be the projection of Q onto the y-u-plane. Let $u = my + c$ be the equation of the tangent line of β at B, where m and c depend on Q. This equation defines a plane in 3-space. If $\overline{\Gamma_0}$ denotes the curve in 3-space over the curve Γ_0 given by (x, y, κ), then this plane touches $\overline{\Gamma_0}$

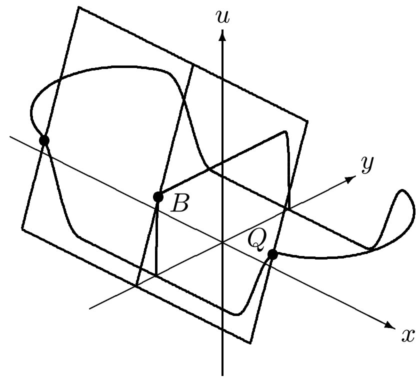

Figure 2: The graphs of $\overline{\Gamma_0}$, β, and of $u = my + c$

in exactly two points, namely in Q and in the symmetric image of Q in left half of Γ_0.

A nonvertical plane of course is the graph of an harmonic function. By construction this plane has zero x-derivative everywhere. From comparison we see that $u_x > 0$ at Q. From here on we conclude the argument as before. Therefore we have proved the following

Theorem 1 *Assume that* (1) *allows a smooth solution provided the initial configuration is C^∞. Then there are convex smooth initial configurations consisting of a straight tube with two end caps that will evolve into nonconvex curves. The flat tube can be arbitrarily short. In particular these initial curves can be chosen to be arbitrarily close in the C^1-norm to a circle.*

The k-dimensional case, $k \geq 3$

We look at a hypersurface Γ_0 and the free boundary problem governed by the evolution law given by

$$\left\{ \begin{array}{rcll} \Delta u & = & 0 & \text{inside of } \Gamma_t\,, \\ u & = & H & \text{on } \Gamma_t\,, \\ v & = & -\frac{\partial u}{\partial n}\,. & \end{array} \right. \tag{2}$$

Here n is the *outer* unit normal to Γ_t, v and H are the normal velocity and the mean curvature of Γ_t, respectively. The signs are chosen in such a way that a sphere has positive curvature, and a shrinking surface has negative velocity.

We use the curve from the 2-dimensional case and rotate about the x-axis. The resulting hypersurface in $\mathbf{R}^k$ is given by

$$\Gamma_0 = \left\{ (x(s), y(s)\omega) \ : \ s \in [0, L],\, \omega \in S^{k-2} \subset \mathbf{R}^{k-1} \right\} \,.$$

Proposition 2 *The principal curvatures of* Γ_0 *are given by*

$$\kappa_1 = \kappa\,, \quad \kappa_i = -\frac{x'}{y}\,, \quad i = 2, \ldots, k-1\,.$$

The proof involves only routine computations of the first and second fundamental forms in local coordinates and will be omitted here.

The mean curvature is

$$H = \sum_{i=1}^{k-1} \kappa_i = \kappa - (k-2)\frac{x'}{y}.$$

Let $\overline{\Gamma_0} = \{(x,u) \in \mathbf{R}^{k+1} \,:\, x \in \Gamma_0,\, u = H(x)\}$. As before we project along the x_1-axis onto the hyperplane in $\mathbf{R}^{k+1}$ perpendicular to the x_1-axis. The resulting manifold β is the graph of $u = H(x_2, \ldots, x_k)$.

Proposition 3 *For a suitable choice of* κ *the graph* β *is concave down.*

The first step for the proof is to note that this graph is rotationally symmetric with respect to $(x_2, \ldots, x_k)$ by construction. Hence it is enough to look at a radial section, say, in the x_2-u-plane. In the sequel we will use y instead of x_2. We have already seen that $u = \kappa(y)$ is concave down. The same is true for $u = \kappa_i(y),\ i = 2, \ldots, k-1$. These curves are given by

$$\left\{ \begin{array}{rcl} y(s) & = & \int_0^s \sin\left(\int_0^\sigma \kappa(t)\,dt\right)\, d\sigma + y_0 \\ \\ \kappa_i(s) & = & \dfrac{\cos\left(\int_0^s \kappa(t)\,dt\right)}{y(s)} \end{array} \right.$$

where the curvature κ is chosen to be the same as in the 2-dimensional case. As $y(s)$ is increasing one can express κ_i as a function of y and then use methods from calculus to show concavity. Therefore H is concave down.

From here on we proceed exactly as in the 2-dimensional case. For a given point $Q \in \overline{\Gamma_0}$ we look at the tangent hyperplane to its projection $B \in \beta$ in x_2-$\ldots$-x_k-u-space, and use the equation $\overline{u} = a_2x_2 + \ldots + a_kx_k + c$ of this hyperplane as the definition of an affine function on $\mathbf{R}^k$. This gives us a supersolution $\overline{u}$ to the harmonic function u of (2) connected with Γ_0. We get the same conclusion on the sign of u_{x_1} in the same way as in the 2-dimensional case. To proceed we need the additional information that u be symmetric about the x_1-axis. This is true because of the symmetry of the domain, the symmetry of the boundary data, and the invariance of the Laplacian under rotations. Therefore it is enough to look at a section, say, in the x_1-x_2-plane. There we have already seen how the information on the sign of u_{x_1} implies that $|u_{x_2}| = |\frac{\partial u}{\partial n}|$ decreases towards the middle.

Theorem 2 *Assume that* (2) *allows a smooth solution provided the initial configuration is C^∞. Then there are convex smooth initial configurations in $\mathbf{R}^k$ consisting of a cylinder with two end caps that will evolve into nonconvex hypersurfaces. The cylinder can be arbitrarily short. In particular these initial hypersurfaces can be chosen to be arbitrarily close in the C^1-norm to a sphere.*

Remark. The proof of the proposition shows that in fact more is true. The only properties of H we used are that the arithmetic mean preserves concavity, and, for positive arguments, is positive and increases in each argument. One can therefore replace H by any suitably smooth function that has these properties and get the same results. In particular one can use $\sqrt[r]{H_r}$, where H_r is the r-th symmetric function of the principal curvatures [7].

References

[1] N. Alikakos, P. Bates & X. Chen, *Convergence of the Cahn-Hilliard equation to the Hele-Shaw Model*, preprint.

[2] X. Chen, *Hele-Shaw problem and area-preserving, curve shortening motion*, Arch. Rat. Mech. Anal., to appear.

[3] M. P. do Carmo, *Differential Geometry of Curves and Surfaces*, Prentice-Hall, New Jersey, 1976.

[4] P. Constantin & M. Pugh, *Global solutions for small data to the Hele-Shaw problem*, preprint.

[5] J. Duchon et R. Robert, *Évolution d'une interface par capillarité et diffusion de volume, existence locale en temps.* C. R. Acad. Sci. Paris Sér. I Math. **298** (1984) 473–476.

[6] M. Gage & R. S. Hamilton, *The heat equation shrinking convex plane curves*, J. Differential Geometry **23** (1986) 69–96.

[7] L. Garding, *An inequality for hyperbolic polynomials.* J. Math. Mech. **8** (1959) 957–965.

[8] D. Gilbarg & N. S. Trudinger, *Elliptic Partial Differential Equations of Second Order*, Springer, Berlin, 1983.

[9] G. Huisken, *Flow by mean curvature of convex surfaces into points*, J. Differential Geometry **20** (1984) 237–260.

[10] W. W. Mullins & R. F. Sekerka, *Morphological Stability of a Particle Growing by Diffusion or Heat Flow*, J. Appl. Phys. **34** (1963) 323–329.

[11] R. S. Pego, *Front migration in the nonlinear Cahn-Hilliard equation.* Proc. R. Soc. Lond., Ser. A **422** (1989) 261–278.

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