A unifying framework for interpolation on general Lissajous-Chebyshev points

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Outline of the talk

▶ Introduction
  ▶ What are Lissajous-Chebyshev points?
  ▶ Preliminary questions towards a unified theory
▶ Interpolation on Lissajous-Chebyshev nodes $\mathbf{LC}_{\kappa}^{(m)}$
  ▶ Some description of the involved Lissajous curves
  ▶ Interpolation and quadrature on $\mathbf{LC}_{\kappa}^{(m)}$
  ▶ Convergence and fast algorithms of the interpolation schemes
Definition of Lissajous-Chebyshev points $\text{LC}_{\kappa}^{(m)}$

We define the sets $\text{LC}_{\kappa}^{(m)}$ with help of the index sets

$$I_{\kappa}^{(m)} = I_{\kappa,0}^{(m)} \cup I_{\kappa,1}^{(m)}$$

with the sets $I_{\kappa,r}^{(m)}$, $r \in \{0,1\}$, given by

$$I_{\kappa,r}^{(m)} = \left\{ i \in \mathbb{N}_0^d \mid \forall j: 0 \leq i_j \leq m_j \text{ and } i_j \equiv r + \kappa_j \mod 2 \right\}.$$
With the Chebyshev-Gauss-Lobatto points given by

$$z_i^{(m)} = \left( z_{i_1}^{(m_1)}, \ldots, z_{i_d}^{(m_d)} \right), \quad z_i^{(m)} = \cos \left( i \pi / m \right).$$

we then define the Lissajous-Chebyshev points as

$$\text{LC}_{\kappa}^{(m)} = \left\{ z_i^{(m)} \mid i \in I^{(m)}_{\kappa} \right\}.$$
Cardinalities of the node sets

We have

\[ \# \mathbf{LC}^{(m)}_{\kappa, \tau} = \# I^{(m)}_{\kappa} = \# I^{(m)}_{\kappa, 0} + \# I^{(m)}_{\kappa, 1} \]

with

\[ \# I^{(m)}_{\kappa, \tau} = \prod_{i \in \{1, \ldots, d\}} \frac{m_i + 2}{2} \times \prod_{i \in \{1, \ldots, d\}} \frac{m_i}{2} \times \prod_{i \in \{1, \ldots, d\}} \frac{m_i + 1}{2} \]

- \( m_i \equiv 0 \mod 2 \) and \( \kappa_i \equiv \tau \mod 2 \)
- \( m_i \equiv 0 \mod 2 \) and \( \kappa_i \not\equiv \tau \mod 2 \)
- \( m_i \equiv 1 \mod 2 \)
Examples

The interpolation nodes $\mathbf{LC}_{\kappa}^{(m)}$ are well-known in the literature

- Morrow-Patterson-Xu points 2D: $\mathbf{LC}_{\kappa}^{(m,m)}$ [10, 11].
- Morrow-Patterson-Xu points 3D: $\mathbf{LC}_{\kappa}^{(m,m,m)}$ [5].
- Padua points: $\mathbf{LC}_{(0,0)}^{(m)}$ for $m = (m, m+1)$ or $m = (m+1, m)$ [3, 4].
- Lissajous nodes in MPI: $\mathbf{LC}_{(0,1)}^{(2m_1,2m_2)}$ with $m_1$, $m_2$ relatively prime [9].
- Degenerate Lissajous curves: $\mathbf{LC}_{0}^{(m)}$, in which $m$ consists of relatively prime numbers [6].

$\mathbf{LC}_{\kappa}^{(m)}$ are also well-known nodes for multivariate quadrature [1].
Observation 1:

- Polynomial interpolation on all of these point sets is very similar.
- Many of these points have a generating Lissajous curve:

\[ \ell^{(8,6)}_{(4,3)}(t) = (\sin 3t, \sin 4t) \]

Non-degenerate Lissajous curve used in Magnetic Particle Imaging [9].

\[ \ell^{(6,5)}_{(0,0)}(t) = (\cos 5t, \cos 6t) \]

Degenerate Lissajous curve generating the Padua points [3, 4].
Observation 2:

- Morrow-Patterson-Xu (MPX) points are more symmetric compared to Padua points. In the literature, there is however no generating curve given for MPX points.
- Interpolation spaces have a slightly more complicated structure [11].

Is there a way to get a single Lissajous curve that connects these points?
Questions considered in this tutorial

- Is there a unified interpolation framework including Padua points, MPX points and Lissajous curves?
- Is there a single generating curve for the MPX points? What are the alternatives?
- Are there fundamental differences in the convergence and the implementation of the different schemes?
Definition of $d$-dimensional Lissajous curves

We will consider $d$-dimensional Lissajous curves

$$\ell_{\kappa, u}^{(m)} : \mathbb{R} \to \mathbb{R}^d$$

in the parametrized form

$$\ell_{\kappa, u}^{(m)}(t) = \left( u_1 \cos \left( \frac{\text{lcm}[m] \cdot t - \kappa_1 \pi}{m_1} \right), \ldots, u_d \cos \left( \frac{\text{lcm}[m] \cdot t - \kappa_d \pi}{m_d} \right) \right),$$

where

- $m = (m_1, \ldots, m_d) \in \mathbb{N}^d$ are ‘frequency dividers’,
- $u \in \{-1, 1\}^d$ are ‘reflection parameters’,
- $\text{lcm}[m]$ is the least common multiple of $m_1, \ldots, m_d$,
- $\kappa = (\kappa_1, \ldots, \kappa_d) \in \mathbb{R}^d$ specifies additional phase shifts.

The definition guarantees that in any case the minimal period of $\ell_{\kappa, u}^{(m)}$ is $2\pi$.

**We know:** If the entries $m_i$ are pairwise relatively prime, then the Lissajous curve $\ell_{\kappa, u}^{(m)}$ generates the points $\text{LC}_{\kappa}^{(m)}$ [6].
If we try to use Lissajous curves to generate the MPX points we get

\[ \ell(4,4)(0,0), \ell(4,4)(0,2), \ell(4,4)(0,4) \]

Using \( \ell^{(4,4)}(t) = (\cos t, \cos t) \) as generating curve.

Using \( \ell^{(4,4)}(t), \ell^{(4,4)}(t), \ell^{(4,4)}(t) \) as generating curves.

**Observation:** For MPX points in general more than 1 generating curve is needed. The number depends on \( m \) and \( \kappa \).
The union of all generating Lissajous curves forms an algebraic variety

\[ C_{\kappa}^{(m)} = \{ x \in [-1, 1]^d \mid (-1)^{\kappa_1} T_{m_1}(x_1) = \ldots = (-1)^{\kappa_d} T_{m_d}(x_d) \} , \]

where \( T_m \) denote the Chebyshev polynomial of first kind of degree \( m \). The variety \( C_{\kappa}^{(m)} \) is called Chebyshev variety.

**Theorem**

*We have*

\[ \text{LC}_{\kappa}^{(m)} = \left\{ x \in [-1, 1]^d \mid (-1)^{\kappa_1} T_{m_1}(x_1) = \ldots = (-1)^{\kappa_d} T_{m_d}(x_d) \in \{ \pm 1 \} \right\} . \]

Note: the elements of \( \text{LC}_{\kappa}^{(m)} \) in the interior of the hypercube \([-1, 1]^d\) are exactly the singular points of the variety \( C_{\kappa}^{(m)} \).
Characterize the Lissajous curves inside $C^{(m)}_{\kappa}$

**Proposition**

Let $m \in \mathbb{N}^d$. There exist (not necessarily uniquely determined) integer vectors $m^\#$, $m^b \in \mathbb{N}^d$ such that the following properties are satisfied:

1. For all $i \in \{1, \ldots, d\}$: $m_i = m_i^b m_i^\#$  \hspace{1cm} (1a)
2. For all $i \in \{1, \ldots, d\}$: $m_i^b$ and $m_i^\#$ are relatively prime. \hspace{1cm} (1b)
3. The numbers $m_1^\#, \ldots, m_d^\#$ are pairwise relatively prime. \hspace{1cm} (1c)
4. We have $\text{lcm}[m] = p[m^\#] = \prod_{i=1}^d m_i^\#$. \hspace{1cm} (1d)

Define the sets

$$H(m^\#) = \{0, \ldots, 2p[m^\#] - 1\} \quad \text{and} \quad R(m^b) = \prod_{i=1}^d \{0, \ldots, m_i^b - 1\}.$$
Proposition

Let $\mathbf{m}, \mathbf{m}^\#, \mathbf{m}^b \in \mathbb{N}^d$ satisfy the conditions (1a)-(1d), then

a) For all $(l, \rho) \in H(m^\#) \times R(m^b)$, there exists a uniquely determined $i \in I(m)$ and a (not necessarily unique) $v \in \{-1, 1\}^d$ such that

$$\forall i \in \{1, \ldots, d\} : \quad i_i \equiv v_i \left( l - 2\rho_i m_i^\# - \kappa_i \right) \mod 2m_i.$$  

Thus, a function $j : H(m^\#) \times R(m^b) \to I(m)$ is well defined by

$$j(l, \rho) = i.$$  

b) Let $M \subseteq \{1, \ldots, d\}$. If $i \in I(m)$ and $z_i(m) \in F_M$, then

$$\#\{ (l, \rho) \in H(m^\#) \times R(m^b) \mid j(l, \rho) = i \} = 2^\#M.$$
We consider the following set of Lissajous curves

\[
\mathcal{L}_{\kappa}^{(m^\#, m^b)} = \left\{ \ell^{(m)}_{(\kappa_1+2\rho_1 m^\#, \ldots, \kappa_d+2\rho_d m^\#)} \middle| \rho \in R^{(m^b)} \right\}.
\]

**Theorem**

Let \( m, m^\#, m^b \in \mathbb{N}^d \) satisfy the conditions (1a)-(1d).

a) Using the sampling points \( t_l^{(m)} \), we have

\[
\mathbf{LC}_{\kappa}^{(m)} = \left\{ \ell(t_l^{(m)}) \middle| \ell \in \mathcal{L}_{\kappa}^{(m^\#, m^b)}, l \in H^{(m^\#)} \right\}.
\]

b) The affine Chebyshev variety \( C_{\kappa}^{(m)} \) can be decomposed as

\[
C_{\kappa}^{(m)} = \bigcup_{\ell \in \mathcal{L}_{\kappa}^{(m^\#, m^b)}} \ell([0, 2\pi)).
\]
Example: MPX points in 2D

One possible decomposition of $\mathbf{m}$ is given by $\mathbf{m}^\# = (m, 1)$, $\mathbf{m}^b = (1, m)$. The respective sets $H(\mathbf{m}^\#)$ and $R(\mathbf{m}^b)$ are given by

$$H(\mathbf{m}^\#) = \{0, \ldots, 2m - 1\} \quad \text{and} \quad R(\mathbf{m}^b) = \{0\} \times \{0, \ldots, m - 1\}.$$
Example: Padua points and Lissajous curves

If $m_1$ and $m_2$ are relatively prime, the decomposition of $m$ is given by $m^\# = (m_1, m_2)$, $m^\flat = (1, 1)$. Then

$$H(m^\#) = \{0, \ldots, 2m_1 m_2 - 1\} \quad \text{and} \quad R(m^\flat) = \{0\} \times \{0\}.$$

$$\ell^{(5,3)}_{(0,0)}(t) = (\cos 3t, \cos 5t) \quad \ell^{(6,5)}_{(0,0)}(t) = (\cos 5t, \cos 6t)$$
Polynomial interpolation on $\mathbf{LC}_{\kappa}^{(m)}$

We are looking for polynomial interpolants of the form

$$P_{\kappa, h}^{(m)}(x) = \sum_{\gamma \in \Gamma_{\kappa}^{(m)}} c_\gamma(h) T_\gamma(x),$$

$$\tilde{P}_{\kappa, h}^{(m)}(x) = \sum_{\gamma \in \tilde{\Gamma}_{\kappa}^{(m)}} \frac{c_\gamma(h)}{\#[\gamma]} T_\gamma(x),$$

such that the following interpolation condition is satisfied:

$$P_{\kappa, h}^{(m)}(z_i^{(m)}) = \tilde{P}_{\kappa, h}^{(m)}(z_i^{(m)}) = h(i), \quad i \in \Omega_{\kappa}^{(m)}. \quad (IP)$$

- $T_\gamma(x) = \prod_{i=1}^{d} \cos(\gamma_i \arccos x_i)$ are multivariate Chebyshev polynomials,
- $\Gamma_{\kappa}^{(m)}, \tilde{\Gamma}_{\kappa}^{(m)}$ are appropriate spectral index sets.
Examples

For the Padua points $\mathbf{LC}_{(0,0)}^{(6,5)}$ we use the index set $\Gamma_{(0,0)}^{(6,5)}$.

For the MPX points $\mathbf{LC}_{(0,0)}^{(5,5)}$, we can use the index set $\Gamma_{(0,0)}^{(5,5)}$. 
General definition of spectral index sets $\overline{\Gamma}^{(m)}_{\kappa}$

For $\mathbf{m} \in \mathbb{N}^d$, $\mathbf{\kappa} \in \mathbb{N}^d$, $r \in \{0,1\}$, we define the cubic index sets

$$\Gamma^{(m)}_{\mathbf{\kappa}, r} = \left\{ \gamma \in \mathbb{N}_0^d \middle| \begin{array}{c} \forall i \text{ with } \kappa_i \equiv r \mod 2 : 2\gamma_i \leq m_i, \\ \forall i \text{ with } \kappa_i \not\equiv r \mod 2 : 2\gamma_i < m_i \end{array} \right\},$$

and the spectral index sets

$$\overline{\Gamma}^{(m)}_{\mathbf{\kappa}} = \left\{ \gamma \in \mathbb{N}_0^d \middle| \begin{array}{c} \forall i \in \{1, \ldots, d\} : \gamma_i \leq m_i, \\ \forall i, j \text{ with } i \neq j : \gamma_i/m_i + \gamma_j/m_j \leq 1, \\ \forall i, j \text{ with } \kappa_i \not\equiv \kappa_j \mod 2 : (\gamma_i, \gamma_j) \neq (m_i/2, m_j/2) \end{array} \right\}.$$  

For $d = 2$, $\overline{\Gamma}^{(m)}_{\mathbf{\kappa}}$ contains all integer vectors inside a triangle.

If $d > 2$, the spectral index set $\overline{\Gamma}^{(m)}_{\mathbf{\kappa}}$ has a polyhedral structure.
Examples in 3D

The spectral index set $\Gamma^{(4,4,4)}_{(0,0,0)}$ for the MPX points.

The spectral index set $\Gamma^{(5,4,2)}_{(0,0,1)}$. 
We introduce a class decomposition \( \Gamma_{\kappa}^{(m)} \) of \( \Gamma_{\kappa}^{(m)} \). We define

\[
K^{(m)}(\gamma) = \left\{ j \in \{1, \ldots, d\} \mid \gamma_j/m_j = \max^{(m)}[\gamma] \right\}
\]

where \( \max^{(m)}[\gamma] = \max \{ \gamma_i/m_i \mid i \in \{1, \ldots, d\} \} \).

Further, using the flip operator

\[
s_j^{(m)}(\gamma) = (\gamma_1, \ldots, \gamma_{j-1}, m_j - \gamma_j, \gamma_{j+1}, \ldots, \gamma_d)
\]

we define the sets \( G^{(m)}(\gamma) = \left\{ s_j^{(m)}(\gamma) \mid j \in K^{(m)}(\gamma) \right\} \).

Now, we introduce the class decomposition \( \Gamma_{\kappa}^{(m)} \) as

\[
\left[ \Gamma_{\kappa}^{(m)} \right] = \left\{ \{ \gamma \} \mid \gamma \in \Gamma_{\kappa,0}^{(m)} \right\} \cup \left\{ G^{(m)}(\gamma) \mid \gamma \in \Gamma_{\kappa,1}^{(m)} \right\}.
\]

The set \( \Gamma_{\kappa}^{(m)} \) denotes a set of representatives of \( \left[ \Gamma_{\kappa}^{(m)} \right] \).
By this definition of the class decomposition \( \Gamma_{\kappa}^{(m)} \) we get

\[
\# \left[ \Gamma_{\kappa}^{(m)} \right] = \# \Gamma_{\kappa,0}^{(m)} + \# \Gamma_{\kappa,1}^{(m)} = \# I_{\kappa,1}^{(m)} + \# I_{\kappa,0}^{(m)} = \# I_{\kappa}^{(m)} = \# L C_{\kappa}^{(m)}
\]

In special cases (as for instance the Padua points) the situation is simpler.

**Proposition**

Let \( \kappa \in \mathbb{Z}^d \). The following statements are equivalent.

i) We have \( \gcd\{m_i, m_j\} \leq 2 \) for all \( i, j \in \{1, \ldots, d\} \) with \( i \neq j \).

ii) All classes in \( \left[ \Gamma_{\kappa}^{(m)} \right] \setminus \{ \mathcal{S}(m)(0) \} \) consist of precisely one element.
The spectral index set $\Gamma^{(4,4)}_{(0,0)}$ for MPX points.

The spectral index set $\Gamma^{(5,4,2)}_{(0,0,1)}$. 
Discrete orthogonality structure

For \( i \in I^{(m)}_{\kappa} \), the weights are given by

\[
w^{(m)}_{\kappa, i} = \frac{2^#M}{(2p[m])} \quad \text{if} \quad z^{(m)}_{i} \in \mathcal{L}C^{(m)}_{\kappa} \cap F_{M}^d.
\]

and the measure \( \omega^{(m)}_{\kappa} \) on the power set of \( I^{(m)}_{\kappa} \) by \( \omega^{(m)}_{\kappa} \{ i \} = w^{(m)}_{\kappa, i} \).

Denote by \( \mathcal{L}(I^{(m)}_{\kappa}) \) the set of the functions \( h : I^{(m)}_{\kappa} \to \mathbb{C} \).

To prove the unisolvence of the interpolation problem (IP), we show that

\[
\chi^{(m)}_{\gamma}(i) = T_{\gamma}(z^{(m)}_{i}) = \prod_{i=1}^{d} \cos(\gamma_i i \pi / m_i), \quad \gamma \in \Gamma^{(m)}_{\kappa},
\]

is an orthogonal basis of the Hilbert space \( \mathcal{L}(I^{(m)}_{\kappa}) \) with respect to

\[
\langle f, g \rangle_{\omega^{(m)}_{\kappa}} = \sum_{i \in I^{(m)}_{\kappa}} f(i) \ g(i) \ w^{(m)}_{\kappa, i}.
\]
Proposition

For \( \gamma \in \mathbb{N}_0^d \) and \( \chi^{(m)}_{\gamma} \in \mathcal{L}(\mathbb{I}_{\kappa}^{(m)}) \) we have

\[
\sum_{i \in \mathbb{I}_{\kappa}^{(m)}} \chi^{(m)}_{\gamma}(i) \ w^{(m)}_{\kappa} \neq 0
\]

if and only if

there exists \( h \in \mathbb{N}_0^d \) with \( \gamma_i = h_i m_i \), \( i = 1, \ldots, d \), and \( \sum_{i=1}^{d} h_i \in 2\mathbb{N}_0 \). (2)

If (2) is satisfied, then

\[
\sum_{i \in \mathbb{I}_{\kappa}^{(m)}} \chi^{(m)}_{\gamma}(i) \ w^{(m)}_{\kappa} = (-1)^{\sum_{i=1}^{d} h_i \kappa_i}.
\]

For the proof of the orthogonality we further need the product formula

\[
\chi^{(m)}_{\gamma} \chi^{(m)}_{\gamma'} = \frac{1}{2^d} \sum_{\boldsymbol{v} \in \{-1,1\}^d} \chi^{(m)}(\gamma_1 + v_1 \gamma'_1, \ldots, \gamma_d + v_d \gamma'_d).
\]
Main interpolation result

We consider $\Pi_{\kappa}(m) = \text{span}\{ T_{\gamma} \mid \gamma \in \Gamma_{\kappa}(m) \}$ and an appropriately defined space $\tilde{\Pi}_{\kappa}(m)$ regarding (anti-)symmetries on the classes $[\gamma]$, see [7].

**Theorem**

For $h \in \mathcal{L}(\mathbf{I}_{\kappa}(m))$, the unique coefficients $c_{\gamma}(h)$ such that the polynomials

$$P_{\kappa,h}(m)(x) = \sum_{\gamma \in \Gamma_{\kappa}(m)} c_{\gamma}(h) T_{\gamma}(x), \quad \tilde{P}_{\kappa,h}(m)(x) = \sum_{\gamma \in \tilde{\Gamma}_{\kappa}(m)} \frac{c_{\gamma}(h)}{\# [\gamma]} T_{\gamma}(x),$$

solve the interpolation problem (IP) in $\Pi_{\kappa}(m)$ or $\tilde{\Pi}_{\kappa}(m)$, respectively, are

$$c_{\gamma}(h) = \frac{1}{\| \chi^{(m)}_{\gamma} \|_{\omega_{\kappa}^{(m)}}^{2}} \langle h, \chi^{(m)}_{\gamma} \rangle_{\omega_{\kappa}^{(m)}}.$$
Efficient computation of the interpolating polynomial

We introduce

\[
g^{(m)}_{\kappa_\gamma}(i) = \begin{cases} w^{(m)}_{\kappa_\gamma}(i) h(i), & \text{if } i \in I^{(m)}_\kappa, \\ 0, & \text{if } i \in J^{(m)} \setminus I^{(m)}_\kappa, \end{cases}\]

and the d-dimensional discrete cosine transform \( \hat{g}^{(m)}_{\kappa_\gamma} \) of \( g^{(m)}_{\kappa_\gamma} \) starting with

\[
\hat{g}^{(m)}_{\kappa_\gamma}(i_2, \ldots, i_d) = \sum_{i_1=0}^{m_1} g^{(m)}_{\kappa_\gamma}(i) \cos(\gamma_1 i_1 \pi / m_1).
\]

and, then proceeding recursively for \( i = 2, \ldots, d \) with

\[
\hat{g}^{(m)}_{\kappa_\gamma, (\gamma_1, \ldots, \gamma_i)}(i_{i+1}, \ldots, i_d) = \sum_{i_1=0}^{m_i} \hat{g}^{(m)}_{\kappa_\gamma, (\gamma_1, \ldots, \gamma_{i-1})}(i_1, \ldots, i_d) \cos(\gamma_i i_{i+1} \pi / m_i).
\]

Then, we have

\[
c_{\gamma}(h) = \| \chi^{(m)}_{\kappa_\gamma} \|^{-2} \omega^{(m)}_{\kappa_\gamma} \hat{g}^{(m)}_{\kappa_\gamma}(\gamma).
\]

Using FFT this can be done in \( \mathcal{O}(p[m] \log p[m]) \) steps.
Matlab toolboxes for interpolation at the nodes $\mathbf{LC}^{(m)}_{\kappa}$ are available at

https://github.com/WolfgangErb/LC2Ditp
https://github.com/WolfgangErb/LC3Ditp


New cubature formulae and hyperinterpolation in three variables. *BIT* 49, 1 (2009), 55–73.


[9] Erb, W., Kaethner, C., Dencker, P., and Ahlborg, M.
