

Plancherel averages: Remarks on a paper by Stanley

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Abstract

Let M_n stand for the Plancherel measure on \mathbb{Y}_n , the set of Young diagrams with n boxes. A recent result of R. P. Stanley (arXiv:0807.0383) says that for certain functions G defined on the set \mathbb{Y} of all Young diagrams, the average of G with respect to M_n depends on n polynomially. We propose two other proofs of this result together with a generalization to the Jack deformation of the Plancherel measure.

1 Introduction

Let \mathbb{Y} denote the set of all integer partitions, which we identify with Young diagrams. For $\lambda \in \mathbb{Y}$, denote by $|\lambda|$ the number of boxes in λ and by $\dim \lambda$ the number of standard tableaux of shape λ . Let also $c_1(\lambda), \dots, c_{|\lambda|}(\lambda)$ be the contents of the boxes of λ written in an arbitrary order (recall that the content of a box is the difference $j - i$ between its column number j and row number i).

For each $n = 1, 2, \dots$, denote by $\mathbb{Y}_n \subset \mathbb{Y}$ the (finite) set of diagrams with n boxes. The well-known *Plancherel measure* on \mathbb{Y}_n assigns weight $(\dim \lambda)^2/n!$ to a diagram $\lambda \in \mathbb{Y}_n$. This is a probability measure. Given a function F on the set \mathbb{Y} of all Young diagrams, let us define the *n th Plancherel average* of F as

$$\langle F \rangle_n = \sum_{\lambda \in \mathbb{Y}_n} \frac{(\dim \lambda)^2}{n!} F(\lambda). \quad (1.1)$$

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In the recent paper [17], R. P. Stanley proves, among other things, the following result ([17, Theorem 2.1]):

Theorem 1.1. *Let $\varphi(x_1, x_2, \dots)$ be an arbitrary symmetric function and set*

$$G_\varphi(\lambda) = \varphi(c_1(\lambda), \dots, c_{|\lambda|}(\lambda), 0, 0, \dots), \quad \lambda \in \mathbb{Y}. \quad (1.2)$$

Then $\langle G_\varphi \rangle_n$ is a polynomial function in n .

The aim of the present note is to propose two other proofs of this result and a generalization, which is related to the *Jack deformation* of the Plancherel measure.

The first proof relies on a claim concerning the shifted (aka interpolation) Schur and Jack polynomials, established in [10] and [11]. Modulo this claim, the argument is almost trivial.

The second proof is more involved but can be made completely self-contained. In particular, no information on Jack polynomials is required. The argument is based on a remarkable idea due to S. Kerov [5] and some considerations from my paper [12].

As indicated by R. P. Stanley, his paper was motivated by a conjecture in the paper [2] by G.-N. Han (see Conjecture 3.1 in [2]). Note also that examples of the Plancherel averages of functions of type (1.2) appeared in S. Fujii et al. [1, Section 3 and Appendix].

2 The algebra \mathbb{A} of regular functions on \mathbb{Y}

For a Young diagram $\lambda \in \mathbb{Y}$, denote by λ_i its i th row length. Clearly, λ_i vanishes for i large enough. Thus, $(\lambda_1, \lambda_2, \dots)$ is the partition corresponding to λ .

Definition 2.1. Let u be a complex variable. The *characteristic function* of a diagram $\lambda \in \mathbb{Y}$ is

$$\Phi(u; \lambda) = \prod_{i=1}^{\infty} \frac{u+i}{u-\lambda_i+i} = \prod_{i=1}^{\ell(\lambda)} \frac{u+i}{u-\lambda_i+i},$$

where $\ell(\lambda)$ is the number of nonzero rows in λ .

The characteristic function is rational and takes the value 1 at $u = \infty$. Therefore, it admits the Taylor expansion at $u = \infty$ with respect to the variable u^{-1} . Likewise, such an expansion also exists for $\log \Phi(u; \lambda)$.

Definition 2.2. Let \mathbb{A} be the unital \mathbb{R} -algebra of functions on \mathbb{Y} generated by the coefficients of the Taylor expansion at $u = \infty$ of the characteristic function $\Phi(u; \lambda)$ (or, equivalently, of $\log \Phi(u; \lambda)$). We call \mathbb{A} the *algebra of regular functions on \mathbb{Y}* . (In [7] and [3], we employed the term *polynomial functions on \mathbb{Y}* .)

The Taylor expansion of $\log \Phi(u; \lambda)$ at $u = \infty$ has the form

$$\log \Phi(u; \lambda) = \sum_{m=1}^{\infty} \frac{p_m^*(\lambda)}{m} u^{-m},$$

where, by definition,

$$p_m^*(\lambda) = \sum_{i=1}^{\infty} [(\lambda_i - i)^m - (-i)^m] = \sum_{i=1}^{\ell(\lambda)} [(\lambda_i - i)^m - (-i)^m], \quad m = 1, 2, \dots, \quad \lambda \in \mathbb{Y}.$$

Thus, the algebra \mathbb{A} is generated by the functions p_1^*, p_2^*, \dots . It is readily verified that these functions are algebraically independent, so that \mathbb{A} is isomorphic to the algebra of polynomials in the variables p_1^*, p_2^*, \dots . Note that $p_1^*(\lambda) = |\lambda|$.

Using the isomorphism between \mathbb{A} and $\mathbb{R}[p_1^*, p_2^*, \dots]$ we define a *filtration* in \mathbb{A} by setting $\deg p_m^*(\cdot) = m$. In more detail, the m th term of the filtration, consisting of elements of degree $\leq m$, $m = 1, 2, \dots$, is the finite-dimensional subspace $\mathbb{A}^{(m)} \subset \mathbb{A}$ defined in the following way:

$$\mathbb{A}^{(0)} = \mathbb{R}1; \quad \mathbb{A}^{(m)} = \text{span}\{(p_1^*)^{r_1} (p_2^*)^{r_2} \dots : 1r_1 + 2r_2 + \dots \leq m\}.$$

The regular functions on \mathbb{Y} (that is, elements of \mathbb{A}) coincide with the *shifted symmetric functions* in the variables $\lambda_1, \lambda_2, \dots$ as defined in [10, Sect. 1]. Thus, we have the canonical isomorphism of filtered algebras $\mathbb{A} \simeq \Lambda^*$, where Λ^* stands for the algebra of shifted symmetric functions. This also establishes an isomorphism of graded algebras $\text{gr } \mathbb{A} \simeq \Lambda$, where Λ denotes the algebra of symmetric functions.

For a diagram $\lambda \in \mathbb{Y}$, denote by $\delta(\lambda)$ the number of its diagonal boxes, by λ' the transposed diagram, and set

$$a_i = \lambda_i - i + \frac{1}{2}, \quad b_i = \lambda'_i - i + \frac{1}{2}, \quad i = 1, \dots, \delta(\lambda). \quad (2.1)$$

We call the numbers (2.1) the *modified Frobenius coordinates* of λ (see [18, (10)]).

Proposition 2.3. *Equivalently, \mathbb{A} may be defined as the algebra of super-symmetric functions in the variables $\{a_i\}$ and $\{-b_i\}$.*

Proof. See [7]. Here I am sketching another proof, which was given in [3, Proposition 1.2].

A simple argument (a version of Frobenius' lemma) shows that

$$\Phi(u - \frac{1}{2}; \lambda) = \prod_{i=1}^{\delta(\lambda)} \frac{u + b_i}{u - a_i}$$

(this identity can also be deduced from formula (2.3) below). It follows

$$\log \Phi(u - \frac{1}{2}; \lambda) = \sum_{m=1}^{\infty} \frac{u^{-m}}{m} \sum_{i=1}^{\delta(\lambda)} (a_i^m - (-b_i)^m),$$

which implies that \mathbb{A} is freely generated by the functions

$$p_m(\lambda) := \sum_{i=1}^{\delta(\lambda)} (a_i^m - (-b_i)^m), \quad m = 1, 2, \dots, \quad (2.2)$$

which are *super-power sums* in $\{a_i\}$ and $\{-b_i\}$. □

Another characterization of regular functions is provided by

Proposition 2.4. \mathbb{A} coincides with the unital algebra generated by the function $\lambda \mapsto |\lambda|$ and the functions $G_\varphi(\lambda)$ of the form (1.2).

Proof. This result is due to S. Kerov. It is pointed out in his note [4], see also [7, proof of Theorem 4]. Here is a detailed proof taken from Kerov's unpublished work notes:

We claim that the algebra \mathbb{A} is freely generated by the functions

$$\widehat{p}_r(\lambda) = \sum_{\square \in \lambda} (c(\square))^r, \quad r = 0, 1, \dots,$$

where the sum is taken over the boxes \square of λ and $c(\square)$ denotes the content of a box. Note that $\widehat{p}_0(\lambda) = |\lambda|$.

Indeed, we start with the relation

$$\Phi(u - \frac{1}{2}; \lambda) = \prod_{i=1}^{\ell(\lambda)} \frac{u + i - \frac{1}{2}}{u - \lambda_i + i - \frac{1}{2}} = \prod_{\square \in \lambda} \frac{u - c(\square) + \frac{1}{2}}{u - c(\square) - \frac{1}{2}}. \quad (2.3)$$

It implies

$$\log \Phi(u - \frac{1}{2}; \lambda) = \sum_{m=1}^{\infty} \frac{u^{-m}}{m} \sum_{\square \in \lambda} \left((c(\square) + \frac{1}{2})^m - (c(\square) - \frac{1}{2})^m \right),$$

or

$$p_m(\lambda) = \sum_{k=0}^{\lfloor \frac{m-1}{2} \rfloor} 2^{-2k} \binom{m}{2k+1} \widehat{p}_{m-1-2k}(\lambda), \quad m = 1, 2, \dots,$$

and our claim follows. □

Remark 2.5. Note a shift of degree: as seen from the above computation, the degree of $\widehat{p}_r(\lambda)$ with respect to the filtration of \mathbb{A} equals $r + 1$.

Remark 2.6. Proposition 2.3 makes it possible to introduce a natural algebra isomorphism between Λ and \mathbb{A} , which sends the power-sums $p_m \in \Lambda$ to the functions $p_m(\lambda)$ defined in (2.2),

Remark 2.7. The algebra \mathbb{A} is stable under the change of the argument $\lambda \mapsto \lambda'$ (transposition of diagrams): this claim is not obvious from the initial definition but becomes clear from Proposition 2.3 or Proposition 2.4.

Finally, note that one more characterization of the algebra \mathbb{A} is given in Section 6.

3 A proof of Theorem 1.1

The *Young graph* has \mathbb{Y} as the vertex set, and the edges are formed by couples of diagrams that differ by a single box. This is a graded graph: its n th level ($n = 0, 1, \dots$) is the subset $\mathbb{Y}_n \subset \mathbb{Y}$. The notation $\mu \nearrow \lambda$ or, equivalently, $\lambda \searrow \mu$ means that λ is obtained from μ by adding a box (so that the couple $\{\mu, \lambda\}$ forms an edge). The quantity $\dim \lambda$ coincides with the number of monotone paths $\emptyset \nearrow \dots \nearrow \lambda$ in the Young graph.

More generally, for any two diagrams $\mu, \lambda \in \mathbb{Y}$ we denote by $\dim(\mu, \lambda)$ the number of monotone paths $\mu \nearrow \dots \nearrow \lambda$ in the Young graph that start at μ and end at λ . If there is no such path, then we set $\dim(\mu, \lambda) = 0$. Equivalently, $\dim(\mu, \lambda)$ is the number of standard tableaux of skew shape λ/μ when $\mu \subseteq \lambda$, and $\dim(\mu, \lambda) = 0$ otherwise.

Let $x^{\downarrow m}$ stand for the m th falling factorial power of x . That is,

$$x^{\downarrow m} = x(x-1)\dots(x-m+1), \quad m = 0, 1, \dots$$

With an arbitrary $\mu \in \mathbb{Y}$ we associate the following function on \mathbb{Y} :

$$F_\mu(\lambda) = n^{\downarrow m} \frac{\dim(\mu, \lambda)}{\dim \lambda}, \quad \lambda \in \mathbb{Y}, \quad n = |\lambda|, \quad m = |\mu|. \quad (3.1)$$

Proposition 3.1. *For any $\mu \in \mathbb{Y}$, the function F_μ belongs to \mathbb{A} and has degree $|\mu|$. Under the isomorphism $\text{gr } \mathbb{A} \simeq \Lambda$, the top degree term of F_μ coincides with the Schur function s_μ .*

Proof. This can be deduced from [7, Theorem 5]. For direct proofs, see [10, Theorem 8.1] and [14, Proposition 1.2]. \square

Remark 3.2. Under the isomorphism between \mathbb{A} and Λ^* , F_μ turns into the *shifted Schur function* s_μ^* , see [10, Definition 1.4]. Under the isomorphism between \mathbb{A} and Λ (Remark 2.6), F_μ is identified with the *Frobenius–Schur function* Fs_μ , see [13], [14, Section 2].

Introduce a notation for the n th Plancherel measure:

$$M_n(\lambda) = \frac{(\dim \lambda)^2}{n!}, \quad \lambda \in \mathbb{Y}_n. \quad (3.2)$$

Thus, the n th Plancherel average of a function F on \mathbb{Y} is

$$\langle F \rangle_n = \sum_{\lambda \in \mathbb{Y}_n} F(\lambda) M_n(\lambda). \quad (3.3)$$

By virtue of Proposition 2.4, Theorem 1.1 follows from

Theorem 3.3. *For any $F \in \mathbb{A}$, $\langle F \rangle_n$ is a polynomial in n of degree at most $\deg F$, where \deg refers to degree with respect to the filtration in \mathbb{A} . Furthermore,*

$$\langle F_\mu \rangle_n = \binom{n}{m} \dim \mu, \quad \mu \in \mathbb{Y}, \quad m := |\mu|. \quad (3.4)$$

Proof. First, let us check (3.4). If $n < m$ then the both sides of (3.4) vanish: the restriction of F_μ to \mathbb{Y}_n is identically 0 and $\binom{n}{m} = 0$. Consequently, we may assume $n \geq m$.

Let (\cdot, \cdot) denote the standard inner product in Λ . The simplest case of Pieri's rule for the Schur functions says that

$$p_1 s_\mu = \sum_{\mu^\bullet: \mu^\bullet \searrow \mu} s_{\mu^\bullet}.$$

It follows that for $\lambda \in \mathbb{Y}_n$

$$\dim(\mu, \lambda) = (p_1^{n-m} s_\mu, s_\lambda), \quad \dim \lambda = (p_1^n, s_\lambda). \quad (3.5)$$

Therefore, using the definition (3.1), we have

$$\begin{aligned} \langle F_\mu \rangle_n &= \frac{n^{\downarrow m}}{n!} \sum_{\lambda \in \mathbb{Y}_n} \dim(\mu, \lambda) \dim \lambda \\ &= \frac{n^{\downarrow m}}{n!} \sum_{\lambda \in \mathbb{Y}_n} (p_1^{n-m} s_\mu, s_\lambda) (p_1^n, s_\lambda) = \frac{n^{\downarrow m}}{n!} (p_1^{n-m} s_\mu, p_1^n) \\ &= \frac{n^{\downarrow m}}{n!} \left(s_\mu, \frac{\partial^{n-m}}{\partial p_1^{n-m}} p_1^n \right) = \frac{n^{\downarrow m}}{m!} (s_\mu, p_1^m) = \binom{n}{m} \dim \mu, \end{aligned}$$

as required.

By virtue of Proposition 3.1, $\deg F_\mu = |\mu|$ and $\{F_\mu\}$ is a basis in \mathbb{A} compatible with the filtration. On the other hand, $\binom{n}{m}$ is a polynomial in n of degree m . Therefore, the first claim of the theorem follows from (3.4). \square

Remark 3.4. Stanley [17, Section 3] shows that the claim of Theorem 1.1 generalizes to functions of the form $G_\varphi H_\psi$, where ψ is an arbitrary symmetric function and

$$H_\psi(\lambda) := \psi(\lambda_1 + |\lambda| - 1, \lambda_2 + |\lambda| - 2, \dots, \lambda_{|\lambda|}, 0, 0, \dots), \quad \lambda \in \mathbb{Y}. \quad (3.6)$$

This apparently stronger result also follows from Theorem 3.3, because (as is readily seen) any function of the form (3.6) belongs to the algebra \mathbb{A} .

4 The Jack deformation of the algebra \mathbb{A}

Here we extend the definitions of Section 2 by introducing the deformation parameter $\theta > 0$. The previous picture corresponds to the particular value $\theta = 1$. We call θ the *Jack parameter*, because of a close relation to Jack symmetric functions. Note that θ is inverse to the parameter α used in Macdonald's book [8] and Stanley's paper [15].

Definition 4.1. The θ -characteristic function of a diagram $\lambda \in \mathbb{Y}$ is defined as

$$\Phi_\theta(u; \lambda) = \prod_{i=1}^{\infty} \frac{u + \theta i}{u - \lambda_i + \theta i} = \prod_{i=1}^{\ell(\lambda)} \frac{u + \theta i}{u - \lambda_i + \theta i}.$$

This is again a rational function in u , regular at infinity and hence admitting the Taylor expansion at $u = \infty$ with respect to u^{-1} .

Definition 4.2. The algebra \mathbb{A}_θ of θ -regular functions on \mathbb{Y} is the unital \mathbb{R} -algebra generated by the coefficients of the Taylor expansion at $u = \infty$ of the function $\Phi_\theta(u; \lambda)$ (or, equivalently, of $\log \Phi_\theta(u; \lambda)$).

The Taylor expansion of $\log \Phi_\theta(u; \lambda)$ at $u = \infty$ has the form

$$\log \Phi_\theta(u; \lambda) = \sum_{m=1}^{\infty} \frac{p_{m;\theta}^*(\lambda)}{m} u^{-m},$$

where, by definition,

$$p_{m;\theta}^*(\lambda) = \sum_{i=1}^{\infty} [(\lambda_i - \theta i)^m - (-\theta i)^m], \quad m = 1, 2, \dots, \quad \lambda \in \mathbb{Y}$$

(as above, summation actually can be taken up to $i = \ell(\lambda)$). Thus, the algebra \mathbb{A}_θ is generated by the functions $p_{1;\theta}^*, p_{2;\theta}^*, \dots$. These functions are algebraically independent.

The filtration in \mathbb{A}_θ is introduced exactly as in the particular case $\theta = 1$. We still have a canonical isomorphism of graded algebras $\text{gr}(\mathbb{A}_\theta) \simeq \Lambda$ and a canonical isomorphism of filtered algebras $\mathbb{A} \simeq \Lambda_\theta^*$, where Λ_θ^* denotes the algebra of θ -shifted symmetric functions [6]. However, for general θ , we do not see a natural way to define an isomorphism between \mathbb{A}_θ and Λ .

5 Jack deformation of Plancherel averages

Recall that $\theta > 0$ is a fixed parameter, which is inverse to Macdonald's [8] parameter α . We consider the Jack deformation $(\cdot, \cdot)_\theta$ of the standard inner product in the algebra Λ of symmetric functions. In the basis $\{p_\lambda\}$ of power-sum functions,

$$(p_\lambda, p_\mu)_\theta = \delta_{\lambda\mu} z_\lambda \theta^{-|\lambda|}, \quad \lambda, \mu \in \mathbb{Y}, \quad (5.1)$$

cf. [8, Chapter VI, Section 10]; the standard notation z_λ is explained in [8, Chapter I, Section 2]. Let $\{P_\lambda\}$ and $\{Q_\lambda\}$ be the biorthogonal bases formed the P and Q Jack symmetric functions (which differ from each other by normalization factors). In Macdonald's notation ([8, Chapter VI, Section 10]), these are $P_\lambda^{(1/\theta)}$ and $Q_\lambda^{(1/\theta)}$. To simplify the notation, we will not include θ into the notation for the Jack functions. When $\theta = 1$, the both versions of the Jack functions turn into the Schur functions s_λ .

Introduce the notation

$$\dim_\theta \lambda = (p_1^n, Q_\lambda)_\theta, \quad \dim'_\theta \lambda = (p_1^n, P_\lambda)_\theta, \quad \lambda \in \mathbb{Y}_n. \quad (5.2)$$

More generally, we set (cf. (3.5))

$$\dim_\theta(\mu, \lambda) = (p_1^{|\lambda|-|\mu|} P_\mu, Q_\lambda)_\theta, \quad \dim'_\theta(\mu, \lambda) = (p_1^{|\lambda|-|\mu|} Q_\mu, P_\lambda)_\theta, \quad (5.3)$$

where we assume $|\mu| \leq |\lambda|$; otherwise the dimension is set to be 0.

Proposition 5.1. *The quantities (5.2) are strictly positive. The quantities (5.3) are strictly positive if $\mu \subseteq \lambda$ and vanish otherwise.*

Proof. The first claim being a particular case of the second one, we focus on the second claim. We employ the formalism described in [6].

The simplest case of Pieri's rule for Jack symmetric functions ([8, Chapter VI, Section 10 and (6.24)(iv)]) says that $p_1 P_\mu$ is a linear combination of the functions $P_{\mu^\bullet}, \mu^\bullet \searrow \mu$, with strictly positive coefficients. The coefficients are just the quantities $\varkappa_\theta(\mu, \mu^\bullet) := (p_1 P_\mu, Q_{\mu^\bullet})_\theta$; let us view them as *formal multiplicities* attached to the edges $\mu \nearrow \mu^\bullet$. More generally, the *weight* of a finite monotone path $\mu \nearrow \cdots \nearrow \lambda$ in the Young graph is defined as the product of the formal multiplicities of edges entering the path. Observe now that $\dim_\theta(\mu, \lambda)$ is the sum of the weights of all monotone paths connecting μ to λ . This proves the claim concerning $\dim_\theta(\mu, \lambda)$. For $\dim'_\theta(\mu, \lambda)$ the argument is the same: we simply swap the P and Q functions. \square

With an arbitrary $\mu \in \mathbb{Y}$ we associate the following function on \mathbb{Y} , cf. (3.1):

$$F_{\mu;\theta}(\lambda) = n^{\downarrow m} \frac{\dim_\theta(\mu, \lambda)}{\dim_\theta \lambda}, \quad \lambda \in \mathbb{Y}, \quad n = |\lambda|, \quad m = |\mu|.$$

Proposition 5.2. *For any $\mu \in \mathbb{Y}$, the function $F_{\mu;\theta}$ just defined belongs to \mathbb{A}_θ . Under the isomorphism $\text{gr } \mathbb{A}_\theta \simeq \Lambda$, the top degree term of $F_{\mu;\theta}$ coincides with the Jack function P_μ .*

Proof. See [11, Section 5]. Note that under the isomorphism $\Lambda_\theta^* \rightarrow \mathbb{A}_\theta$, $F_{\mu;\theta}$ coincides with the image of the *shifted Jack function* P_μ^* . \square

Definition 5.3. The *Jack deformation of the Plancherel measure* with parameter θ on the set \mathbb{Y}_n (or *Jack-Plancherel measure*, for short) is defined by

$$M_{n;\theta}(\lambda) = \frac{(p_1^n, Q_\lambda)_\theta (p_1^n, P_\lambda)_\theta}{(p_1^n, p_1^n)_\theta}, \quad \lambda \in \mathbb{Y}_n. \quad (5.4)$$

By Proposition 5.1, the quantity $M_{n;\theta}(\lambda)$ is always positive. Since $\{P_\lambda\}$ and $\{Q_\lambda\}$ are biorthogonal bases, the sum of the quantities (5.4) over $\lambda \in \mathbb{Y}_n$ equals 1. Therefore, $M_{n;\theta}$ is a probability measure. Note that the above definition agrees with that given in [5, Section 7] and [9, Section 3.3.2].

Because

$$(p_1^n, p_1^n)_\theta = z_{(1^n)} \theta^{-n} = \frac{n!}{\theta^n},$$

(5.4) can be rewritten as

$$M_{n;\theta}(\lambda) = \frac{\theta^n (p_1^n, Q_\lambda)_\theta (p_1^n, P_\lambda)_\theta}{n!} = \frac{\theta^n \dim_\theta \lambda \dim'_\theta \lambda}{n!}, \quad \lambda \in \mathbb{Y}_n. \quad (5.5)$$

Clearly, for $\theta = 1$ the definition coincides with (3.2).

Remark 5.4. From the Jack version of the duality map $\Lambda \rightarrow \Lambda$ ([8, Chapter VI, (10.17)]) it can be seen that under the involution $\lambda \mapsto \lambda'$ the measure $M_{n;\theta}$ is transformed into $M_{n;\theta^{-1}}$.

Given a function F on \mathbb{Y} , its n th *Jack–Plancherel average* is defined by analogy with (3.3):

$$\langle F \rangle_{n;\theta} = \sum_{\lambda \in \mathbb{Y}_n} F(\lambda) M_{n;\theta}(\lambda). \quad (5.6)$$

Here is a generalization of Theorem 3.3:

Theorem 5.5. For any $F \in \mathbb{A}_\theta$, $\langle F \rangle_{n;\theta}$ is a polynomial in n of degree at most $\deg F$, where \deg refers to degree with respect to the filtration in \mathbb{A}_θ . Furthermore,

$$\langle F_{\mu;\theta} \rangle_{n;\theta} = \theta^m \binom{n}{m} \dim_\theta \mu.$$

Proof. The argument relies on Proposition 5.2 and is the same as in the proof of Theorem 3.2, with minor obvious modifications. In particular, we use the fact that the adjoint to multiplication by p_1 is equal to $\theta^{-1} \partial / \partial p_1$. For reader’s convenience, we repeat the main computation:

$$\begin{aligned} \langle F_{\mu;\theta} \rangle_{n;\theta} &= \theta^n \frac{n^{\downarrow m}}{n!} \sum_{\lambda \in \mathbb{Y}_n} \dim_\theta(\mu, \lambda) \dim'_\theta \lambda \\ &= \theta^n \frac{n^{\downarrow m}}{n!} \sum_{\lambda \in \mathbb{Y}_n} (p_1^{n-m} P_\mu, Q_\lambda)_\theta (p_1^n, P_\lambda)_\theta = \theta^n \frac{n^{\downarrow m}}{n!} (p_1^{n-m} P_\mu, p_1^n)_\theta \\ &= \theta^n \frac{n^{\downarrow m}}{n!} (P_\mu, (\theta^{-1} \partial / \partial p_1)^{n-m} p_1^n)_\theta = \theta^m \frac{n^{\downarrow m}}{m!} (P_\mu, p_1^m)_\theta = \theta^m \binom{n}{m} \dim_\theta \mu. \end{aligned}$$

□

6 Kerov’s interlacing coordinates

Let $\lambda \in \mathbb{Y}$ be a Young diagram drawn according to the “English picture” [8, Chapter I, Section 1], that is, the first coordinate axis (the row axis) is directed downwards and the second coordinate axis (the column axis) is directed to the right. Consider the border line of λ as the directed path coming from $+\infty$ along the second (horizontal) axis, next turning several times alternately down and to the left, and finally going away to $+\infty$ along the first (vertical) axis. The corner points on this path are of two types: the *inner corners*, where the path switches from the horizontal direction to the vertical one, and the *outer corners* where the direction is switched from vertical to horizontal. Observe that the inner and outer corners always interlace and the number of inner corners always exceeds by 1 that of outer corners. Let $2d - 1$ be the total number of the corners and (r_i, s_i) , $1 \leq i \leq 2d - 1$, be their coordinates. Here the odd and even indices i refer to the inner and outer corners, respectively.

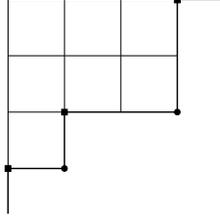


Figure 1. The corners of the diagram $\lambda = (3, 3, 1)$.

For instance, the diagram $\lambda = (3, 3, 1)$ shown on the figure has $d = 3$, three inner corners $(r_1, s_1) = (0, 3)$, $(r_3, s_3) = (2, 1)$, $(r_5, s_5) = (3, 0)$, and two outer corners $(r_2, s_2) = (2, 3)$, $(r_4, s_4) = (3, 1)$.

As above, θ is assumed to be a fixed strictly positive parameter. The numbers

$$x_1 := s_1 - \theta r_1, \quad y_1 := s_2 - \theta r_2, \quad \dots, \quad y_{d-1} := s_{2d-2} - \theta r_{2d-2}, \quad x_d := s_{2d-1} - \theta r_{2d-1} \quad (6.1)$$

form two interlacing sequences of integers

$$x_1 > y_1 > x_2 > \dots > y_{d-1} > x_d$$

satisfying the relation

$$\sum_{i=1}^d x_i - \sum_{j=1}^{d-1} y_j = 0. \quad (6.2)$$

For instance, if $\lambda = (3, 3, 1)$ as in the example above, then

$$x_1 = 3, \quad y_1 = 3 - 2\theta, \quad x_2 = 1 - 2\theta, \quad y_2 = 1 - 3\theta, \quad x_3 = -3\theta.$$

Definition 6.1. The two interlacing sequences

$$X = (x_1, \dots, x_d), \quad Y = (y_1, \dots, y_{d-1}) \quad (6.3)$$

as defined above are called the (θ -dependent) *Kerov interlacing coordinates* of a Young diagram λ . (Note that in the case $\theta = 1$, Kerov's (X, Y) coordinates are similar to Stanley's " (p, q) coordinates" introduced in [16]: the two coordinate systems are related by a simple linear transformation.)

Let u be a complex variable. Given a Young diagram λ , we set

$$\mathbf{H}(u; \lambda) = \frac{u \prod_{j=1}^{d-1} (u - y_j)}{\prod_{i=1}^d (u - x_i)},$$

and

$$\mathbf{p}_m(\lambda) = \sum_{i=1}^d x_i^m - \sum_{j=1}^{d-1} y_j^m, \quad m = 1, 2, \dots,$$

where $X = \{x_i\}$, $Y = \{y_j\}$, and d are as in Definition 6.1. Obviously,

$$\log \mathbf{H}(u; \lambda) = \sum_{m=1}^{\infty} \frac{\mathbf{p}_m(\lambda)}{m} u^{-m}.$$

Note that $\mathbf{p}_1(\lambda) \equiv 0$ because of (6.2).

Proposition 6.2. *The following relation holds*

$$\mathbf{H}(u; \lambda) = \frac{\Phi(u - \theta; \lambda)}{\Phi(u; \lambda)}.$$

Proof. See [12, Proposition 6.3]. □

From this result one deduces:

Proposition 6.3. *The functions $\mathbf{p}_m(\lambda)$ belong to the algebra \mathbb{A}_θ . More precisely, we have*

$$\mathbf{p}_m = \theta \cdot m \cdot p_{m-1; \theta}^* + \dots, \quad m = 2, 3, \dots,$$

where dots stand for lower degree terms, which are a linear combination of elements $p_{l; \theta}^*$ with $1 \leq l \leq m - 2$.

Proof. See [12, Proposition 6.5]. □

Corollary 6.4. *The functions $\{\mathbf{p}_2, \mathbf{p}_3, \dots\}$ form a system of algebraically independent generators of the algebra \mathbb{A}_θ , compatible with the filtration. More precisely, under the identification of \mathbb{A}_θ with the algebra of polynomials $\mathbb{R}[\mathbf{p}_2, \mathbf{p}_3, \dots]$, the filtration is determined by setting*

$$\deg \mathbf{p}_m = m - 1, \quad m = 2, 3, \dots$$

Thus, the algebra \mathbb{A}_θ of θ -regular functions coincides with the algebra of *super-symmetric functions* in Kerov's θ -dependent interlacing coordinates.

Consider the expansion in partial fractions for $u^{-1}\mathbf{H}(u; \lambda)$:

$$\frac{\prod_{j=1}^{d-1} (u - y_j)}{\prod_{i=1}^d (u - x_i)} = \sum_{i=1}^d \frac{\pi_i^\uparrow}{u - x_i}.$$

Here the coefficients π_i^\uparrow are given by the formula

$$\pi_i^\uparrow = \pi_i^\uparrow(\lambda) = \frac{\prod_{j=1}^{d-1} (x_i - y_j)}{\prod_{l: l \neq i} (x_i - x_l)}, \quad i = 1, \dots, d.$$

Observe that the boxes that may be appended to λ are associated, in a natural way, with the inner corners of the boundary of λ . Consequently, we may also associate these boxes with the x 's: $\square_i \leftrightarrow x_i$.

It is ready to check that the coefficients π_i^\uparrow are strictly positive and sum up to 1. Introduce the notation

$$p_{n;\theta}^\uparrow(\lambda, \lambda \cup \square_i) = \pi_i^\uparrow(\lambda), \quad 1 \leq i \leq d, \quad \lambda \in \mathbb{Y}_n$$

(the quantities $\pi_i^\uparrow(\lambda)$ in the right-hand side depend on θ through (6.1)). We regard $p_{n;\theta}^\uparrow$ as a transition function acting from \mathbb{Y}_n to \mathbb{Y}_{n+1} . The system $\{p_{n;\theta}^\uparrow\}_{n=0,1,\dots}$ determines a model of random growth of Young diagrams: an inhomogeneous Markov chain on \mathbb{Y} whose state at time $n = 0, 1, \dots$ is a diagram from \mathbb{Y}_n . Every trajectory of this Markov chain is an infinite monotone path in \mathbb{Y} starting at \emptyset .

Denote by $M'_{n;\theta}$ the marginal distribution of this Markov chain after n steps. That is, $M'_{n;\theta}$ is the probability measure on \mathbb{Y}_n defined by the recursion

$$M'_{n+1;\theta}(\nu) = \sum_{\lambda \in \mathbb{Y}_n: \lambda \nearrow \nu} M'_{n;\theta}(\lambda) p_{n;\theta}^\uparrow(\lambda, \nu) \quad (6.4)$$

with the initial condition $M'_{0;\theta}(\emptyset) = 1$.

Proposition 6.5. $M'_{n;\theta}$ coincides with the Jack–Plancherel measure $M_{n;\theta}$ as defined in (5.4)

Proof. This is one of the main results of Kerov [5] (see Section 7 in [5]). For $\theta = 1$, it allows a direct elementary verification. For general θ , the proof given in [5] is more delicate; it uses the hook-type formulas for $\dim_\theta \lambda$ and $\dim'_\theta \lambda$ (see [5, Section 6] and [15, Section 5]). \square

If we agree to take (6.4) as the initial definition of the Jack deformation of the Plancherel measure, then (as will be seen) we may completely eliminate the Jack polynomials from our considerations.

Let us restate the first claim of Theorem 5.5 in terms of the measure $M'_{n;\theta}$:

Theorem 6.6. Let $\langle \cdot \rangle'_{n;\theta}$ stand for the expectation with respect to the measure $M'_{n;\theta}$. For any $F \in \mathbb{A}_\theta$, $\langle F \rangle'_{n;\theta}$ is a polynomial in n of degree at most $\deg F$.

We will deduce Theorem 6.6 from the following claim.

Let ∂ denote the operator acting in the space of functions on \mathbb{Y} as

$$(\partial F)(\lambda) = -F(\lambda) + \sum_{\nu: \nu \searrow \lambda} p_{n;\theta}^\uparrow(\lambda, \nu) F(\nu), \quad \lambda \in \mathbb{Y}, \quad n = |\lambda|. \quad (6.5)$$

Theorem 6.7. The operator ∂ defined by (6.5) preserves the algebra \mathbb{A}_θ and reduces degree by 1.

Reduction of Theorem 6.6 to Theorem 6.7. By virtue of (6.4),

$$\langle F \rangle'_{n+1;\theta} - \langle F \rangle'_{n;\theta} = \langle \partial F \rangle'_{n;\theta}, \quad n = 0, 1, \dots$$

Since $\langle 1 \rangle'_{n;\theta} = 1$, the claim of Theorem 6.6 is obtained by induction on $\deg F$. \square

Proof of Theorem 6.7. The claim of Theorem 6.7 can be obtained by a degeneration from a much more general claim, [12, Theorem 7.1(ii)]. In the notation of [12], the degeneration consists in letting certain parameters z and z' go to infinity.

An alternative possibility is to adapt the approach of [12] to the present situation by eliminating these parameters at all, which substantially simplifies the computations. Here is a sketch of the argument; for more detail we refer to [12].

Introduce the functions $\mathbf{h}_0(\lambda), \mathbf{h}_1(\lambda), \dots$ on \mathbb{Y} from the decomposition

$$\mathbf{H}(u; \lambda) = \sum_{m=0}^{\infty} \mathbf{h}_m(\lambda) u^{-m}$$

and note that

$$\mathbf{h}_0(\lambda) \equiv 1, \quad \mathbf{h}_1(\lambda) \equiv 0.$$

The functions $\mathbf{h}_2, \mathbf{h}_3, \dots$ are algebraically independent generators of the algebra \mathbb{A}_θ .

For a partition $\rho = (\rho_1, \rho_2, \dots)$, set

$$\mathbf{h}_\rho = \mathbf{h}_{\rho_1} \mathbf{h}_{\rho_2} \dots$$

Because of $\mathbf{h}_1 = 0$, we will assume in what follows that ρ does not have parts ρ_i equal to 1 (otherwise $\mathbf{h}_\rho = 0$). Then the elements \mathbf{h}_ρ form a linear basis in \mathbb{A}_θ , consistent with filtration:

$$\deg \mathbf{h}_\rho = |\rho| - \ell(\rho),$$

where $|\rho| = \sum \rho_i$ and $\ell(\rho)$ is the number of nonzero parts in ρ . This is related to the fact that $\deg \mathbf{h}_m = m - 1$ for $m = 2, 3, \dots$

We aim at computing the action of ∂ on the basis elements \mathbf{h}_ρ . For any $k = 1, 2, \dots$, we have

$$\prod_{l=1}^k \mathbf{H}(u_l; \lambda) = \sum_{\rho: \ell(\rho) \leq k} m_\rho(u_1^{-1}, \dots, u_k^{-1}) \mathbf{h}_\rho(\lambda),$$

where m_ρ is the monomial symmetric function. Thus, we may view finite products $\prod_l \mathbf{H}(u_l; \lambda)$ as generating series for the basis elements \mathbf{h}_ρ . It is convenient to consider first the action of ∂ on these generating series.

The argument in [12, Section 7.2] shows that $1 + \partial$ acts on $\mathbf{H}(u_1; \lambda) \dots \mathbf{H}(u_k; \lambda)$ as multiplication by the series

$$F^\uparrow(u_1, \dots, u_k; \lambda) := \sum_{i=1}^d \pi_i^\uparrow(\lambda) \prod_{l=1}^k \frac{(u_l - x_i)(u_l - x_i + \theta - 1)}{(u_l - x_i - 1)(u_l - x_i + \theta)}.$$

This series belongs to $\mathbb{A}_\theta[[u_1^{-1}, \dots, u_l^{-1}]]$, because of the fundamental identity

$$\sum_{i=1}^d \pi_i^\uparrow(\lambda) x_i^m = \mathbf{h}_m(\lambda), \quad m = 0, 1, \dots,$$

see [12, Lemma 6.11]. It follows that ∂ maps \mathbb{A}_θ into itself.

A more detailed analysis (see [12, Section 7.3]) shows the following. Introduce the linear map $f \rightarrow \langle f \rangle^\uparrow$ from $\mathbb{R}[x]$ to \mathbb{A}_θ by setting

$$\langle x^m \rangle^\uparrow = \mathbf{h}_m.$$

Next, write the decomposition

$$\prod_{l=1}^k \frac{(u_l - x)(u_l - x + \theta - 1)}{(u_l - x - 1)(u_l - x + \theta)} = \sum_{\sigma} a_{\sigma}(x) m_{\sigma}(u_1^{-1}, \dots, u_k^{-1}),$$

where σ ranges over partitions with $\ell(\sigma) \leq k$ and $a_{\sigma}(x)$ are appropriate polynomials. Finally, let $c_{\sigma\tau}^\rho$ be the structure constants of the algebra Λ in the basis of monomial symmetric functions:

$$m_{\sigma} m_{\tau} = \sum_{\rho} c_{\sigma\tau}^{\rho} m_{\rho}$$

(note that $c_{\sigma\tau}^\rho$ vanishes unless $|\rho| = |\sigma| + |\tau|$). Then we have

$$(1 + \partial)\mathbf{h}_{\rho} = \sum_{\sigma, \tau: |\sigma| + |\tau| = |\rho|} c_{\sigma\tau}^{\rho} \langle a_{\sigma}(x) \rangle^\uparrow \mathbf{h}_{\tau}, \quad (6.6)$$

see [12, Lemma 7.4]. Note that

$$a_{\sigma}(x) = a_{\sigma_1}(x) a_{\sigma_2}(x) \dots, \quad (6.7)$$

where

$$a_s(x) = (s-1)\theta x^{s-2} + \frac{(s-1)(s-2)}{2} \theta(1-\theta)x^{s-3} + \dots, \quad s \geq 2 \quad (6.8)$$

$$\langle a_0(x) \rangle^\uparrow \equiv 1, \quad \langle a_1(x) \rangle^\uparrow \equiv 0 \quad (6.9)$$

(see [12, Lemma 7.3]). It follows, in particular, that we may assume that in (6.6), σ does not have parts equal to 1.

Identify \mathbb{A}_θ with the polynomial algebra $\mathbb{R}[\mathbf{h}_2, \mathbf{h}_3, \dots]$. Using the argument of [12, Lemma 7.12], one deduces from formulas (6.6), (6.7), (6.8), and (6.9) that

$$\partial = \theta \frac{\partial}{\partial \mathbf{h}_2} + \dots$$

where the dots stand for terms of degree ≤ -2 (that is, operators in \mathbb{A}_θ reducing degree at least by 2). This concludes the proof, since the operator $\partial/\partial \mathbf{h}_2$ reduces degree by 1 (recall that \mathbf{h}_2 has degree 1). \square

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