Fekete Points as Norming Sets

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To my friend and long time collaborator, Norm Levenberg, on the occasion of his sixtieth birthday.

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Abstract

Suppose that $K \subset \mathbb{R}^d$ is compact. Fekete points of degree $n$ are those points $F_n \subset K$ that maximize the determinant of the interpolation matrix for polynomial interpolation of degree $n$. We discuss some special cases where we can show that Fekete points (of uniformly higher degree) are norming sets for $K$, i.e., for any $c > 1$, there exists a constant $C > 0$ such that $\|p\|_K \leq C\|p\|_{F_n}$, for all polynomials of degree at most $n$. It is conjectured that this is true for “general” $K$.

1 Introduction

Suppose that $K \subset \mathbb{R}^d$ is compact. We let $P_n(K)$ denote the space of polynomials of degree $\leq n$, restricted to $K$ and $N_n(K) = \dim(P_n(K))$. Often, when no ambiguity is possible, we will abbreviate, $N_n(K) = N_n$, or even $N_n(K) = N$. Also, in case $m \geq 0$ is not an integer, we will let

$$N_m(K) = N_m := N_{\lfloor m \rfloor}(K).$$

We note that if $K$ is polynomially determining, i.e., $p(x) = 0$ for $\forall x \in K$ implies that $p \equiv 0$, then

$$N_n(K) = \binom{n + d}{d} - \binom{n - 2 + d}{d}.$$

Otherwise the dimension may be smaller than this binomial expression. Indeed, for for $K = S^{d-1} \subset \mathbb{R}^d$, the unit sphere $P_n(K)$ is the space of spherical harmonics of degree at most $n$ and then

$$N_n(K) = \binom{n + d}{d} - \binom{n - 2 + d}{d}.$$

The corresponding polynomial interpolation problem may be formulated as follows. Given $x_1, x_2, \ldots, x_N$ points in $K$ and values $z_1, z_2, \ldots, z_N \in \mathbb{R}$, find $p \in P_n(K)$ such that $p(x_i) = z_i$, $i = 1, \ldots, N$. Its solution is accomplished by choosing a basis $\{p_1, p_2, \ldots, p_N\}$ for $P_n(K)$, writing $p = \sum_{j=1}^{N} a_j p_j$ and considering the associated linear system

$$\begin{bmatrix}
  p_1(x_1) & p_2(x_1) & \cdots & p_N(x_1) \\
p_1(x_2) & p_2(x_2) & \cdots & p_N(x_2) \\
  \vdots & \vdots & \ddots & \vdots \\
p_1(x_N) & p_2(x_N) & \cdots & p_N(x_N)
\end{bmatrix} \begin{bmatrix}
  a_1 \\
a_2 \\
  \vdots \\
a_N
\end{bmatrix} = \begin{bmatrix}
  z_1 \\
z_2 \\
  \vdots \\
z_N
\end{bmatrix}$$

corresponding to $p(x_i) = z_i$, $1 \leq i \leq N$.

Hence, the interpolation problem has a unique solution for any set of values $z_i$ iff the associated, so-called vandermonde determinant

$$\text{vdm}(x_1, x_2, \ldots, x_N) := \begin{vmatrix}
p_1(x_1) & p_2(x_1) & \cdots & p_N(x_1) \\
p_1(x_2) & p_2(x_2) & \cdots & p_N(x_2) \\
  \vdots & \vdots & \ddots & \vdots \\
p_1(x_N) & p_2(x_N) & \cdots & p_N(x_N)
\end{vmatrix}$$

is non-zero. If this is the case then one may form the so-called fundamental (cardinal) Lagrange polynomials,

$$l_i(x) := \frac{\text{vdm}(x_1, \ldots, x_{i-1}, x, x_{i+1}, \ldots, x_N)}{\text{vdm}(x_1, \ldots, x_N)}, \quad 1 \leq i \leq N.$$
These are cardinal in the sense that \( \ell_j(x_j) = \delta_{ij} \). Further, the interpolation projection \( \pi : C(K) \to \mathcal{P}_n(K) \) is given by

\[
\pi(f)(x) = \sum_{i=1}^{n} f(x_i) \ell_i(x)
\]

with operator norm

\[
\|\pi\| = \max_{x \in K} \sum_{i=1}^{n} |\ell_i(x)|,
\]

otherwise known as the Lebesgue constant.

Points \( f_1, f_2, \ldots, f_N \in K \) are said to be Fekete points of degree \( n \) if they maximize \( vdm(x_1, \ldots, x_n) \) over \( K^N \). Collecting Fekete points for degrees \( n = 1, 2, \ldots \) we get a Fekete array, \( F_1, F_2, \ldots \). We note that they need not be unique!

Fekete points have the basic properties that \( \max_{x \in K} |\ell_i(x)| = 1 \) and that the Lebesgue constant \( \|\pi\| = \max_{x \in K} \sum_{i=1}^{n} |\ell_i(x)| \leq N \).

Consequently, for \( p \in \mathcal{P}_n(K) \),

\[
\|p\|_K \leq N \|p\|_{P_n}.
\]

Here, for \( X \subset \mathbb{C}^d \), compact, and \( f \in C(X) \),

\[
\|f\|_X := \max_{x \in K} |f(x)|.
\]

In words, the maximum norm of a polynomial of degree at most \( n \) on all of \( K \) is at most \( N \) times its norm on \( F_n \). A Norming Set is one for which this upper bound factor \( N \) may be replaced by a constant. Specifically, an array of subsets \( X_n \subset K \), \( n = 1, 2, \cdots \) is a Norming Set if there exists a constant \( C \) such that

\[
\|p\|_K \leq C \|p\|_{X_n}, \quad \forall p \in \mathcal{P}_n(K), \quad n = 1, 2, \cdots.
\]

Clearly, \( \#(X_n) \geq N (= \dim(\mathcal{P}_n(K))) \) and a Norming Set is said to be optimal if \( \#(X_n) = O(N) \).

The first theorem in this regard is that of Ehlich and Zeller [7].

**Theorem 1.1** (Ehlich-Zeller 1964). For any \( \alpha > 1 \) the Chebyshev points of degree \( \lfloor \alpha \rfloor \) form an optimal Norming Set for \([-1, 1]\).

The proof is simple, yet informative, based on the fact that the Chebyshev points are well-spaced with respect to the arc-cosine metric and uses an appropriate Markov-Bernstein inequality for the derivatives of polynomials. A rather general result, based on the so-called Dubiner distance is as follows.

**Definition 1.** Suppose that \( K \subset \mathbb{R}^d \) is compact. Then the Dubiner distance between any two points \( x, y \in K \) is defined as

\[
d_K(x, y) := \sup_{n \geq 1, p \in \mathcal{P}_n(K), \|p\|_K = 1} \frac{1}{n} |\cos^{-1}(p(x)) - \cos^{-1}(p(y))|.
\]

The Dubiner distance was introduced by Dubiner in [6] and extensively studied in [4] and [5]. In particular for \( K = [-1, 1] \subset \mathbb{R}^1 \),

\[
d_K(x, y) = |\cos^{-1}(x) - \cos^{-1}(y)|
\]

is the arc-cosine metric.

**Proposition 1.2.** (see [3] and [10, Prop. 1]) Suppose that \( K \subset \mathbb{R}^d \) is compact and that \( X_n \subset K \) is a subset with the property that there is some \( \alpha < \pi/2 \),

\[
\min_{y \in X_n} d_K(x, y) \leq \frac{\alpha}{n}, \quad \forall x \in K.
\]

Then, for all \( p \in \mathcal{P}_n(K) \),

\[
\|p\|_K \leq \sec(\alpha) \|p\|_{X_n}.
\]

**Proof.** Suppose that \( x \in K \) is such that \( \|p(x)\| = \|p\|_K \), which we may assume without loss to be \( \|p\|_K = 1 \). We may further assume, by normalizing by \(-1\) if necessary, that \( p(x) = 1 \). By assumption there exists a point \( y \in X_n \) such that \( d_K(x, y) \leq \alpha/n \). Hence

\[
\frac{1}{n} \cos^{-1}(p(y)) = \frac{1}{n} |\cos^{-1}(p(y))|
\]

\[
= \frac{1}{n} |\cos^{-1}(p(x)) - \cos^{-1}(p(y))|
\]

\[
\leq d_K(x, y)
\]

\[
\leq \frac{\alpha}{n}
\]

from which it follows that

\[
\cos^{-1}(p(y)) \leq \alpha < \pi/2
\]

and, in particular, \( p(y) > 0 \).

Consequently, as \( \cos^{-1} \) is decreasing,

\[
p(y) \geq \cos(\alpha)
\]
and thus
\[ ||p||_{\infty} = 1 \leq \frac{1}{\cos(\alpha)} |p(y)| \leq \sec(\alpha)||p||_{\infty}, \quad \square \]

**Remark.** In the Ehlich-Zeller case, \( X_n \) is the set of Chebyshev points of degree \( m := [an] \) (the zeros of \( T_m(x) \)). It is elementary to verify that for every \( x \in K = [-1, 1] \) there is a point \( y \in X_n \) such that
\[ d_k(x, y) \leq \frac{\pi}{2m} \leq \frac{\pi}{2n}, \]
i.e., Proposition 1.2 applies with \( \alpha := \pi/(2a) < \pi/2 \) and the N\orming Constant \( C = \sec(\pi/(2a)) \). \( \square \)

Proposition 1.2 may also be used to prove an analogous result for the Fekete points for \( K = [-1, 1] \).

**Proposition 1.3.** Suppose that \( K = [-1, 1] \) and that \( a > 3/2 \). Then the Fekete points of degree \( m := [an] \), \( F_m \), form a N\orming Set with norming constant \( C = \sec(\pi/(4a)) \).\( \square \)

**Proof.** The proof will be a simple consequence of Sündermann’s Lemma ([12, Lemma 1]) on the spacing of the Fekete points for the interval.

**Lemma 1.4.** (Sündermann) Let \( f_k = \cos(\theta_k) \), \( 1 \leq k \leq (m + 1) \) denote the Fekete points of degree \( m \) for the interval \([-1, 1]\), in decreasing order. Then
\[ \frac{(j - 1)\pi}{m + 1/2} \leq \theta_j \leq \frac{(j - 1/2)\pi}{m + 1/2}, \quad j = 1, \ldots, (m + 1). \]

**Proof.** As the Sündermann paper [12] is not easily accessible, we will reproduce his proof here. First note that for \( \omega(x) := \prod_{k=1}^{m+1} (x - f_k) \), we may write the Lagrange polynomials as
\[ \ell_k(x) = \frac{\omega(x)}{(x - f_k)\omega'(f_k)}, \quad k = 1, \ldots, (m + 1). \]

Then, from the facts that \( f_1 = +1 \), \( f_{m+1} = -1 \), and at the interior points \( \max_{x \in [-1, 1]} |\ell_k'(x)| = 1 \) and hence \( \ell_k'(x_k) = 0 \), \( 2 \leq k \leq m \), it follows easily that
\[ (1 - x^2)\omega''(x) + n(n + 1)\omega(x) = 0. \]

We note that it then follows that \( \omega(x) = c(x^2 - 1)P_{m-1}'(x) \) for some constant \( c \) and where \( P_m(x) \) is the classical Legendre polynomial of degree \( m - 1 \).

For \( u(\theta) := (\sin(\theta))^{-1/2}\omega(\cos(\theta)) \) we consequently have
\[ u''(\theta) + \left( \frac{m + 1/2}{2} \right)^2 - \frac{3}{4\sin^2(\theta)} u(\theta) = 0. \]

Now compare \( u(\theta) \) with a solution of the differential equation
\[ v''(\theta) + \left( \frac{m + 1/2}{2} \right)^2 v(\theta) = 0. \]

Consider first \( 2 \leq k \leq (m - 1) \) and the particular solution
\[ v(\theta) = \sin((m + 1/2)(\theta - \theta_1)), \]
\[ \theta_1 = \frac{\pi}{(m + 1/2)}, \quad k = 2, \ldots, (m - 1). \]

We claim that (2) also holds for \( k = 1 \) and \( k = m \). To see this, note that \( f_2 = \cos(\theta_2) \) is the largest zero of \( P_m'(x) \). By [13, Thm. 6.21.1] it follows that this is smaller than the largest zero of \( T_m'(x) \), i.e., \( \theta_2 > \pi/m \). But as \( f_2 = +1 \), \( f_1 = 0 \), and hence
\[ \theta_2 - \theta_1 = \theta_2 - \pi/m > \pi/(m + 1/2). \]

The \( k = m \) case follows by symmetry.

Summation of the inequalities (2) for \( k = 1 \) to \( k = j - 1 \) yields \( \theta_j \geq (j - 1)\pi/(m + 1/2) \) and by summation from \( k = j \) through \( m \) we obtain \( \theta_j \leq (j - 1/2)\pi/(m + 1/2) \). \( \square \)

Continuing with the proof of the Proposition, the Sündermann Lemma implies that
\[ \theta_{j+1} - \theta_j \leq \frac{(j + 1/2)\pi}{m + 1/2} - \frac{(j - 1)\pi}{m + 1/2} = \frac{(3/2)\pi}{m + 1/2}, \quad j = 1, \cdots, m \]
from which it follows that for all \( x \in [-1, 1] \) there exists a Fekete point \( f_j \in F_m \) of degree \( m \) such that
\[ d_k(x, f_j) \leq \frac{(3/4)\pi}{m + 1/2} \leq \frac{(3/4)\pi}{an + 1/2} \leq \frac{a}{n}, \]
\[ \square \]
for $\alpha := 3\pi/(4a) < \pi/2$ for $a > 3/2$. □

Remark. It is likely that the Proposition holds for any $a > 1$, but a proof would require a refinement of the Sündermann Lemma.
□

It is also interesting to note that a very simple argument shows that Fekete points for degree $m = \lceil \log(n) \rceil$, i.e., with $a$ replaced by $\log(n)$, are always a near optimal Norming Set.

Proposition 1.5. ([2]) Suppose that $K \subset \mathbb{R}^d$ is a compact set for which there is an integer $s \leq d$ such that $N_s(K) = O(n^s)$ (as is the case for compact subsets of algebraic varieties). Then the Fekete points $F_m$ of degree $m = n[\log(n)]$ and $(X_n) = O((n \log(n))^s)$ form a Norming Set for $K$.

Proof. First note that for $\deg(p) \leq n$, $\deg(p[\log(n)]) \leq m$, and hence

$$\|p\|_K^{\log(n)} = \|p[\log(n)]\|_K \leq \|p[\log(n)]\|_{F_m} = \|(F_m)p[\log(n)]\|_{F_m}$$

and hence

$$\|p\|_K \leq (\|F_m\|)^{\log(n)}\|p\|_{F_m}.$$  

Now note that

$$(\#(X_n)^{\log(n)}) \leq O((n \log(n))^\log(n))$$

where $(n \log(n))^{\log(n)} \rightarrow e^c$ as $n \rightarrow \infty$, and hence is bounded. □

2 The Unit Sphere

Marzo and Ortega-Cerdà [9] have shown, as a special case of a more general result, that Fekete points of degree $\lceil an \rceil$ form a Norming set for polynomials of degree at most $n$ on the unit sphere.

Theorem 2.1 (Marzo and Ortega-Cerdà - 2010 [9]). For any $a > 1$ the Fekete points of degree $m := \lceil an \rceil$ form a Norming Set for $K = S^{d-1} \subset \mathbb{R}^d$, the unit sphere.

Proof. We note that in the case of $K = S^{d-1}$, as already shown by Dubiner [6] (cf. [4, 5]), the Dubiner distance is just geodesic distance on the sphere:

$$d_k(x, y) = \cos^{-1}(x \cdot y), \ x, y \in S^{d-1}.$$  

Now, the key ingredients of their proof are:

1. The discrete equally-weighted measure based on the Fekete points is a bounded proxy for integrals of polynomials squared. Specifically, there is a constant $C > 0$ such that

$$\frac{1}{N_k} \sum_{i=1}^{N_k} P^2(f_k) \leq C \int_{S^{d-1}} P^2(x)d\sigma(x),$$

for all $P \in P_n(K)$, $n = 1, 2, \ldots$, where $d\sigma(x)$ is surface area measure on the sphere, normalized to be a probability measure.

2. For every point $A \in S^{d-1}$ and every degree $n$, there is a peaking polynomial $P_n(x) \in P_n(K)$ such that $P_n(A) = 1$ and that

$$\int_{S^{d-1}} P^2_n(x) d\sigma(x) = O(N^{-1}).$$

Assuming these properties for the time being, their proof goes as follows.

Given $Q \in P_n(S^{d-1})$, let $A \in S^{d-1}$ be such that

$$|Q(A)| = \|Q\|_{S^{d-1}}.$$  

Further, let $P_n(x)$ be the peaking polynomial for $A \in S^{d-1}$ of degree $m := \lceil (a - 1)n/2 \rceil$ postulated by Ingredient 2. It is important to note the specific degree of $P_n$. Then

$$R(x) = R_n(x) := Q(x)P^2_n(x)$$

is a polynomial of degree at most $\lceil an \rceil$ and has the property that $\|Q\|_{S^{d-1}} = |Q(A)| = |R(A)|$.

We let $\{f_1, f_2, \ldots, f_{N_k}\}$ denote a set of Fekete points for degree $\lceil an \rceil$ and $\ell_k(x)$ the associated Lagrange polynomials. Then

$$\|Q\|_{S^{d-1}} = |R(A)|$$

$$= \sum_{k=1}^{N_k} R(f_k)\ell_k(A)$$

$$= \sum_{k=1}^{N_k} Q(f_k)P^2_n(f_k)\ell_k(A)$$

$$\leq \sum_{k=1}^{N_k} Q(f_k)P^2_n(f_k)$$
as \(\|\ell_4\|_K = 1\) for the Fekete points. Hence,

\[
\|Q\|_{\ell^g\rightarrow 1} \leq \left\{ \max_{1 \leq i \leq N_n} |Q(f_i)| \right\} \sum_{k=1}^{N_n} P_k^2(f_k) = \left\{ \max_{1 \leq i \leq N_n} |Q(f_i)| \right\} \frac{1}{N_n} \sum_{k=1}^{N_n} P_k^2(f_k) \leq \left\{ \max_{1 \leq i \leq N_n} |Q(f_i)| \right\} \frac{N_n}{N_{(a-1)n}} \int_{S^d-1} P_k^2(x) d\sigma(x)
\]

by Ingredient 1.

Consequently, by the integral property of the peaking polynomial \(P_n\),

\[
\|Q\|_{\ell^g\rightarrow 1} \leq C \left\{ \max_{1 \leq i \leq N_n} |Q(f_i)| \right\} \frac{N_n}{N_{(a-1)n}} \leq C' \left\{ \max_{1 \leq i \leq N_n} |Q(f_i)| \right\}
\]

for some constant \(C'\), using the fact that \(N_{an}/N_{(a-1)n}\) is bounded. \(\square\)

For completeness sake we will provide the details of their proofs of the two Ingredients above.

**Proposition 2.2.** ([8, Cor. 4.6]) There is a constant \(C > 0\) such that for \(n = 1, 2, \cdots,\)

\[
\frac{1}{N_n} \sum_{k=1}^{N_n} P_k^2(f_k) \leq C \int_{S^d-1} P_k^2(x) d\sigma(x),
\]

for all \(P \in \mathcal{P}_n(K)\), where \(d\sigma(x)\) is surface area measure on the sphere, normalized to be a probability measure and \(F_n := \{f_1, f_2, \cdots, f_{N_n}\}\) is a set of Fekete points for degree \(n\).

**Proof.** We first note that Fekete points are well-spaced with respect to the Dubiner distance. Indeed, as shown by Dubiner [6],

\[
d_k(f_i, f_j) \geq \frac{\pi}{2n}, \ i \neq j.
\]

The proof is quite simple – one just notes that

\[
d_k(f_i, f_j) = \sup_{n \geq 1, \ p \in \mathcal{P}_n(K), \ \|p\|_1 = 1} \frac{1}{n} |\cos^{-1}(p(f_i)) - \cos^{-1}(p(f_j))|
\]

\[
\geq \frac{1}{n} |\cos^{-1}(\ell(f_i)) - \cos^{-1}(\ell(f_j))|
\]

\[
= \frac{1}{n} |\cos^{-1}(1) - \cos^{-1}(0)|
\]

\[
= \frac{\pi}{2n}.
\]

We will make use of the following notation:

- For \(z \in \mathbb{R}^d\), \(B_r(z) := \{x \in \mathbb{R}^d : |x-z| \leq r\}\) will denote the Euclidean ball of radius \(r\) centred at \(z\), and
- For \(z \in S^{d-1}\), \(B_r(z) := \{x \in S^{d-1} : d_k(x, z) \leq r\}\) will denote the spherical cap of radius \(r\) centred at \(z\).

We note that

\[
\text{vol}_d(B_r(z)) = C_d r^d, \ \text{for some dimensional constant} \ C_d, \ \text{and} \quad (4)
\]

\[
\text{vol}_{d-1}(B_r(z)) \approx C'_{d-1} r^{d-1}, \ z \in S^{d-1} \quad (5)
\]

where here we mean that \(\text{vol}_d(B_r(z))/r^d\) is bounded above and below by (positive) dimensional constants. We note also that \(\text{vol}_d(B_r(z))\) is the same for any \(z \in S^{d-1}\).

We make use of the following simple geometric facts.

**Lemma 2.3.** Suppose that \(x, y \in K = S^{d-1}\) and that \(u \in \mathbb{R}^d\) has Euclidean norm \(|u| = r > 0\). Then

1. \(d_k(x, y) \leq \frac{\pi}{2} |x-y|,\)
2. \(\left| \frac{u}{|u|} - x \right| \leq \frac{1}{\sqrt{r}} |u-x|\).
Proof. To see 1., note that this is equivalent to
\[
\theta^2 \leq \frac{\pi^2}{4} 2(1 - \cos(\theta)), \quad \cos(\theta) = x \cdot y \in [0, \pi]
\]
\[\iff \theta^2 \leq \pi^2 \sin^2(\theta/2)\]
\[\iff \sin(\theta/2) \geq \frac{2}{\pi} \left(\frac{\theta}{2}\right), \quad \theta/2 \in [0, \pi/2],\]
a well-known elementary inequality.

To see 2., just note that this is equivalent to
\[
\left|\frac{u}{|u|} - x\right|^2 \leq \frac{1}{r} |u - x|^2
\]
\[\iff 2\left(1 - \frac{u \cdot x}{|u|}\right) \leq \frac{1}{r} |u|^2 - 2(u \cdot x) + 1\]
\[\iff 2r(1 - \cos(\theta)) \leq r^2 - 2r \cos(\theta) + 1, \quad \cos(\theta) = (u \cdot x)/|u|\]
\[\iff 4r \sin^2(\theta/2) \leq (r^2 - 2r + 1) + 4r \sin^2(\theta/2)
\]
\[= (r - 1)^2 + 4r \sin^2(\theta/2). \quad \blacksquare\]

Now, from the spacing (3) we may easily conclude that for every \(0 < c\) there is a constant \(C = C(c) > 0\) such that for every \(0 \neq u \in \mathbb{R}^d\) and \(n = 1, 2, \ldots\)
\[\#(F_u \cap B_{c/n}(u)) \leq C.\quad (6)\]

To see this, first note that by 2. of Lemma 2.3,
\[B_{c/n}(u) \cap S^{d-1} \subset B_{c/n}(u/|u|) \cap S^{d-1}\]
where \(c' := c/\sqrt{|u|}\) and that then, by 1.,
\[(B_{c/n}(u) \cap S^{d-1}) \subset (B_{c/n}(u/|u|) \cap S^{d-1}) \subset B_{c'/n}(u/|u|)\]
where \(c'' := (\pi/2)c'\).

Suppose now that there are \(m\) distinct Fekete points \(f_1, \ldots, f_m \in B_{c/n}(u)\). Necessarily then \(f_1, \ldots, f_m \in B_{c'/n}(x)\) where \(x := u/|u| \in S^{d-1}\).

Choose \(a < 1\) so that \(ac < \pi/2\). Then, we have
\[\mathcal{B}_{u/c}(f_j) \cap \mathcal{B}_{a/c}(f_k) = \emptyset, \quad j \neq k.\]

Also, there is constant \(R_0 = R_0(c)\) so that
\[\text{vol}_{d-1}(\mathcal{B}_{u/c}(f_j) \cap \mathcal{B}_{a/c}(f_j)) \geq R_0 \text{vol}_{d-1}(\mathcal{B}_{a/c}(f_j)), \quad j = 1, \ldots, m.\]

Hence
\[\text{vol}_{d-1}(\mathcal{B}_{c/n}(x)) \geq R_0 \text{vol}_{d-1}\left(\bigcup_{j=1}^m \mathcal{B}_{a/c}(f_j)\right)\]
\[\geq m C_0(ac/n)^{d-1}\] (for some constant \(C_0\))
and so
\[m \leq \text{vol}_{d-1}(\mathcal{B}_{c/n}(x))/(C_0(ac/n)^{d-1}) \leq C.\]

There is a further technical inequality that we will need. For \(0 < c < 1\) we let
\[T_{c,n} := \{x \in \mathbb{R}^d : ||x|| - 1 \leq c/n\}\]
denote the tubular neighbourhood of the unit sphere \(S^{d-1}\), of “radius" \(c/n\). Then, given a polynomial \(P \in \mathcal{P}_n(S^{d-1})\) it has a harmonic extension to all of \(\mathbb{R}^d\). We denote this extension also by \(P\).

Corollary 4.3 of [8] asserts (as a special case of a more general result) that there is a constant \(C\) such that
\[\int_{T_{c,n}} P^2(x) dx \leq \frac{C}{n} \int_{S^{d-1}} P^2(x) d\sigma(x),\quad (7)\]

Their proof of this relies on the following lemma.

Lemma 2.4. ([8, Lemma 4.2]) For \(r > 0\) let \(S^{d-1} \subset \mathbb{R}^d\) denote the sphere of radius \(r\), centred at the origin. Then for \(\rho > 1\) and \(P \in \mathcal{P}_n(S^{d-1})\) and any \(|r - 1| \leq \rho/n\) there exists a constant \(C\), depending only on \(\rho\) and \(d\), such that
\[\int_{S^{d-1}} P^2(x) d\sigma(x) \leq C \int_{S^{d-1}} P^2(x) d\sigma(x).\]
Proof. Changing variables \( x' = rx \), we have
\[
\int_{g_{d-1}} P^2(x) d\sigma(x) = \int_{g_{d-1}} P^2(rx) d\sigma(x)
\]
(as the measures are both normalized to be probability measures).

We continue with the conclusion of the proof of Proposition 2.2. Indeed, by subharmonicity, there is a constant \( h \)
where \( h \) is a harmonic, homogeneous polynomial of degree \( k \), and so
\[
\int_{g_{d-1}} P^2(rx) d\sigma(x) \leq \max\{1, r\}^{2\deg(P)} \int_{g_{d-1}} P^2(x) d\sigma(x)
\]
from which the result follows easily. To see this, expand
\[
P(x) = \sum_{k=0}^{n} a_k h_k(x)
\]
where \( h_k(x) \) is a harmonic, homogeneous polynomial of degree \( k \), as is always possible to do. The \( h_k(x) \) are mutually orthogonal and so
\[
\int_{g_{d-1}} P^2(rx) d\sigma(x) = \int_{g_{d-1}} \left( \sum_{k=0}^{n} a_k h_k(rx) \right)^2 d\sigma(x)
\]
\[
= \int_{g_{d-1}} \left( \sum_{k=0}^{n} a_k r^k h_k(x) \right)^2 d\sigma(x)
\]
\[
= \sum_{k=0}^{n} a_k^2 r^{2k} \left\{ \int h_k^2(x) d\sigma(x) \right\}
\]
\[
\leq \max\{1, r\}^{2n} \sum_{k=0}^{n} a_k^2 \left\{ \int h_k^2(x) d\sigma(x) \right\}
\]
\[
= \max\{1, r\}^{2n} \int_{g_{d-1}} P^2(x) d\sigma(x). \quad \square
\]

We now state and prove (7) as a lemma.

Lemma 2.5. ([8, Cor. 4.3]) There is a constant \( C \) such that for any harmonic polynomial \( P(x) \) of degree at most \( n \),
\[
\int_{T_{c,n}} p^2(x) dx \leq \frac{C}{n} \int_{g_{d-1}} p^2(x) d\sigma(x).
\]

Proof. First note that there is a dimensional constant \( C_d \) such that
\[
\int_{T_{c,n}} P^2(x) dx = C_d \int_{r=1-c/n}^{r=1+c/n} r^{d-1} P^2(x) d\sigma(x)
\]
where again \( d\sigma(x) \) is normalized to be a probability measure., and hence by the preceeding Lemma,
\[
\int_{T_{c,n}} P^2(x) dx = C_d \int_{r=1-c/n}^{r=1+c/n} \left\{ \int_{g_{d-1}} r^{d-1} P^2(x) d\sigma(x) \right\} dr
\]
\[
\leq C \int_{r=1-c/n}^{r=1+c/n} \left\{ \max\{1, r\}^{2n} \int_{g_{d-1}} P^2(x) d\sigma(x) \right\} dr
\]
\[
\leq C \frac{2c}{n} (1 + c/n)^{2n} \int_{g_{d-1}} P^2(x) d\sigma(x)
\]
\[
\leq C e^{2n} \frac{2c}{n} \int_{g_{d-1}} P^2(x) d\sigma(x). \quad \square
\]

We continue with the conclusion of the proof of Proposition 2.2. Indeed, by subharmonicity, there is a constant \( C \) such that for all harmonic polynomials \( P(x) \) of degree at most \( n \) and \( x \in S^{d-1} \), we have
\[
|P(x)|^2 \leq C n^d \int_{\mathbb{R}^{d-1}} P^2(x) dm(x)
\]
where, as before, $B(z, 1/n)$ denotes the Euclidean ball of radius $1/n$ centred at $z$ and $dm(x)$ denotes Lebesgue measure on $\mathbb{R}^d$. Hence

$$
\frac{1}{N_n} \sum_{k=1}^{N_n} p^2(f_k) \leq C \frac{1}{N_n} \sum_{k=1}^{N_n} \left\{ \int_{B(0,1/n)} p^2(x) dm(x) \right\}
$$

$$
\leq C n^d \int_{C_{1,n}} p^2(x) \left\{ \frac{1}{N_n} \sum_{k=1}^{N_n} x_{B(0,1/n)}(x) \right\} dm(x)
$$

$$
= C n^d \int_{C_{1,n}} p^2(x) \left\{ \frac{1}{N_n} \sum_{k=1}^{N_n} x_{B(0,1/n)}(f_k) \right\} dm(x)
$$

$$
\leq C n^d \int_{C_{1,n}} p^2(x) \left\{ \frac{C}{N_n} \right\} dm(x) \text{ (by (6))}
$$

$$
\leq C \frac{n^d}{N_n(K)} \int_{C_{1,n}} p^2(x) dm(x)
$$

$$
\leq C \frac{n^d}{N_n(K)} \frac{1}{n} \int_{S^{d-1}} p^2(x) d\sigma(x) \text{ (by Lemma 2.5)}
$$

$$
\leq C \int_{S^{d-1}} p^2(x) d\sigma(x)
$$

as $N_n(K) = O(n^{d-1})$. □

For Ingredient 2, we let for $x, y \in S^{d-1}$, $K_n(x, y)$ denote the reproducing kernel for polynomials of degree at most $n$ with respect to the measure $d\sigma(x)$ on $S^{d-1}$. As is well known (see e.g. [11, p. 69])

$$
K_n(x, y) \equiv N_n, x \in S^{d-1}.
$$

Then, let

$$
P_n(A) := \frac{1}{N_n} K_n(A, A).
$$

We have $P_n(A) = N_n/N_n = 1$ and

$$
\int P_n(x)^2 d\sigma(x) = \frac{1}{N_n^2} \int K_n(A, x) K_n(A, x) d\sigma(x)
$$

$$
= \frac{1}{N_n^2} K_n(A, A) = \frac{1}{N_n}
$$

as required. □

**Concluding Remarks.** We emphasize that the results of Marzo and Ortega-Cerdà are for the comparison of general $L_p$ norms of polynomials with the corresponding discrete $L_p$ norms based on Fekete points. We have extracted the essentials of their proofs necessary for the $L_\infty$ case, in which we are primarily concerned.

We conjecture that Fekete points of degree $\lceil \alpha n \rceil$, $\alpha > 1$, are norming sets for general “sufficiently regular” compact sets $K \subset \mathbb{R}^d$. Indeed it would be sufficient to show that $K$ has the analogous properties of Ingredients 1 and 2 above. The cases of $K$ a ball or simplex will be discussed in a forthcoming paper.

**References**


