An addition theorem on the cyclic group $\mathbb{Z}_{p^\alpha q^\beta}$

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Submitted: May 22, 2005; Accepted: Apr 30, 2006; Published: May 12, 2006
Mathematics Subject Classifications: 11B75, 20K99

Abstract

Let $n > 1$ be a positive integer and p be the smallest prime divisor of n. Let S be a sequence of elements from $\mathbb{Z}_n = \mathbb{Z}/n\mathbb{Z}$ of length $n + k$ where $k \geq \frac{n}{p} - 1$. If every element of $\mathbb{Z}_n$ appears in S at most k times, we prove that there must be a subsequence of S of length n whose sum is zero when n has only two distinct prime divisors.

1 Introduction

Let G be an additive abelian group and $S = \{a_i\}_{i=1}^m$ be a sequence of elements from G. Denote $\sigma(S) = \sum_{i=1}^m a_i$. We say S is zero-sum if $\sigma(S) = 0$. For each integer $1 \leq r \leq m$, we denote

$$\sum_r S = \{a_{i_1} + a_{i_2} + \cdots + a_{i_r} \ : \ 1 \leq i_1 < i_2 < \cdots < i_r \leq m\}.$$

Let $h(S)$ denote the maximal multiplicity of the terms of S.

In 1961, Erdős-Ginzburg-Ziv [1] proved the following theorem.

EGZ Theorem *If S is a sequence of elements from $\mathbb{Z}_n$ of length $2n-1$, then $0 \in \sum_n S$.*

The inverse problem to EGZ Theorem is how to describe the structure of a sequence S in $\mathbb{Z}_n$ with $0 \notin \sum_n S$. Recently W. D. Gao [2] made a conjecture as follows and proved it for $n = p^l$ for any prime p and any integer $l > 1$.

Conjecture *Let $n > 1$, k be positive integers and p be the smallest prime divisor of n. Let S be a sequence of elements from $\mathbb{Z}_n$ of length $n + k$ with $k \geq \frac{n}{p} - 1$. If $0 \notin \sum_n S$ then $h(S) > k$.*

In this paper we shall prove the Conjecture for n which has only two distinct prime divisors.

Theorem 1 *The above* Conjecture *is true for $n = p^\alpha q^\beta$ where p, q are distinct primes and α, β are positive integers.*

2 Proof of Theorem 1

For any subset A of an abelian group G let $H(A)$ denote the maximal subgroup of G such that $A + H(A) = A$. What we state below is a classical theorem of Kneser [3].

Kneser's Theorem *Let G be a finite abelian group. Let $A_1, A_2, \ldots, A_n$ be nonempty subsets of G. Then*

$$|A_1 + A_2 + \cdots + A_n| \geq \sum_{i=1}^{n} |A_i + H| - (n-1)|H|,$$

where $H = H(A_1 + A_2 + \cdots + A_n)$.

Lemma 1 *Let $k \geq 2, n = p_1^{\alpha_1} p_2^{\alpha_2}$ be integers where p_1, p_2 are distinct primes and α_1, α_2 are positive integers. Let S be a sequence of elements from $\mathbb{Z}_n = \mathbb{Z}/n\mathbb{Z}$ of length $n + k$. If $h(S) \leq k$ then $H(\sum_k S) \neq \{0\}$.*

Proof Suppose that $H(\sum_k S) = \{0\}$. Let N_i be the subgroup of $\mathbb{Z}_n$ with $|N_i| = p_i$ for $i = 1, 2$. Then $\sum_k S + N_i \not\subseteq \sum_k S$ for $i = 1, 2$. And so there exist subsequences $\{a_j^{(i)}\}_{j=1}^{k} (i = 1, 2)$ of S such that

$$\sum_{j=1}^{k} a_j^{(i)} + N_i \not\subseteq \sum_k S, \quad i = 1, 2.$$

We can assume that $a_j^{(1)} = a_j^{(2)}$ for $1 \leq j \leq l$ and $a_j^{(1)} \neq a_r^{(2)}$ for $l < j, r \leq k$. Then

$$\{a_1^{(1)}, a_2^{(1)}, \cdots, a_l^{(1)}, a_{l+1}^{(1)}, \cdots, a_k^{(1)}, a_{l+1}^{(2)}, \cdots, a_k^{(2)}\}$$

is a subsequence of S. Now we distribute the terms of S into k subsets $A_1, A_2, \ldots, A_k$. At first, we put $a_j^{(1)}$ into A_j for $1 \leq j \leq l$ and $a_j^{(1)}, a_j^{(2)}$ into A_j for $l < j \leq k$. Then the other terms of S are put into $A_1, A_2, \cdots, A_k$ such that each A_i does not include identical terms. Since $h(S) \leq k$, we can do it. Therefore

$$\sum_{j=1}^{k} a_j^{(1)} \in A_1 + A_2 + \cdots + A_k,$$

and

$$\sum_{j=1}^{k} a_j^{(2)} = \sum_{j=1}^{l} a_j^{(1)} + \sum_{j=l+1}^{k} a_j^{(2)} \in A_1 + A_2 + \cdots + A_k.$$

As $A_1 + A_2 + \cdots + A_k \subseteq \sum_k S$, we have

$$\sum_{j=1}^{k} a_j^{(i)} + N_i \not\subseteq A_1 + A_2 + \cdots + A_k, \quad i = 1, 2.$$

It follows that

$$N_i \not\subseteq H(A_1 + A_2 + \cdots + A_k), \quad i = 1, 2.$$

Since every nontrivial subgroup of $\mathbb{Z}_n$ contains either N_1 or N_2, we must have $H(A_1 + A_2 + \cdots + A_k) = \{0\}$. As a result, Kneser's Theorem implies

$$|A_1 + A_2 + \cdots + A_k| \geq \sum_{j=1}^{k} |A_j| - (k-1) = n + 1,$$

contradicting $A_1 + A_2 + \cdots + A_k \subseteq \mathbb{Z}_n$.

Now the proof is complete.

Lemma 2 (Gao, [2]) *Let G be a cyclic group of order n. Let S be a sequence of elements from G of length $n + k$ where $k \geq \frac{n}{p} - 1$ and p is the smallest prime divisor of n. Then*

$$\sum_n S \bigcap H \neq \emptyset$$

for any nontrivial subgroup H of G.

Proof For any nontrivial subgroup H of G, let $\varphi : G \to G/H$ be the natural homomorphism. Then $\varphi(S)$ is a sequence of elements from G/H of length $n + k$. Since $|H| \geq p$,

$$n + k \geq n + \frac{n}{p} - 1 \geq |H||G/H| + |G/H| - 1,$$

using EGZ Theorem repeatedly, we can find $|H|$ disjoint zero-sum subsequences of $\varphi(S)$, each of which has length $|G/H|$. Thus we find a subsequence of S with length $|H||G/H| = n$, whose sum is in H, i.e., $\sum_n S \cap H \neq \emptyset$. We are done.

Proof of Theorem 1 Suppose that $h(S) \leq k$. By Lemma 1, $H = H(\sum_k S) \neq \{0\}$. Thus Lemma 2 implies that $\sum_n S \cap H \neq \emptyset$. Therefore we have a subsequence $\{a_i\}_{i=1}^k$ of S such that $\sigma(S) - \sum_{i=1}^k a_i \in H$. And so

$$\sigma(S) \in \sum_{i=1}^{k} a_i + H \subseteq \sum_k S + H = \sum_k S.$$

It follows that $0 \in \sum_n S$. This ends the proof.

Acknowledgment. I would like to thank W. D. Gao for his report in which he introduced his conjecture.

References

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