

# HYPERGEOMETRIC SERIES ACCELERATION VIA THE WZ METHOD

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*Dedicated to Herb Wilf on his one million-first birthday*

ABSTRACT. Based on the WZ method, some series acceleration formulas are given. These formulas allow us to write down an infinite family of parametrized identities from any given identity of WZ type. Further, this family, in the case of the Zeta function, gives rise to many accelerated expressions for  $\zeta(3)$ .

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We recall [Z] that a discrete function  $A(n,k)$  is called *Hypergeometric* (or *Closed Form (CF)*) *in two variables* when the ratios  $A(n+1, k)/A(n, k)$  and  $A(n, k+1)/A(n, k)$  are both rational functions. A discrete 1-form  $\omega = F(n, k)\delta k + G(n, k)\delta n$  is a *WZ 1-form* if the pair  $(F, G)$  of CF functions satisfies  $F(n+1, k) - F(n, k) = G(n, k+1) - G(n, k)$ .

We use:  $N$  and  $K$  for the *forward shift operators* on  $n$  and  $k$ , respectively.  $\Delta_n := N - 1$ ,  $\Delta_k := K - 1$ .

Consider the WZ 1-form  $\omega = F(n, k)\delta k + G(n, k)\delta n$ . Then, we define the sequence  $\omega_s$ ,  $s = 1, 2, 3, \dots$  of new WZ 1-forms:  $\omega_s := F_s\delta k + G_s\delta n$ ; where

$$F_s(n, k) = F(sn, k) \quad \text{and} \quad G_s(n, k) = \sum_{i=0}^{s-1} G(sn + i, k).$$

**Proposition:**  $\omega_s$  is WZ, for all  $s$ .

**Proof:** (a)  $\omega_s$  is closed:

$$\begin{aligned} \Delta_n F_s &= F(s(n+1), k) - F(sn, k) \\ &= \sum_{i=0}^{s-1} \left( F(sn + i + 1, k) - F(sn + i, k) \right) \\ &= \sum_{i=0}^{s-1} \left( G(sn + i, k + 1) - G(sn + i, k) \right) \\ &= \sum_{i=0}^{s-1} G(sn + i, k + 1) - \sum_{i=0}^{s-1} G(sn + i, k) \\ &= \Delta_k G_s. \end{aligned}$$

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Note that since  $\omega$  is a WZ, it has the form ([Z], p.590):

$$(*) \quad \omega = f(n, k)(P(n, k)\delta k + Q(n, k)\delta n)$$

for some CF  $f$  and some polynomials  $P$  and  $Q$ .

(b)  $\omega_s$  has the form (\*):

Indeed,  $\omega_s$  can be rewritten as:

$$\begin{aligned} \omega_s &= f(sn, k) \left( P(sn, k)\delta k + \sum_{i=0}^{s-1} \frac{f(sn+i, k)}{f(sn, k)} Q(sn+i, k)\delta n \right) \\ &= f(sn, k)(P(sn, k)\delta k + R(n, k)\delta n); \end{aligned}$$

where  $R(n,k)$  is a rational function and  $f(sn,k)$  is still CF. Hence after pulling out a common denominator, we see that  $\omega_s$  too has the form (\*). This proves the Proposition.  $\square$

**Theorem 1:** ([Z], Theorem 7, p.596) For any WZ pair  $(F,G)$

$$\sum_{n=0}^{\infty} G(n, 0) = \sum_{n=1}^{\infty} (F(n, n-1) + G(n-1, n-1)) - \lim_{n \rightarrow \infty} \sum_{k=0}^{n-1} F(n, k),$$

whenever both side converge.

**Formula 1:**

$$(1) \quad \sum_{n=0}^{\infty} G(n, 0) = \sum_{n=0}^{\infty} \left( F(s(n+1), n) + \sum_{i=0}^{s-1} G(sn+i, n) \right) - \lim_{n \rightarrow \infty} \sum_{k=0}^{n-1} F(sn, k).$$

**Proof:** Apply Theorem 1 above on  $\omega_s$ . Alternatively, integrate  $\omega$  along the boundary contour  $\partial\Omega_s$  of the region  $\Omega_s = \{(n, k) : sn \geq k\}$ .  $\square$

**Formula 2:** We also have that

$$(2) \quad \sum_{k=0}^{\infty} F(0, k) - \lim_{n \rightarrow \infty} \sum_{k=0}^n F(n, k) = \sum_{n=0}^{\infty} G(n, 0) - \lim_{k \rightarrow \infty} \sum_{n=0}^k G(n, k),$$

whenever both side converge.

**Proof:** Integrate  $\omega$  along the boundary contour  $\partial\Omega_0$  of the region  $\Omega_0 = \{(n, k) : n \geq 0, k \geq 0\}$ .  $\square$

**Remark:** By shear symmetry, a formulation similar to (2) can be given in ‘k’. And a combination leads to:

**Formula 3:** For  $\omega_{s,t} = F_{s,t}\delta k + G_{s,t}\delta n$ ; where

$$F_{s,t}(n, k) = \sum_{j=0}^{t-1} F(sn, tk + j) \quad \text{and} \quad G_{s,t}(n, k) = \sum_{i=0}^{s-1} G(sn + i, tk), \quad \text{we have}$$

$$(3) \quad \sum_{n=0}^{\infty} G(n, 0) = \sum_{n=0}^{\infty} \left( \sum_{j=0}^{t-1} F(s(n+1), tn + j) + \sum_{i=0}^{s-1} G(sn + i, tn) \right) - \lim_{n \rightarrow \infty} \sum_{k=0}^{n-1} F_{s,t}(n, k).$$

Analogous statements hold in several variables. To wit:

for the WZ 1-form in 3 variables,  $\omega_{s,t,r} := F_{s,t,r}\delta k + G_{s,t,r}\delta n + H_{s,t,r}\delta a$ ; where

$$F_{s,t,r}(n, k, a) = \sum_{j=0}^{t-1} F(sn, tk + j, ra), \quad G_{s,t,r}(n, k, a) = \sum_{i=0}^{s-1} G(sn + i, tk, ra) \quad \text{and}$$

$$H_{s,t,r}(n, k, a) = \sum_{u=0}^{r-1} H(sn, tk, ra + u),$$

**Formula 4:**

$$\sum_{n=0}^{\infty} H(0, 0, n) = \sum_{n=0}^{\infty} \left( \sum_{u=0}^{r-1} H(s(n+1), t(n+1), rn + u) + \sum_{j=0}^{t-1} F(s(n+1), tn + j, rn) + \sum_{i=0}^{s-1} G(sn + i, tn, rn) \right) - \lim_{a \rightarrow \infty} \sum_{k=0}^{a+1} F_{s,t,r}(a + 1, k, a) - \lim_{a \rightarrow \infty} \sum_{n=0}^{a+1} G_{s,t,r}(n, a + 1, a).$$

In [A], formula (1) was used to give a list of series acceleration for  $\zeta(3)$  (where  $F(n, k)$  is given and its companion  $G(n, k)$  is produced by the amazing Maple Package EKHAD accompanying [PWZ]). A small Maple Package `accel` applying (3) is available at [http://www.math.temple.edu/~tewodros\\_zeilberg](http://www.math.temple.edu/~tewodros_zeilberg).

For example: with  $F(n, k) = (-1)^k \frac{n!^6(2n-k-1)!k!^3}{2(n+k+1)!^2(2n)!^3}$ ,  $s=1$  and  $t=1$  `accel` produces the following pretty formula:

$$(**) \quad \zeta(3) = \sum_{n=0}^{\infty} (-1)^n \frac{n!^{10}(205n^2 + 250n + 77)}{64(2n + 1)!^5}.$$

Greg Fee and Simon Plouffe used (\*\*) in their evaluation of  $\zeta(3)$  to *520,000 digits* (available at <http://www.cecm.sfu.ca/projects/ISC/records.html>).

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## REFERENCES

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The package EKHAD is available by the www at <http://www.math.temple.edu/~zeilberg/programs.html>
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