Abstract. Let $K$ be a finite extension of $\mathbb{Q}_p$. We use the theory of $(\varphi, \Gamma)$-modules in the Lubin-Tate setting to construct some corestriction-compatible families of classes in the cohomology of $V$, for certain representations $V$ of $\text{Gal}(\overline{\mathbb{Q}}_p/K)$. If in addition $V$ is crystalline, we describe these classes explicitly using Bloch-Kato’s exponential maps. This allows us to generalize Perrin-Riou’s period map to the Lubin-Tate setting.

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Introduction

Let $K$ be a finite extension of $\mathbb{Q}_p$ and let $G_K = \text{Gal}(\overline{\mathbb{Q}}_p/K)$. In this article, we use the theory of $(\varphi, \Gamma)$-modules in the Lubin-Tate setting to construct some classes in $H^1(K, V)$, for “$F$-analytic” representations $V$ of $G_K$. If in addition $V$ is crystalline, we describe these classes explicitly using Bloch and Kato’s exponential maps and generalize Perrin-Riou’s period map to the Lubin-Tate setting.

We now describe our constructions in more detail, and introduce some notation which is used throughout this paper. Let $F$ be a finite Galois extension of $\mathbb{Q}_p$, with ring of integers $\mathcal{O}_F$ and maximal ideal $\mathfrak{m}_F$, let $\pi$ be a uniformizer of $\mathcal{O}_F$ and let $k_F = \mathcal{O}_F/\pi$ and $q = \text{Card}(k_F)$. Let $LT$ be the Lubin-Tate formal group $[LT65]$ attached to $\pi$. We fix a coordinate $T$ on LT, so that for each $a \in \mathcal{O}_F$ the multiplication-by-$a$ map is given by a power series $[a](T) = aT + O(T^2) \in \mathcal{O}_F[T]$. Let $\log_{LT}(T)$ denote the attached logarithm and $\exp_{LT}(T)$ its inverse for the composition. Let $\chi_{LT} : G_F \to \mathcal{O}_F^\times$ be the attached Lubin-Tate character.

If $K$ is a finite extension of $F$, let $K_n = K(LT[\pi^n])$ and $K_\infty = \bigcup_{n \geq 1} K_n$ and $\Gamma_K = \text{Gal}(K_\infty/K)$.

Let $A_F$ denote the set of power series $\sum_{i \in \mathbb{Z}} a_i T^i$ with $a_i \in \mathcal{O}_F$ such that $a_i \to 0$ as $i \to -\infty$ and let $B_F = A_F[1/\pi]$, which is a field. It is endowed with a Frobenius map $\varphi_q : f(T) \mapsto f([\chi_{LT}(g)](T))$ and an action of $\Gamma_F$ given by $g : f(T) \mapsto f([\varphi_q(T)](T))$. If $K$ is a finite extension of $F$, the theory of the field of norms ([FW79a, FW79b] and [Win83]) provides us with a finite unramified extension $B_K$ of $B_F$. Recall [Fon90] that a $(\varphi, \Gamma)$-module over $B_K$ is a finite dimensional $B_K$-vector space endowed with a compatible Frobenius map $\varphi_q$ and action of $\Gamma_K$. We say that a $(\varphi, \Gamma)$-module over $B_K$ is étale if it has a basis in which $\text{Mat}(\varphi_q) \in \text{GL}_d(A_K)$. The relevance of these objects is explained by the result below (see [Fon90], [KR99]).

Theorem. There is an equivalence of categories between the category of $F$-linear representations of $G_K$ and the category of étale $(\varphi, \Gamma)$-modules over $B_K$.

Let $B_F^\dagger$ denote the set of power series $f(T) \in B_F$ that have a non-empty domain of convergence. The theory of the field of norms again provides us ([Mat95]) with a finite extension $B_K^\dagger$ of $B_F^\dagger$. We say that a $(\varphi, \Gamma)$-module over $B_K^\dagger$ is overconvergent if it has a basis in which $\text{Mat}(\varphi_q) \in \text{GL}_d(B_K^\dagger)$ and $\text{Mat}(g) \in \text{GL}_d(B_K^\dagger)$ for all $g \in \Gamma_K$. If $F = \mathbb{Q}_p$, every étale $(\varphi, \Gamma)$-module over $B_K$ is overconvergent ([CC98]). If $F \neq \mathbb{Q}_p$, this is no longer the case ([FX13]).
Let us say that an $F$-linear representation $V$ of $G_K$ is $F$-analytic if for all embeddings $\tau : F \to \overline{Q}_p$, with $\tau \neq \text{Id}$, the representation $C_p \otimes_F V$ is trivial (as a semilinear $C_p$-representation of $G_K$). The following result is known [Ber16].

**Theorem.** If $V$ is an $F$-analytic representation of $G_K$, it is overconvergent.

Another source of overconvergent representations of $G_K$ is the set of representations that factor through $\Gamma_K$ (see §1.3). Our first result is the following (Theorem 1.3.1).

**Theorem A.** If $V$ is an overconvergent representation of $G_K$, there exists an $F$-analytic representation $X_{an}$ of $G_K$, a representation $Y_{\Gamma}$ of $G_K$ that factors through $\Gamma_K$, and a surjective $G_K$-equivariant map $X_{an} \otimes_F Y_{\Gamma} \to V$.

We next focus on $F$-analytic representations. Let $B^\dagger_{\text{rig}, F}$ denote the Robba ring, which is the ring of power series $f(T) = \sum_{i \geq 0} a_i T^i$ with $a_i \in F$ such that there exists $\rho < 1$ such that $f(T)$ converges for $\rho < |T| < 1$. We have $B^\dagger_{\text{rig}, F} \subset B^\dagger_{\text{rig}, K}$. The theory of the field of norms again provides us with a finite extension $B^\dagger_{\text{rig}, K}$ of $B^\dagger_{\text{rig}, F}$. If $V$ is an $F$-linear representation of $G_K$, let $D(V)$ denote the $(\varphi, \Gamma)$-module over $B_K$ attached to $V$. If $V$ is overconvergent, there is a well defined $(\varphi, \Gamma)$-module $D^i(V)$ over $B^\dagger_{rig, K}$ attached to $V$, such that $D(V) = B_K \otimes_{B^\dagger_{\text{rig}, K}} D^i(V)$. We call $D^i_{\text{rig}}(V)$ the $(\varphi, \Gamma)$-module over $B^\dagger_{\text{rig}, K}$ attached to $V$, given by $D^i_{\text{rig}}(V) = B^\dagger_{\text{rig}, K} \otimes_{B^\dagger_{\text{rig}, K}} D^i(V)$.

The ring $B^\dagger_{\text{rig}, K}$ is a free $\varphi_q(B^\dagger_{\text{rig}, K})$-module of degree $q$. This allows us to define [FX13] a map $\psi_q : B^\dagger_{\text{rig}, K} \to B^\dagger_{\text{rig}, K}$ that is a $\Gamma_K$-equivariant left inverse of $\varphi_q$, and likewise, if $V$ is an overconvergent representation of $G_K$, a map $\psi_q^1 : D^1_{\text{rig}}(V) \to D^1_{\text{rig}}(V)$ that is a $\Gamma_K$-equivariant left inverse of $\varphi_q$.

The main result of this article is the construction, for an $F$-analytic representation $V$ of $G_K$, of a collection of maps

$$h^1_{K_n, V} : D^1_{\text{rig}}(V)^{\psi_q = 1} \to H^1(K_n, V),$$

having a certain number of properties. For example, these maps are compatible with corestriction: $\text{cor}^n_{K_{n+1}/K_n} \circ h^1_{K_{n+1}, V} = h^1_{K_n, V}$ if $n \geq 1$. Another property is that if $F = \mathbf{Q}_p$ and $\pi = p$ (the cyclotomic case), these maps coincide with those constructed in [CC99] (and generalized in [Ber03]).

If now $K = F$ and $V$ is a crystalline $F$-analytical representation of $G_F$, we give explicit formulas for $h^1_{K_n, V}$ using Bloch and Kato’s exponential maps [BK90].

Let $V$ be as above, let $D_{\text{cris}}(V) = (B_{\text{cris}, F} \otimes_F V)^{G_F}$ (note that because the $\otimes$ is over $F$, this is the identity component of the usual $D_{\text{cris}}$) and let $t_\pi = \log_{\text{LT}}(T)$. Let $\{u_n\}_{n \geq 0}$ be a compatible sequence of primitive $\pi^n$-torsion points of LT. Let $B^+_{\text{rig}, F}$ denote the positive part of the Robba ring, namely the ring of power series $f(T) = \sum_{i \geq 0} a_i T^i$ with $a_i \in F$ such that $f(T)$ converges for $0 \leq |T| < 1$. If $n \geq 0$, we have a map $\varphi_q^{-n} : B^+_{\text{rig}, F} \to F_n[T_\pi]$ given by $f(T) \mapsto f(u_n \otimes \exp_{\text{LT}}(t_\pi / \pi^n))$. Using the results of [KR09], we prove that
there is a natural \((\varphi, \Gamma)\)-equivariant inclusion \(D^1_{\rig}(V)^{\psi_q=1} \to B^*_{\rig,F}[1/t_{\pi}] \otimes_F D_{\rig}(V)\). This provides us, by composition, with maps \(\varphi^{-n} : D^1_{\rig}(V)^{\psi_q=1} \to F_n \otimes_F D_{\rig}(V)\) where \(\partial_V\) is the “coefficient of \(t_{\pi}^n\)” map. Finally, theorem C, and Kato’s exponential \(\exp_{F_n,V} \colon F_n \otimes_F D_{\rig}(V) \to H^1(F_n,V)\) and its dual \(\exp^*_{F_n,V}(1) \colon H^1(F_n,V) \to F_n \otimes_F D_{\rig}(V)\) (the subscript \(V^*(1)\) denotes the dual of \(V\) twisted by the cyclotomic character, but is merely a notation here).

The first result is as follows (theorem 3.3.1).

**Theorem B.** If \(V\) is as above and \(y \in D^1_{\rig}(V)^{\psi_q=1}\), then

\[
\exp^*_{F_n,V}(1)(h^1_{F_n,V}(y)) = \begin{cases} 
q^{-n}\partial_V(\varphi^{-n}_q(y)) & \text{if } n \geq 1 \\
(1-q^{-1}\varphi_q^{-1})\partial_V(y) & \text{if } n = 0.
\end{cases}
\]

Let \(\nabla = \frac{d}{d t_{\pi}}\) - \(d/dx\), let \(\nabla_i = \nabla - i\) if \(i \in \mathbb{Z}\) and let \(h \geq 1\) be such that \(\Fil^{-h}D_{\rig}(V) = D_{\rig}(V)\). We prove that if \(y \in (B^+_{\rig,F} \otimes_F D_{\rig}(V))^{\psi_q=1}\), then \(\nabla_{h-1} \cdots \nabla_0(y) \in D^1_{\rig}(V)^{\psi_q=1}\), and we have the following result (theorem 3.3.2).

**Theorem C.** If \(V\) is as above and \(y \in (B^+_{\rig,F} \otimes_F D_{\rig}(V))^{\psi_q=1}\), then

\[
h^1_{F_n,V}(\nabla_{h-1} \cdots \nabla_0(y)) = (-1)^{h-1}(h-1)! \begin{cases} 
\exp_{F_n,V}(q^{-n}\partial_V(\varphi^{-n}_q(y))) & \text{if } n \geq 1 \\
\exp_{F,V}(1-q^{-1}\varphi_q^{-1})\partial_V(y) & \text{if } n = 0.
\end{cases}
\]

Using theorems B and C, we give in §3.5 a Lubin-Tate analogue of Perrin-Riou’s “big exponential map” [PR94] using the same method as that of [Ber03] which treats the cyclotomic case. It will be interesting to compare this big exponential map with the “big logarithms” constructed in [Fou05] and [Fou08]. It is also instructive to specialize theorem C to the case \(V = F(\chi_\pi)\), which corresponds to “Lubin-Tate” Kummer theory. Recall that if \(L\) is a finite extension of \(F\), Kummer theory gives us a map \(\delta : \LT(m_L) \to H^1(L,F(\chi_\pi))\). When \(L\) varies among the \(F_n\), these maps are compatible: the diagram

\[
\begin{array}{ccc}
\LT(m_{F_{n+1}}) & \xrightarrow{\delta} & H^1(F_{n+1},V) \\
\Tr_{F_{n+1}/F_n}^\LT & \downarrow & \downarrow \text{cor}_{F_{n+1}/F_n} \\
\LT(m_{F_n}) & \xrightarrow{\delta} & H^1(F_n,V)
\end{array}
\]

commutes. Let \(S\) denote the set of sequences \(\{x_n\}_{n \geq 1}\) with \(x_n \in m_{F_n}\) and such that \(\Tr_{F_{n+1}/F_n}(x_{n+1}) = [q/\pi](x_n)\) for \(n \geq 1\). We prove that \(S\) is big, in the sense that (if \(F \neq \mathbb{Q}_p\)) the projection on the \(n\)-th coordinate map \(S \otimes_{\mathcal{O}_F} F \to F_n\) is onto (this would not be the case if we did not have the factor \(q/\pi\) in the definition of \(S\)). Furthermore, we prove that if \(x \in S\), there exists...
a power series \( f(T) \in \left( \mathbb{B}_{\text{rig}, \pi}^{+} \right)^{\psi_{n}=1} \) such that \( f(u_{n}) = \log_{\text{LT}}(x_{n}) \) for \( n \geq 1 \).

We have \( d/dt_{\pi}(f(T)) \in \left( \mathbb{B}_{\text{rig}, \pi}^{+} \right)^{\psi_{n}=1} \) and the following holds (theorem 3.4.5), where \( u \) is the basis of \( F(\chi_{n}) \) corresponding to the choice of \( \{ u_{n} \}_{n \geq 0} \).

**Theorem D.** We have

\[
h_{F_{n}, \psi}(d/dt_{\pi}(f(T)) \cdot u) = (q/\pi)^{-n} \cdot \delta(x_{n}) \quad \text{for all} \quad n \geq 1.
\]

In the cyclotomic case, there is [Col79] a power series \( \text{Col}_{x}(T) \) such that \( \text{Col}_{x}(u_{n}) = x_{n} \) for \( n \geq 1 \). We then have \( f(T) = \log \text{Col}_{x}(T) \), and theorem D is proved in [CC99]. In the general Lubin-Tate case, we do not know whether there is a “Coleman power series” of which \( f(T) \) would be the \( \log_{\text{LT}} \). This seems like a non-trivial question.

It would be interesting to compare our results with those of [SV17]. The authors of [SV17] also construct some classes in \( H^{1}(\mathbb{A}_{\mathbb{Q}}^{\text{f}}) \) via the Lubin-Tate character \( \chi \). Theorems 3.4.5 and 3.4.6 extend our constructions to representations of the form \( V \otimes_{\mathbb{F}} Y_{T} \) with \( V \) analytic and \( Y_{T} \) factoring through \( \Gamma_{K} \), and in particular recover the explicit reciprocity law of [Tsu04].

1 **Lubin-Tate \((\varphi, \Gamma)\)-modules**

In this chapter, we recall the theory of Lubin-Tate \((\varphi, \Gamma)\)-modules and classify overconvergent representations.

1.1 **Notation**

Let \( F \) be a finite Galois extension of \( \mathbb{Q}_{p} \) with ring of integers \( \mathcal{O}_{F} \), and residue field \( k_{F} \). Let \( \pi \) be a uniformizer of \( \mathcal{O}_{F} \). Let \( d = [F : \mathbb{Q}_{p}] \) and \( e \) be the ramification index of \( F/\mathbb{Q}_{p} \). Let \( q = p^{e} \) be the cardinality of \( k_{F} \) and let \( F_{0} = W(k_{F})[1/\pi] \) be the maximal unramified extension of \( \mathbb{Q}_{p} \) inside \( F \). Let \( \sigma \) denote the absolute Frobenius map on \( F_{0} \).

Let \( \text{LT} \) be the Lubin-Tate formal \( \mathcal{O}_{F} \)-module attached to \( \pi \) and choose a co-
ordinate \( T \) for the formal group law, such that the action of \( \pi \) on \( \text{LT} \) is given by \( \pi(T) = T^{q} + \pi T \). If \( a \in \mathcal{O}_{F} \), let \( [a](T) \) denote the power series that gives the action of \( a \) on \( \text{LT} \). Let \( \log_{\text{LT}}(T) \) denote the attached logarithm and \( \exp_{\text{LT}}(T) \) its inverse. If \( K \) is a finite extension of \( F \), let \( K_{n} = K(\text{LT}[\pi^{n}]) \) and let \( K_{\infty} = \bigcup_{n \geq 1} K_{n} \). Let \( H_{K} = \text{Gal}(\mathbb{Q}_{p}/K_{\infty}) \) and \( \Gamma_{K} = \text{Gal}(K_{\infty}/K) \). By Lubin-Tate theory (see [LT65]), \( \Gamma_{K} \) is isomorphic to an open subgroup of \( \mathcal{O}_{F}^{\times} \) via the Lubin-Tate character \( \chi_{\pi} : \Gamma_{K} \to \mathcal{O}_{F}^{\times} \).

Let \( n(K) \geq 1 \) be such that if \( n \geq n(K) \), then \( \chi_{\pi} : \Gamma_{K_{n}} \to 1 + \pi^{n} \mathcal{O}_{F} \) is an isomorphism, and \( \log_{p} : 1 + \pi^{n} \mathcal{O}_{F} \to \pi^{n} \mathcal{O}_{F} \) is also an isomorphism.

Since \( \log_{\text{LT}}(T) \) converges on the open unit disk, it can be seen as an element of \( \mathbb{B}_{\text{rig}, \pi}^{+} \) and we denote it by \( t_{\pi} \). Recall that \( g(t_{\pi}) = \chi_{\pi}(g) \cdot t_{\pi} \pi \) if \( g \in \Gamma_{K} \) and that \( g(t_{\pi}) = \pi \cdot t_{\pi} \pi \). Let \( \varphi = d/dt_{\pi} \) so that \( \varphi f(T) = a(T) \cdot df(T)/dT \), where \( a(T) = (d \log_{\text{LT}}(T)/dT)^{-1} \in \mathcal{O}_{F}[T]^{\times} \).

We have \( \varphi \circ g = \chi_{\pi}(g) \cdot g \circ \varphi \) if \( g \in \Gamma_{K} \) and \( \varphi \circ \varphi = \pi \cdot \varphi \circ \varphi \).
Recall that $B_{\text{rig},F}^i$ denotes the Robba ring, the ring of power series $f(T) = \sum_{i \in \mathbb{Z}} a_i T^i$ with $a_i \in F$ such that there exists $\rho < 1$ such that $f(T)$ converges for $\rho < |T| < 1$. We have $B_F^+ \subset B_{\text{rig},F}^i$ and by writing a power series as the sum of its plus part and its minus part, we get $B_{\text{rig},F}^i = B_{\text{rig},F}^i + B_F^i$.

Each ring $R \in \{ B_{\text{rig},F}^i, B_{\text{rig},F}^+, B_F^+, B_F \}$ is equipped with a Frobenius map $\varphi_q: f(T) \mapsto f([\pi](T))$ and an action of $\Gamma_F$ given by $g: f(T) \mapsto f([\chi_q(g)](T))$.

Moreover, the ring $R$ is a free $\varphi_q(R)$-module of rank $q$, and we define $\psi_q: R \mapsto R$ by the formula $\varphi_q(\psi_q(f)) = 1/q \cdot \text{Tr}_{R/\varphi_q(R)}(f)$. The map $\psi_q$ has the following properties (see for instance §2A of [FX13] and §1.2.3 of [Col16]): $\psi_q(x \cdot \varphi_q(y)) = \psi_q(x) \cdot y$, the map $\psi_q$ commutes with the action of $\Gamma_F$, $\partial \circ \psi_q = \pi^{-1} \cdot \psi_q \circ \partial$ and if $f(T) \in B_{\text{rig},F}^i$ then $\varphi_q \circ \psi_q(f) = 1/q \cdot \sum_{z \in \mathbb{L}(\mathbb{R})} f(T \cdot z)$. If $M$ is a free $R$-module with a semilinear Frobenius map $\varphi_q$ such that $\text{Mat}(\varphi_q)$ is invertible, then any $m \in M$ can be written as $m = \sum_i r_i \cdot \varphi_q(m_i)$ with $r_i \in R$ and $m_i \in M$ and the map $\psi_q: m \mapsto \sum_i \psi_q(r_i) \cdot m_i$ is then well-defined. This applies in particular to the rings $B_{\text{rig},K}, B_{\text{rig},K}^+, B_K^+, B_K$ and to the $(\varphi, \Gamma)$-modules over them.

1.2 Construction of Lubin-Tate $(\varphi, \Gamma)$-modules

A $(\varphi, \Gamma)$-module over $B_K$ (or over $B_K^+$) is a finite dimensional $B_K$-vector space $D$ (or a finite dimensional $B_K^+$-vector space or a free $B_{\text{rig},K}^+$-module of finite rank respectively), along with a semilinear Frobenius map $\varphi_q$ whose matrix (in some basis) is invertible, and a continuous, semilinear action of $\Gamma_K$ that commutes with $\varphi_q$.

We say that a $(\varphi, \Gamma)$-module $D$ over $B_K$ is étale if $D$ has a basis in which $\text{Mat}(\varphi_q) \in \text{GL}_d(A_K)$. Let $B$ be the $p$-adic completion of $\bigcup_{M/F} B_M$ where $M$ runs through the finite extensions of $F$. By specializing the constructions of [Fon90], Kisin and Ren prove the following theorem (theorem 1.6 of [KR09]).

**Theorem 1.2.1.** The functors $V \mapsto D(V) = (B \otimes_F V)^{H_K}$ and $D \mapsto (B \otimes_{B_K} D)^{\varphi_q = 1}$ give rise to mutually inverse equivalences of categories between the category of $F$-linear representations of $G_K$ and the category of étale $(\varphi, \Gamma)$-modules over $B_K$.

We say that a $(\varphi, \Gamma)$-module $D$ is overconvergent if there exists a basis of $D$ in which the matrices of $\varphi_q$ and of all $g \in \Gamma_K$ have entries in $B_K^+$. This basis then generates a $B_K^+$-vector space $D^+$ which is canonically attached to $D$. If $V$ is a $p$-adic representation, we say that it is overconvergent if $D(V)$ is overconvergent, and then $D^+(V)$ denotes the corresponding $(\varphi, \Gamma)$-module over $B_K^+$. The main result of [CC98] states that if $F = \mathbb{Q}_p$, then every étale $(\varphi, \Gamma)$-module over $B_K$ is overconvergent (the proof is given for $\pi = p$, but it is easy to see that it works for any uniformizer). If $F \neq \mathbb{Q}_p$, some simple examples (see [FX13]) show that this is no longer the case.

Recall that an $F$-linear representation of $G_K$ is $F$-analytic if $C_F \otimes_F V$ is the trivial $C_F$-semilinear representation of $G_K$ for all embeddings $\tau \neq \text{Id} \in \text{Gal}(F/\mathbb{Q}_p)$. Documenta Mathematica 22 (2017) 999–1030
This definition is the natural generalization of Kisin and Ren’s notion of $F$-crystalline representation. Kisin and Ren then show that if $K \subset F_{\infty}$, and if $V$ is a crystalline $F$-analytic representation of $G_K$, the $(\varphi, \Gamma)$-module attached to $V$ is overconvergent (see §3.3 of [KR09]; they actually prove a stronger result, namely that the $(\varphi, V)$ is overconvergent (see §3.3 of [KR09]). The purpose of this section is to prove a conjecture of Colmez that describes all overconvergent representations of $G_K$. Any representation $V$ of $G_K$ that factors through $\Gamma_K$ is overconvergent, since $H_K$ acts trivially on $V$ so that $D(V) = B_K \otimes_F V$ and therefore $D(V)$ has a basis in which $\text{Mat}(\varphi_q) = \text{Id}$ and $\text{Mat}(g) \in \text{GL}_d(O_F)$ if $g \in \Gamma_K$. If $X$ is $F$-analytic and $Y$ factors through $\Gamma_K$, $X \otimes_F Y$ is therefore overconvergent. We prove that any overconvergent representation of $G_K$ is a quotient (and therefore also a subobject, by dualizing) of some representation of the form $X \otimes_F Y$ as above.

THEOREM 1.3.1. If $V$ is an overconvergent representation of $G_K$, there exists an $F$-analytic representation $X$ of $G_K$, a representation $Y$ of $G_K$ that factors through $\Gamma_K$, and a surjective $G_K$-equivariant map $X \otimes_F Y \to V$.

Proof. Recall (see §3 of [Ber16]) that if $r > 0$, then inside $B_{\text{rig},K}^1$ we have the subring $B_{\text{rig},K}^{1r}$ of elements defined on a fixed annulus whose inner radius depends on $r$ and whose outer radius is 1, and that $(\varphi, \Gamma)$-modules over $B_{\text{rig},K}^{1r}$ can be defined over $B_{\text{rig},K}^{1s}$ if $r$ is large enough, giving us a module $D_{\text{rig}}^{1r}(V)$. We also have rings $B_{K}^{[r,s]}$ of elements defined on a closed annulus whose radii depend on $r \leq s$. One can think of an element of $B_{\text{rig},K}^{1r}$ as a compatible family
of elements of \( \{ B_{D, r}^I \} \), where \( I \) runs over a set of closed intervals whose union is \([ r, +\infty ]\). In the rest of the proof, we use this principle of gluing objects defined on closed annuli to get an object on the annulus corresponding to \( B_{D, r}^I \).

Choose \( r > 0 \) large enough such that \( D_{D, r}^I(V) \) is defined, and \( s \geq qr \). Let \( D_{D, r}^{[r,s]}(V) = B_{K}^{[r,s]} \otimes_{B_{rig, r}^I} D_{D, r}^I(V) \). If \( a \in O_{F} \), and if \( \text{val}_a(a) \geq n \) for \( n = n(r, s) \) large enough, the series \( \exp(a \nabla) \) converges in the operator norm to an operator on the Banach space \( D_{D, r}^{[r,s]}(V) \). This way, we can define a twisted action of \( \Gamma_K \) on \( D_{D, r}^{[r,s]}(V) \), by the formula \( h \star x = \exp(\log_p(\chi_p(h)) \cdot \nabla)(x) \). This action is now \( F \)-analytic by construction.

Since \( s \geq qr \), the modules \( D_{D, r}^{[q^m r, q^m s]}(V) \) for \( m \geq 0 \) are glued together (using the idea explained above) by \( \varphi_q \) and we get a new action of \( \Gamma_K \) on \( D_{D, r}^{[r, +\infty]}(V) \) and hence on \( D_{D, r}^I(V) \). Since \( \varphi_q \) is unchanged, this new \( (\varphi, \Gamma) \)-module is étale, and therefore corresponds to a representation \( W \) of \( G_K \). The representation \( W \) is \( F \)-analytic by theorem 1.2.2, and its restriction to \( H_K \) is isomorphic to \( V \).

Let \( X = \text{ind}_{G_K}^{G_K} W \). By Mackey’s formula, \( X|_{H_K} \) contains \( W|_{H_K} \simeq V|_{H_K} \) as a direct summand. The space \( Y = \text{Hom}(\text{ind}_{G_K}^{G_K} W, V) \otimes_{H_K} \) is therefore a nonzero representation of \( \Gamma_K \), and there is an element \( y \in Y \) whose image is \( V \). The natural map \( X \otimes_F Y \to V \) is therefore surjective. Finally, \( X \) is \( F \)-analytic since \( W \) is \( F \)-analytic.

By dualizing, we get the following variant of theorem 1.3.1.

**Corollary 1.3.2.** If \( V \) is an overconvergent representation of \( G_K \), there exists an \( F \)-analytic representation \( X \) of \( G_K \), a representation \( Y \) of \( G_K \) that factors through \( \Gamma_K \), and an injective \( G_K \)-equivariant map \( V \to X \otimes_F Y \).

### 1.4 Extensions of \((\varphi, \Gamma)\)-modules

In this section, we prove that there are no non-trivial extensions between an \( F \)-analytic \((\varphi, \Gamma)\)-module and the twist of an \( F \)-analytic \((\varphi, \Gamma)\)-module by a character that is not \( F \)-analytic. This is not used in the rest of the paper, but is of independent interest.

If \( \delta : \Gamma_K \to O_F^\times \) is a continuous character, and \( g \in \Gamma_K \), let \( w_{\delta}(g) = \log \delta(g)/\log \chi_p(g) \). Note that \( \delta \) is \( F \)-analytic if and only if \( w_{\delta}(g) \) is independent of \( g \in \Gamma_K \).

We define the first cohomology group \( H^1(D) \) of a \((\varphi, \Gamma)\)-module \( D \) as in §4 of [FX13]. Let \( D \) be a \((\varphi, \Gamma)\)-module over \( B_{rig, K}^I \). Let \( G \) denote the semigroup \( \varphi_q^{-1} \times \Gamma_K \) and let \( Z^1(D) \) denote the set of continuous functions \( f : G \to D \) such that \( (h - 1) f(g) = (g - 1) f(h) \) for all \( g, h \in G \). Let \( B^1(D) \) be the subset of \( Z^1(D) \) consisting of functions of the form \( g \mapsto (g - 1) g, \ y \in D \) and let \( H^1(D) = Z^1(D)/B^1(D) \). If \( g \in G \) and \( f \in Z^1 \), then \( [h \mapsto (g - 1)(h)] \) is \( [h \mapsto (h - 1) f(g)] \) in \( B^1 \). The natural actions of \( \Gamma_K \) and \( \varphi_q \) on \( H^1 \) are therefore trivial.
If $D_0$ and $D_1$ are two $(\varphi, \Gamma)$-modules, then $\text{Hom}(D_1, D_0) = \text{Hom}_{B^{\text{rig}, K}-\text{mod}}(D_1, D_0)$ is a free $B^{\text{rig}, K}$-module of rank $\text{rk}(D_0) \text{rk}(D_1)$ which is easily seen to be itself a $(\varphi, \Gamma)$-module. The space $H^1(\text{Hom}(D_1, D_0))$ classifies the extensions of $D_1$ by $D_0$. More precisely, if $D$ is such an extension and if $s : D_1 \to D$ is a $B^{\text{rig}, K}$-linear map that is a section of the projection $D \to D_1$, then $g \mapsto s - g(s)$ is a cocycle on $G$ with values in $\text{Hom}(D_1, D_0)$ (the element $g(s) \in \text{Hom}(D_1, D)$ being defined by $g(s)(g(x)) = g(s(x))$ for all $g \in G$ and all $x \in D_1$). The class of this cocycle in the quotient $H^1(\text{Hom}(D_1, D_0))$ does not depend on the choice of the section $s$, and every such class defines a unique extension of $D_1$ by $D_0$ up to isomorphism.

**Theorem 1.4.1.** If $D$ is an $F$-analytic $(\varphi, \Gamma)$-module, and if $\delta : \Gamma_K \to O_F^\times$ is not locally $F$-analytic, then $H^1(D(\delta)) = \{0\}$.

**Proof.** If $g \in \Gamma_K$ and $x(\delta) \in D(\delta)$ with $x \in D$, we have

$$\nabla_g(x(\delta)) = \nabla(x(\delta)) + w_\delta(g) \cdot x(\delta).$$

If $g, h \in \Gamma_K$, this implies that $\nabla_g(x(\delta)) - \nabla_h(x(\delta)) = (w_\delta(g) - w_\delta(h)) \cdot x(\delta)$. If $\overline{f} \in H^1(D(\delta))$ and $g \in \Gamma_K$, then $g(\overline{f}) = \overline{f}$ and therefore $\nabla_g(\overline{f}) = 0$. The formula above shows that if $k \in \Gamma_K$, then $\nabla_g(f(k)) - \nabla_h(f(k)) = (w_\delta(g) - w_\delta(h)) \cdot f(k)$, so that $0 = (\nabla_g - \nabla_h)(\overline{f}) = (w_\delta(g) - w_\delta(h)) \cdot \overline{f}$, and therefore $\overline{f} = 0$ if $\delta$ is not locally analytic.  

## 2 Analytic cohomology and Iwasawa theory

In this chapter, we explain how to construct classes in the cohomology groups of $F$-analytic $(\varphi, \Gamma)$-modules. This allows us to define our maps $h_{K_n, V}^1$.

### 2.1 Analytic cohomology

Let $G$ be an $F$-analytic semigroup and let $M$ be a Fréchet or LF space with a pro-$F$-analytic ($\S 2$ of [Ber16]) action of $G$. Recall that this means that we can write $M = \varprojlim \varprojlim M_{ij}$ where $M_{ij}$ is a Banach space with a locally analytic action of $G$. A function $f : G \to M$ is said to be pro-$F$-analytic if its image lies in $\varprojlim M_{ij}$ for some $i$ and if the corresponding function $\overline{f} : G \to M_{ij}$ is locally $F$-analytic for all $j$. The analytic cohomology groups $H^i_{\text{an}}(G, M)$ are defined and studied in §4 of [FX13] and §5 of [Col16]. In particular, we have $H^1_{\text{an}}(G, M) = M^G$ and $H^2_{\text{an}}(G, M) = \mathbb{Z}^1_{\text{an}}(G, M)/B^1_{\text{an}}(G, M)$ where $\mathbb{Z}^1_{\text{an}}(G, M)$ is the set of pro-$F$-analytic functions $f : G \to M$ such that $(g - 1)f(h) = (h - 1)f(g)$ for all $g, h \in G$ and $B^1_{\text{an}}(G, M)$ is the set of functions of the form $g \mapsto (g - 1)m$.

Let $M$ be a Fréchet space, and write $M = \varprojlim M_n$ with $M_n$ a Banach space such that the image of $M_{n+1}$ in $M_n$ is dense for all $j \geq 0$.

**Proposition 2.1.1.** We have $H^1_{\text{an}}(G, M) = \varprojlim H^1_{\text{an}}(G, M_n)$. 

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Proof. By definition, we have an exact sequence
\[ 0 \to B^1_{an}(G, M_n) \to Z^1_{an}(G, M_n) \to H^1_{an}(G, M_n) \to 0. \]
It is clear that $B^1_{an}(G, M) = \lim_{n \to \infty} B^1_{an}(G, M_n)$ and that $Z^1_{an}(G, M) = \lim_{n \to \infty} Z^1_{an}(G, M_n)$, since these spaces are spaces of functions on $G$ satisfying certain compatible conditions. The Banach spaces $B^1_{an}(G, M_n)$ satisfy the Mittag-Leffler condition: $B^1_{an}(G, M_n) = M_n/M_n^G$ and the image of $M_n$ in $M_n$ is dense for all $j \geq 0$. This implies that the sequence
\[ 0 \to \lim_{n \to \infty} B^1_{an}(G, M_n) \to \lim_{n \to \infty} Z^1_{an}(G, M_n) \to \lim_{n \to \infty} H^1_{an}(G, M_n) \to 0 \]
is exact, and the proposition follows. □

In this paper, we mainly use the semigroups $\Gamma_K$, $\Gamma_K \times \Phi$ where $\Phi = \{ \varphi^n, n \in \mathbb{Z}_{\geq 0} \}$ and $\Gamma_K \times \Psi$ where $\Psi = \{ \psi^n, n \in \mathbb{Z}_{\geq 0} \}$. The semigroups $\Phi$ and $\Psi$ are discrete and the $F$-analytic structure comes from the one on $\Gamma_K$.

**Definition 2.1.2.** Let $G$ be a compact group and let $H$ be an open subgroup of $G$. We have the corestriction map $\text{cor} : H^1_{an}(H, M) \to H^1_{an}(G, M)$, which satisfies $\text{cor} \circ \text{res} = [G : H]$. This map has the following equivalent explicit descriptions (see §2.5 of [Ser94] and §II.2 of [CC99]). Let $X \subset G$ be a set of representatives of $G/H$ and let $f \in Z^1_{an}(H, M)$ be a cocycle.

1. By Shapiro’s lemma, $H^1_{an}(H, M) = H^1_{an}(G, \text{ind}^G_H M)$ and $\text{cor}$ is the map induced by $i \mapsto \sum x \in X x \cdot i(x^{-1})$;
2. if $M \subset N$ where $N$ is a $G$-module and if there exists $n \in N$ such that $f(h) = (h - 1)(n)$, then $\text{cor}(f)(g) = (g - 1)(\sum x \in X x n)$;
3. if $g \in G$, let $\tau_g : X \to X$ be the permutation defined by $\tau_g(x) = gxH$. We have $\text{cor}(f)(g) = \sum x \in X \tau_g(x) \cdot f(\tau_g(x)^{-1}gx)$.

If $g \in \Gamma_K$, let $\ell(g) = \log_p \chi(x)(g)$. If $M$ is a Fréchet space with a pro-$F$-analytic action of $\Gamma_K$ and if $g \in \Gamma_K$ is such that $\chi(x)(g) \in 1 + 2p\mathcal{O}_F$, then $\lim_{n \to \infty} (g^p^n - 1)/(p^n \ell(g))$ converges to an operator $\nabla$ on $M$, which is independent of $g$ thanks to the $F$-analyticity assumption. If $c : \Gamma_K \to M$ is an $F$-analytic map, let $c'(1)$ denote its derivative at the identity.

**Proposition 2.1.3.** If $M$ is a Fréchet space with a pro-$F$-analytic action of $\Gamma_K$, the map $c \to c'(1)$ induces an isomorphism $H^1_{an}(\Gamma_K, M) = (M/\nabla M)^{\Gamma_K}$, under which $\text{cor}_{L/K}$ corresponds to $\text{Tr}_{L/K}$.

**Proof.** Assume for the time being that $M$ is a Banach space. We first show that the map induced by $c \to c'(1)$ is well-defined and lands in $(M/\nabla M)^{\Gamma_K}$. The map $c \to c'(1)$ from $Z^1_{an}(\Gamma_K, M) \to M$ is well-defined, and if $c(g) = (g - 1)m$, then $c'(1) = \nabla m$ so that there is a well-defined map $H^1_{an}(\Gamma_K, M) \to M/\nabla M$. If
If \( y \in \Gamma_K \) then \( (h-1)c'(1) = \lim_{y \to 1} (h-1)c(y)/\ell(y) = \lim_{y \to 1} (g-1)c(g)/\ell(g) = \nabla c(h) \) so that the image of \( e \mapsto c'(1) \) lies in \( (M/\nabla M)^{\Gamma_K} \).

The formula for the corestriction follows from the explicit descriptions above: if \( h \in \Gamma_L \) then \( \tau_h(x) = x \) so that \( \text{cor}(c)(h) = \sum_{x \in X} x \cdot c(h) \) and

\[
\text{cor}(c)'(1) = \lim_{h \to 1} \text{cor}(c)(h)/\ell(h) = \sum_{x \in X} x \cdot c'(1) = \text{Tr}_{L/K}(c'(1)).
\]

We now show that the map is injective. If \( c'(1) = \nabla m \), then the derivative of \( g \mapsto c(g) \) at \( g = 1 \) is zero and hence \( c(g) = (g-1)m \) on some open subgroup \( \Gamma_L \) of \( \Gamma_K \) and \( c = [L : K]^{-1} \text{cor}_{L/K} \circ \text{res}_{K/L}(c) = 0. \)

We finally show that the map is surjective. Suppose now that \( y \in (M/\nabla M)^{\Gamma_K} \).

The formula \( g \mapsto (\exp(\ell(g)\nabla) - 1)/\nabla \cdot y \) defines an analytic cocycle \( c_L \) on some open subgroup \( \Gamma_L \) of \( \Gamma_K \). The image of \( [L : K]^{-1}c_L \) under \( \text{cor}_{L/K} \) gives a cocycle \( c \in H^1_{\text{an}}(\Gamma_K, M) \) such that \( c'(1) = y \).

We now let \( M = \lim_{\leftarrow n} M_n \) be a Fréchet space. The map \( H^1_{\text{an}}(\Gamma_K, M) \to (M/\nabla M)^{\Gamma_K} \) induced by \( c \mapsto c'(1) \) is well-defined, and in the other direction we have the map \( y \mapsto c_y : \)

\[
(M/\nabla M)^{\Gamma_K} \to \lim_{\leftarrow n}(M_n/\nabla M_n)^{\Gamma_K} \to \lim_{\leftarrow n}H^1_{\text{an}}(\Gamma_K, M_n) \to H^1_{\text{an}}(\Gamma_K, M).
\]

These two maps are inverses of each other.  

**Remark 2.1.4.** Compare with the following theorem (see [Tam15], corollary 21): if \( G \) is a compact \( p \)-adic Lie group and if \( M \) is a locally analytic representation of \( G \), then \( H^1_{\text{an}}(G, M) = H^1(\text{Lie}(G), M)^G \).

### 2.2 Cohomology of \((\varphi, \Gamma)\)-modules

If \( V \) is an \( F \)-analytic representation, let \( H^1_{\text{an}}(K, V) \subset H^1(K, V) \) classify the \( F \)-analytic extensions of \( F \) by \( V \). Let \( D \) denote an \( F \)-analytic \((\varphi, \Gamma)\)-module over \( B^\dagger_{\text{rig},K} \), such as \( D^\dagger_{\text{rig}}(V) \).

**Proposition 2.2.1.** If \( V \) is \( F \)-analytic, then \( H^1_{\text{an}}(K, V) = H^1_{\text{an}}(\Gamma_K \times \Phi, D^\dagger_{\text{rig}}(V)) \).

**Proof.** The group \( H^1_{\text{an}}(\Gamma_K \times \Phi, D^\dagger_{\text{rig}}(V)) \) classifies the \( F \)-analytic extensions of \( B^\dagger_{\text{rig},K} \) by \( D^\dagger_{\text{rig}}(V) \), which correspond to \( F \)-analytic extensions of \( F \) by \( V \) by theorem 1.2.2.

**Theorem 2.2.2.** If \( D \) is an \( F \)-analytic \((\varphi, \Gamma)\)-module over \( B^\dagger_{\text{rig},K} \) and \( i = 0, 1 \), then \( H^1_{\text{an}}(\Gamma_K, D^{\psi_i = 0}) = 0. \)

**Proof.** Since \( B^\dagger_{\text{rig},F} \subset B^\dagger_{\text{rig},K} \), the \( B^\dagger_{\text{rig},K} \)-module \( D \) is a free \( B^\dagger_{\text{rig},F} \)-module of finite rank. Let \( R_F \) denote \( B^\dagger_{\text{rig},F} \) and let \( R_F[p] \) denote \( C_p \otimes F B^\dagger_{\text{rig},F} \) the Robba
ring with coefficients in $C_p$. There is an action of $G_F$ on the coefficients of $R_{C_p}$ and $R_{C_p}^G = R_F$.

Theorem 5.5 of [Col16] says that $H^i_{an}(\Gamma_K, (R_{C_p} \otimes R_F)D)_{\psi_1 = 0} = 0$. For $i = 0$, this implies our claim. For $i = 1$, it says that if $c : \Gamma_K \to D_{\psi_1 = 0}$ is an $F$-analytic cocycle, there exists $m \in (R_{C_p} \otimes R_F)D_{\psi_1 = 0}$ such that $c(g) = (g-1)m$ for all $g \in \Gamma_K$. If $\alpha \in G_F$, then $c(g) = (g-1)\alpha(m)$ as well, so that $\alpha(m) = m \in ((R_{C_p} \otimes R_F)D_{\psi_1 = 0})^G = D_{\psi_1 = 0}$.

\section*{Corollary 2.2.3}

The groups $H^i_{an}(\Gamma_K \times \Phi, D)$ and $H^i_{an}(\Gamma_K \times \Psi, D)$ are isomorphic for $i = 0, 1$.

\section*{Proof}
For $i = 0$, then we have an inclusion $D_{\psi_1 = 0} = D_{\psi_1 = 0}$. If $x \in D_{\psi_1 = 0}$, then $x = \varphi_1(x) \in D_{\psi_1 = 0}$ by theorem 2.2.2, so that $x = \varphi_1(x)$ and the above inclusion is an equality.

Now let $i = 1$. If $f \in Z^1_{an}(\Gamma_K \times \Phi, D)$, let $Tf \in Z^1_{an}(\Gamma_K \times \Psi, D)$ be the function defined by $Tf(g) = f(g)$ when $g \in \Gamma_K$ and $Tf(g) = -\varphi(g)(f(\varphi(g)))$. If $f \in Z^1_{an}(\Gamma_K \times \Psi, D)$ and $g \in \Gamma_K$, then $(\varphi(g)\varphi(g)(f(\varphi(g))) = 0$ and the map $g \mapsto (\varphi(g)\varphi(g)(f) \in D_{\psi_1 = 0}$ and the map $g \mapsto (\varphi(g)\varphi(g)(f) \in D_{\psi_1 = 0}$. By theorem 2.2.2, applied once for existence and once for unicity, there is a unique $m_f \in D_{\psi_1 = 0}$ such that $m_f(g) = (g-1)m_f$. Let $Uf \in Z^1_{an}(\Gamma_K \times \Phi, D)$ be the function defined by $Uf(g) = f(g)$ if $g \in \Gamma_K$ and $Uf(g) = -\varphi(g)(f(\varphi(g))) + m_f$.

It is straightforward to check that $U$ and $T$ are inverses of each other (even at the level of the $Z^1_{an}$) and that they descend to the $H^1_{an}$.

\section*{Theorem 2.2.4}

The map $f \mapsto f(\psi_q)$ from $Z^1_{an}(\Gamma_K \times \Psi, D)$ to $D$ gives rise to an exact sequence:

$$0 \to H^1_{an}(\Gamma_K, D_{\psi_1 = 0}) \to H^1_{an}(\Gamma_K \times \Psi, D) \to \left(\frac{D}{\psi_1 - 1}\right)_{\Gamma_K}$$

\section*{Proof}
If $f \in Z^1_{an}(\Gamma_K \times \Psi, D)$ and $g \in \Gamma_K$, then $(g-1)f(\psi_q) = (\psi_q - 1)f(g) \in (\psi_q - 1)D$ so that the image of $f$ is in $(D/(\psi_q - 1))_{\Gamma_K}$. The other verifications are similar.

\subsection*{2.3 The space $D/(\psi_q - 1)$}

By theorem 2.2.4 in the previous section, the cokernel of the map $H^1_{an}(\Gamma_K, D_{\psi_1 = 0}) \to H^1_{an}(\Gamma_K \times \Psi, D)$ injects into $(D/(\psi_q - 1))_{\Gamma_K}$. It can be useful to know that this cokernel is not too large. In this section, we bound $D/(\psi_q - 1)$ when $D = B_{rig,F}^1$, with the action of $\varphi_q$ twisted by $a^{-1}$, for some $a \in F_x$.

\section*{Theorem 2.3.1}

If $a \in F_x$, then $\psi_q - a : B_{rig,F}^1 \to B_{rig,F}^1$ is onto unless $a = q^{-1}p^m$ for some $m \in \mathbb{Z}_{\geq 1}$, in which case $B_{rig,F}^1/(\psi_q - a)$ is of dimension 1.
In order to prove this theorem, we need some results about the action of $\psi$ on $B_{\text{rig}, F}^1$. Recall that the map $\partial = d/dt_\pi$ was defined in §1.1.

**Lemma 2.3.2.** If $a \in F^\times$, then $a\varphi_q - 1 : B_{\text{rig}, F}^+ \to B_{\text{rig}, F}^+$ is an isomorphism, unless $a = \pi^{-m}$ for some $m \in \mathbb{Z}_{\geq 0}$, in which case

$$\ker(a\varphi_q - 1 : B_{\text{rig}, F}^+ \to B_{\text{rig}, F}^+) = Ft_\pi^m$$

$$\text{im}(a\varphi_q - 1 : B_{\text{rig}, F}^+ \to B_{\text{rig}, F}^+) = \{f(T) \in B_{\text{rig}, F}^+ \mid \partial^m(f)(0) = 0\}.$$  

**Proof.** This is lemma 5.1 of [FX13].

**Lemma 2.3.3.** If $m \in \mathbb{Z}_{\geq 0}$, there is an $h(T) \in (B_{\text{rig}, F}^+)^{\psi_m=0}$ such that $\partial^m(h)(0) \neq 0$.

**Proof.** We have $\psi_q(T) = 0$ by (the proof of) proposition 2.2 of [FX13]. If there was some $m_0$ such that $\partial^m(T)(0) = 0$ for all $m \geq m_0$, then $T$ would be a polynomial in $t_\pi$, which it is not. This implies that there is a sequence $\{m_i\}$ of integers with $m_i \to +\infty$, such that $\partial^{m_i}(T)(0) \neq 0$, and we can take $h(T) = \partial^{m_i-m}(T)$ for any $m_i \geq m$.

**Corollary 2.3.4.** If $a \in F^\times$, then $\psi_q - a : B_{\text{rig}, F}^+ \to B_{\text{rig}, F}^+$ is onto.

**Proof.** If $f(T) \in B_{\text{rig}, F}^+$ and if we can write $f = (1-a\varphi_q)g$, then $f = (\psi_q - a)(\varphi_q(g))$. If this is not possible, then by lemma 2.3.2 there exists $m \geq 0$ such that $a = \pi^{-m}$ and $\partial^m(f)(0) \neq 0$. Let $h$ be the function provided by lemma 2.3.3. The function $f - (\partial^m(f)(0)/\partial^m(h)(0)) \cdot h$ is in the image of $1-a\varphi_q$ by lemma 2.3.2, and $h = (\psi_q - a)(-a^{-1}h)$ since $\psi_q(h) = 0$. This implies that $f$ is in the image of $\psi_q - a$.

**Lemma 2.3.5.** If $a^{-1} \in q \cdot \mathcal{O}_F$, then $\psi_q - a : B_{\text{rig}, F}^+ \to B_{\text{rig}, F}^+$ is onto.

**Proof.** We have $B_{\text{rig}, F}^1 = B_{\text{rig}, F}^+ + B_{\text{rig}, F}^{-}$ (by writing a power series as the sum of its plus part and of its minus part) and by corollary 2.3.4, $\psi_q - a : B_{\text{rig}, F}^+ \to B_{\text{rig}, F}^+$ is onto. Take $f(T) \in B_{\text{rig}, F}^+$, choose some $r > 0$ and let $B_{F}^{(0,r)}$ be the set of $f(T) \in B_{\text{rig}, F}^+$ that converge and are bounded on the annulus $0 < \text{val}_p(x) \leq r$. It follows from proposition 1.4 of [Col16] that if $n \geq 0$, then $\psi_q^n(f) \in B_{F}^{(0,r)}$ and by proposition 2.4(d) of [FX13], the sequence $(q/\pi \cdot \psi_q)^n(f)$ is bounded in $B_{F}^{(0,r)}$. The series $\sum_{n \geq 0} a^{-1-n}\psi_q^n(f)$ therefore converges in $B_{F}^{(0,r)}$, and we can write $f = (\psi_q - a)g$ where $g = a^{-1}(1 - a^{-1}\psi_q)^{-1}f = \sum_{n \geq 0} a^{-1-n}\psi_q^n(f)$.  

Let $\text{Res} : B_{\text{rig}, F}^+ \to F$ be defined by $\text{Res}(f) = a_{-1}$ where $f(T)dt_\pi = \sum_{n \in \mathbb{Z}} a_n T^n dt_\pi$. The following lemma combines propositions 2.12 and 2.13 of [FX13].

**Lemma 2.3.6.** The sequence $0 \to F \to B_{\text{rig}, F}^+ \xrightarrow{\partial} B_{\text{rig}, F}^+ \xrightarrow{\text{Res}} F \to 0$ is exact, and $\text{Res}(\psi_q(f)) = \pi/q \cdot \text{Res}(f)$.
Proof of theorem 2.3.1. Since $\partial \circ \psi_q = \pi^{-1} \psi_q \circ \partial$, the map $\partial$ induces a map:

$$\frac{B^\dagger_{\text{rig}, F}}{\psi_q - a} \xrightarrow{\partial} \frac{B^\dagger_{\text{rig}, F}}{\psi_q - a\pi}.$$

(Der)

Take $x \in B^\dagger_{\text{rig}, F}$ such that $\text{Res}(x) = 1$. We have $\text{Res}((\psi_q - a\pi)x) = \pi/q - a\pi$. If $a \neq q^{-1}$, this is non-zero and if $f \in B^\dagger_{\text{rig}, F}$, proposition 2.3.6 allows us to write $f = \partial g + \text{Res}(f)/(\pi/q - a\pi) \cdot (\psi_q - a\pi)x$. This implies that (Der) is onto if $a \neq q^{-1}$.

Combined with lemma 2.3.5, this implies that $B^\dagger_{\text{rig}, F}/(\psi_q - a) = 0$ if $a$ is not of the form $q^{-1}\pi^m$ for some $m \in \mathbb{Z}_{\geq 1}$.

When $a = q^{-1}$, we have an exact sequence

$$\frac{B^\dagger_{\text{rig}, F}}{\psi_q - q^{-1}} \xrightarrow{\partial} \frac{B^\dagger_{\text{rig}, F}}{\psi_q - q^{-1}\pi} \xrightarrow{\text{Res}} F \rightarrow 0,$$

which now implies that $B^\dagger_{\text{rig}, F}/(\psi_q - q^{-1}\pi) = F$, generated by the class of $x$.

We now assume again that $a \neq q^{-1}$ and compute the kernel of (Der). If $f \in B^\dagger_{\text{rig}, F}$ is such that $\partial f = (\psi_q - a\pi)g$, then $\text{Res}(\partial f) = \text{Res}(\psi_q - a\pi)g = (\pi/q - a\pi)\text{Res}(g)$, so that $\text{Res}(g) = 0$ and we can write $g = \partial h$. We have $\partial(f - (\psi_q - a)h) = 0$, so that $f = (\psi_q - a)h + c$, with $c \in F$. By corollary 2.3.4, there exists $b \in B^\dagger_{\text{rig}, F}$ such that $(\psi_q - a)(b) = c$, so that $f = (\psi_q - a)(h + b)$ and (Der) is bijective. We then have, by induction on $m \geq 1$, that $B^\dagger_{\text{rig}, F}/(\psi_q - q^{-1}\pi^m) = F$, generated by the class of $\partial^m(x)$. \[ \square \]

Remark 2.3.7. More generally, we expect that the following holds: if $D$ is a $(\varphi, \Gamma)$-module over $B^\dagger_{\text{rig}, K}$, the $F$-vector space $D/(\psi_q - 1)$ is finite dimensional.

2.4 The operator $\Theta_b$

The power series $F(X) = X/(\exp(X) - 1)$ belongs to $\mathbb{Q}_p[X]$ and has a nonzero radius of convergence. If $M$ is a Banach space with a locally $F$-analytic action of $\Gamma_K$ and $h \in \Gamma_K$ is close enough to 1, then

$$\frac{\nabla}{h - 1} = \frac{\nabla}{\exp(\ell(h)\nabla) - 1} = \ell(h)^{-1} F(\ell(h)\nabla)$$

converges to a continuous operator on $M$. If $g \in \Gamma_K$, we then define

$$\frac{\nabla}{1 - g} = \frac{\nabla}{1 - g^n} \cdot \frac{1 - g^n}{1 - g}.$$

This operator is independent of the choice of $n$ such that $g^n$ is close enough to 1, and can be seen as an element of the locally $F$-analytic distribution algebra acting on $M$. 

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If $M$ is a Fréchet space, write $M = \varinjlim_i M_i$ and define operators $\sum_1^{\infty} \gamma_i$ on each $M_i$ as above. These operators commute with the maps $M_j \to M_i$ (because $n$ can be taken large enough for both $M_i$ and $M_j$). This defines an operator $\sum_1^{\infty} \gamma_i$ on $M$ itself. The definition of $\sum_1^{\infty} \gamma_i$ extends to an LF space with a pro-$F$-analytic action of $\Gamma_K$.

Assume that $K$ contains $F_1$ and let $r(K) = f + \text{val}_p([K : F_1])$. For example, $p^{r(F_n)} = q^n$ if $n \geq 1$. Assume further that $K$ contains $F_{n}(K)$, so that $\chi : \Gamma_K \to \mathcal{O}_{K}^{*}$ is injective and its image is a free $\mathbb{Z}_p$-module of rank $d$. If $b = (b_1, \ldots, b_d)$ is a basis of $\Gamma_K$ (that is, $\Gamma_K = b_1\mathbb{Z}_p \cdots b_d\mathbb{Z}_p$), then let $\ell^* (b) = \ell (b_1) \cdots \ell (b_d)/p^{r(K)}$ and

$$\Theta_b = \ell^* (b) \cdot \nabla^d \left( (b_1 - 1) \cdots (b_d - 1) \right).$$

**Lemma 2.4.1.** If $K = F_n$ and $m \geq 0$ and $x \in F_{m+n}$, then

$$\Theta_b(x) = q^{-m-n} \cdot \text{Tr}_{F_{m+n}/F_n} (x).$$

**Proof.** Since $\nabla = \lim_{k \to \infty} (b_i^k - 1)/p^k \ell (b_i)$, we have

$$\Theta_b = \lim_{k \to \infty} \frac{1}{q^{np^{kd}}} \cdot \frac{(b_1^k - 1) \cdots (b_d^k - 1)}{(b_1 - 1) \cdots (b_d - 1)}.$$

The set \{$(b_1^{a_1} \cdots b_d^{a_d})$\} with $0 \leq a_i \leq p^k - 1$ runs through a set of representatives of $\Gamma_n/\Gamma_n^k = \Gamma_n/\Gamma_{n+ek}$ so that

$$\frac{1}{q^{np^{kd}}} \cdot \frac{(b_1^k - 1) \cdots (b_d^k - 1)}{(b_1 - 1) \cdots (b_d - 1)} = \frac{1}{q^{np^{kd}}} \Tr_{F_{n+ek}/F_n} = \frac{1}{q^{n+ek}} \cdot \Tr_{F_{n+ek}/F_n}.$$

The lemma follows from taking $k$ large enough so that $ek \geq m$. ■

For $i \in \mathbb{Z}$, let $\nabla_i = \nabla - i$.

**Lemma 2.4.2.** If $b$ is a basis of $\Gamma_{F_n}$ and if $f(T) \in (B_{\text{rig}, F}^+)_{\psi_q=0}$, then $\Theta_b(f(T)) \in (t_{\pi}/\varphi_q^0(T)) \cdot B_{\text{rig}, F}^+$, and if $h \geq 2$ then $\nabla_{h-1} \circ \cdots \nabla_1 \circ \Theta_b(f(T)) \in (t_{\pi}/\varphi_q^0(T))^h \cdot B_{\text{rig}, F}^+$.

**Proof.** If $m \geq 1$, then by lemma 2.4.1 and using repeatedly the fact (see §1.1) that $\varphi_q \circ \psi_q(f) = 1/q \cdot \sum_{z \in \mathbb{LT}[x]} f(T \odot z)$,

$$\Theta_b(f(u_{m+n})) = 1/q^{m+n} \cdot \text{Tr}_{F_{m+n}/F_n} f(u_{m+n}) = \psi_q^m (f)(u_n) = 0.$$

This proves the first claim, since an element $f(T) \in B_{\text{rig}, F}^+$ is divisible by $t_{\pi}/\varphi_q^0(T)$ if and only if $f(u_{m+n}) = 0$ for all $m \geq 1$. The second claim follows easily. ■

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Let $D$ be a $\varphi_q$-module over $F$. Let $\varphi_q^{-n}: B_{\rig}^+\otimes F D \to F_n((\tau)) \otimes_F D$ be the map

$$\varphi_q^{-n}: t_{\tau}^{-h} f(T) \otimes x \mapsto \pi^n t_{\tau}^{-h} f(u_n \oplus \exp_{LT}(t_{\tau}/\pi^n)) \otimes \varphi_q^{-n}(x).$$

If $f(t_{\tau}) \in F_n((t_{\tau})) \otimes_F D$, let $\partial_D(f) \in F_n \otimes_F D$ denote the coefficient of $t_{\tau}^0$.

**Lemma 2.4.3.** If $y \in (B_{\rig}^+[1/t_{\tau}] \otimes_F D)^{\psi_q=1}$ and if $m \geq n$, then

$$q^{-m} \text{Tr}_{F_m/F_n} \partial_D(\varphi_q^{-m}(y)) = \begin{cases} q^{-n} \partial_D(\varphi_q^{-n}(y)) & \text{if } n \geq 1 \\ (1 - q^{-1}\varphi_q^{-1})\partial_D(y) & \text{if } n = 0. \end{cases}$$

**Proof.** If $y = t_{\tau}^{-\ell} \sum_{k=0}^{+\infty} a_k T^k \in B_{\rig}^+[1/t_{\tau}] \otimes_F D$, then (by definition of $\varphi_q^{-m}$)

$$\varphi_q^{-m}(y) = \pi^m t_{\tau}^{-\ell} \sum_{k=0}^{+\infty} \varphi_q^{-m}(a_k)(u_m \oplus \exp_{LT}(t_{\tau}/\pi^m))^k,$$

and $\psi_q(y) = y$ means that:

$$\varphi_q(y)(T) = \frac{1}{q} \sum_{[\tau](\omega) = 0} y(T \oplus \omega).$$

If $m \geq 2$, the conjugates of $u_m$ under $\text{Gal}(F_m/F_{m-1})$ are the $\{\omega \oplus u_m\}_{[\tau](\omega) = 0}$ so that:

$$\text{Tr}_{F_m/F_{m-1}} \partial_D(\varphi_q^{-m}(y))$$

$$= \partial_D \left( \sum_{[\tau](\omega) = 0} \pi^m t_{\tau}^{-\ell} \sum_{k=0}^{+\infty} \varphi_q^{-m}(a_k)(\omega \oplus u_m \oplus \exp_{LT}(t_{\tau}/\pi^m))^k \right)$$

$$= \partial_D \left( \varphi_q^{-m} \left( \sum_{[\tau](\omega) = 0} y(T \oplus \omega) \right) \right)$$

$$= q \partial_D(\varphi_q^{-(m-1)}(y)).$$

For $m = 1$, the computation is similar, except that the conjugates of $u_1$ under $\text{Gal}(F_1/F)$ are the $\omega$, where $[\tau](\omega) = 0$ but $\omega \neq 0$, which results in:

$$\text{Tr}_{F_1/F} \partial_D(\varphi_q^{-1}(y)) = \partial_D \left( \varphi_q^{-1} \left( \sum_{[\tau](\omega) = 0} y(T \oplus \omega) \right) \right) = \partial_D(qy - \varphi_q^{-1}(y)).$$

$\square$
2.5 Construction of extensions

Let $D$ be an $F$-analytic $(\varphi, \Gamma)$-module over $B_{\mathrm{rig}, K}^\dagger$. The space $D^{\psi_{\pi}=1}$ is a closed subspace of $D$ and therefore an LF space. Take $K$ such that $K$ contains $F_{n(K)}$ and let $b$ be a basis of $\Gamma_K$.

**Proposition 2.5.1.** If $y \in D^{\psi_{\pi}=1}$, there is a unique cocycle $c_k(y) \in Z^1_{\mathrm{an}}(\Gamma_K, D^{\psi_{\pi}=1})$ such that for all $1 \leq j \leq d$ and $k \geq 0$, we have

$$c_k(y)(b_j^k) = \ell^*(b) \cdot \frac{b_j^k - 1}{b_j - 1} \cdot \nabla^{d-1}(y).$$

We then have $c_k(y)'(1) = \Theta_b(y)$.

**Proof.** There is obviously one and only one continuous cocycle satisfying the conditions of the proposition. It is $Q_p$-analytic, and in order to prove that it is $F$-analytic, we need to check that the directional derivatives are independent of $j$. We have

$$\lim_{k \to 0} \frac{c_k(y)(b_j^k)}{\ell(b_j^k)} = \ell^*(b) \cdot \frac{\nabla^d}{\prod(b_i - 1)}(y) = \Theta_b(y),$$

which is indeed independent of $j$, and thus $c_k(y)'(1) = \Theta_b(y)$.

**Lemma 2.5.2.** If $n \geq n(K)$ and $L = K_n$ and $M = K_{n+e}$ and $b$ is a basis of $\Gamma_L$, then $b^p$ is a basis of $\Gamma_M$ and $\mathrm{cor}_{M/L}(c_{p^e}) = c_b(y)$.

**Proof.** The Lubin-Tate character maps $\Gamma_L$ to $1 + \pi^n \mathcal{O}_F$, and $\Gamma_M = \Gamma_L^p$ because $(1 + \pi^n \mathcal{O}_F)^p = 1 + \pi^{n+e} \mathcal{O}_F$. Since $\{b_1^{k_1} \cdots b_d^{k_d}\}$ with $0 \leq k_i \leq p - 1$ is a set of representatives for $\Gamma_L/\Gamma_M$, and since $[M : L] = q^e = p^d$, the explicit formula for the corestriction (definition 2.1.2) implies (here and elsewhere $[x]$ is the smallest integer $\geq x$)

$$\mathrm{cor}_{M/L}(c_{p^e}(y))(b_j^k)$$

$$= \sum_{0 \leq k_1, \ldots, k_d \leq p - 1} b_1^{k_1} \cdots b_d^{k_d} \cdot \ell^*(b^p) \cdot \frac{b_j^p [b_j^{-k} - 1]}{b_j^{p-1}} \cdot \nabla^{d-1}(y)$$

$$= \ell^*(b) \left( \sum_{k_j=0}^{p-1} b_j^{p-1} - 1 \right) \cdot \left( \prod_{i \neq j} \frac{b_i^{p-1} - 1}{b_i - 1} \right) \cdot \nabla^{d-1}(y)$$

$$= c_b(y)(b_j^k).$$

This proves the lemma.
Lemma 2.5.3. If $a$ and $b$ are two bases of $\Gamma_K$, then $c_a(y)$ and $c_b(y)$ are cohomologous.

Proof. If $\alpha_1, \ldots, \alpha_d$ and $\beta_1, \ldots, \beta_d$ are in $F^\times$, the Laurent series

$$\frac{\alpha_1 \cdots \alpha_d \cdot T^{d-1}}{(\exp(\alpha_1 T) - 1) \cdots (\exp(\alpha_d T) - 1)} - \frac{\beta_1 \cdots \beta_d \cdot T^{d-1}}{(\exp(\beta_1 T) - 1) \cdots (\exp(\beta_d T) - 1)}$$

is the difference of two Laurent series, each having a simple pole at 0 with equal residues, and therefore belongs to $F[T]$. Let $a$ and $b$ be two bases of $\Gamma_K$ and $y \in D^{e_0=1}$.

Let $N$ be a $\Gamma_K$-stable Fréchet subspace of $D$ that contains $y$ and write $N = \varprojlim M_j$. Since $M = M_j$ is $F$-analytic, we have $g = \exp(\ell(g)\nabla)$ on $M$ for $g$ in some open subgroup of $\Gamma_K$. Let $k \gg 0$ be large enough such that $a_i^k$ and $b_i^k$ are in this subgroup, and let $\alpha_i = p^k\ell(a_i)$ and $\beta_i = p^k\ell(b_i)$. Taking $k$ large enough (depending on $M$), we can assume moreover that the power series $T/(\exp(T) - 1)$ applied to the operators $\alpha_i\nabla$ and $\beta_i\nabla$ converges on $M$. The element

$$w = \left( \frac{\alpha_1 \cdots \alpha_d \cdot \nabla^{d-1}}{(\exp(\alpha_1 \nabla) - 1) \cdots (\exp(\alpha_d \nabla) - 1)} - \frac{\beta_1 \cdots \beta_d \cdot \nabla^{d-1}}{(\exp(\beta_1 \nabla) - 1) \cdots (\exp(\beta_d \nabla) - 1)} \right)(y)$$

of $M$ is well defined. By proposition 2.5.1, we have

$$c_{a^k}(y)'(1) - c_{b^k}(y)'(1) = (\Theta_{a^k}(y) - \Theta_{b^k}(y) = p^{-r(L)}\nabla(w)$$

where $L$ is the extension of $K$ such that $\Gamma_L = \Gamma_K^d$. Thus, for $g$ close enough to 1, we have $c_{a^k}(y)(g) - c_{b^k}(y)(g) = (g - 1)(p^{-r(L)}w)$. Lemma 2.5.2 now implies by corestricting that this holds for all $g$, and, by corestricting again, that $c_a(y)$ and $c_b(y)$ are cohomologous in $M$. By varying $M$, we get the same result in $N$, which implies the proposition.

Lemma 2.5.4. If $L/K$ is a finite extension contained in $K_\infty$, and if $b$ is a basis of $\Gamma_K$ and $a$ is a basis of $\Gamma_L$, then $\text{cor}_{L/K} c_a(y) = c_b(y)$.

Proof. The groups $\Gamma_K$ and $\Gamma_L$ are both free $\mathbb{Z}_p$-modules of rank $d$, so that by the elementary divisors theorem, we can change the bases $a$ and $b$ in such a way that there exists $e_1, \ldots, e_d$ with $a_i = b_i^{e_i}$. Since $\{b_1^{k_1} \cdots b_d^{k_d}\}$ with $0 \leq k_i \leq p^{e_i} - 1$ is a set of representatives for $\Gamma_K/\Gamma_L$, and since $[L : K] = p^{e_1 + \cdots + e_d},$ the explicit formula for the corestriction implies
Proof. This follows from the definition and from lemma 2.5.4 above.

Definition 2.5.5. Let \( h_{1,K,V} : D_{\text{rig}}^1(V)^{\varphi_q=1} \to H_{1,\text{an}}^1(K,V) \) denote the map obtained by composing \( y \mapsto \overline{c}_y(y) \) with \( H_{1,\text{an}}^1(\Gamma_K, D_{\text{rig}}^1(V)^{\varphi_q=1}) \to H_{1,\text{an}}^1(\Gamma_K \times \Psi, D_{\text{rig}}^1(V)) \) (theorem 2.2.4) and with \( H_{1,\text{an}}^1(\Gamma_K \times \Psi, D_{\text{rig}}^1(V)) \simeq H_{1,\text{an}}^1(K,V) \) (proposition 2.2.1 and corollary 2.2.3).

Proposition 2.5.6. We have \( \text{cor}_{M/L} h_{1,K,V} = h_{L,V}^1 \) if \( M/L \) is a finite extension contained in \( K_{\infty}/K_{n(K)} \). In particular, \( \text{cor}_{K_{n+1}/K_n} \circ h_{K_{n+1},V} = h_{K_n,V}^1 \) if \( n \geqslant n(K) \).

Proof. This follows from the definition and from lemma 2.5.4 above.

Remark 2.5.7. Proposition 2.5.6 allows us to extend the definition of \( h_{1,K,V} \) to all \( K \), without assuming that \( K \) contains \( F_{n(K)} \), by corestricting.

Some of the constructions of this section are summarized in the following theorem. Recall (see §3 of [Ber16]) that there is a ring \( B_{\text{rig}}^1 \) that contains \( B_{\text{rig},F}^1 \), is equipped with a Frobenius map \( \varphi_q \) and an action of \( G_F \) and such that \( V = (B_{\text{rig}}^1 \otimes B_{\text{rig},F}^1, D_{\text{rig}}^1(V))^{\varphi_q=1} \).

Theorem 2.5.8. If \( y \in D_{\text{rig}}^1(V)^{\varphi_q=1} \) and \( K \) contains \( K_{n(K)} \) and \( b \) is a basis of \( \Gamma_K \), then

1. there is a unique \( c_b(y) \in Z_{1,\text{an}}(\Gamma_K, D_{\text{rig}}^1(V)^{\varphi_q=1}) \) such that for \( k \in \mathbb{Z}_p \),

\[
c_b(y)(b_k^j) = \ell^*(b) \cdot \frac{b_j^k - 1}{b_j - 1} \cdot \frac{\nabla^{d-1}}{\prod_{i \neq j} (b_i - 1)}(y);
\]
2. there is a unique $m_p \in D_{rig}^\dagger(V)^\psi_1 = 0$ such that $(\varphi g - 1)c_b(y)(g) = (g - 1)m_p$ for all $g \in \Gamma_K$;

3. the $(\varphi, \Gamma)$-module corresponding to this extension has a basis in which

$$\text{Mat}(g) = \begin{pmatrix} * & c_b(y)(g) \\ 0 & 1 \end{pmatrix}$$

if $g \in \Gamma_K$, and

$$\text{Mat}(\varphi g) = \begin{pmatrix} * & m_p \\ 0 & 1 \end{pmatrix};$$

4. if $z \in \widehat{B}_{rig}^\dagger \otimes_F V$ is such that $(\varphi g - 1)z = m_p$, then the cocycle

$$g \mapsto c_b(y)(g) - (g - 1)z$$

defined on $G_K$ has values in $V$ and represents $h_{K,V}^1(y)$ in $H_{an}^1(K,V)$.

**Proof.** Items (1), (2) and (3) are reformulations of the constructions of this chapter. Let us prove (4). Let us write the $(\varphi, \Gamma)$-module corresponding to the extension in (3) as $D' = D_{rig}^\dagger(V) \oplus B_{rig,F}^\dagger \cdot e$. It is an étale $(\varphi, \Gamma)$-module that comes from the $p$-adic representation

$$V' = (\widehat{B}_{rig}^\dagger \otimes B_{rig,F}^\dagger)^{\varphi_1 = 1}.$$ We have $V' = V \oplus F \cdot (e - z)$ as $F$-vector spaces since $\varphi(e - z) = e - z$. If $g \in G_K$, then

$$g(e - z) = e + c_b(y)(g) - g(z) = e - z + c_b(y)(g) - (g - 1)z.$$ This proves (4). \qed

Let $F = \mathbb{Q}_p$ and $\pi = p = q$, and let $V$ be a representation of $G_K$. In §II.1 of [CC99], Cherbonnier and Colmez define a map $\text{Log}^\dagger_{V^*_1} : D(\varphi_1 = 1) \rightarrow H_{an}^1(K,V)$, which is an isomorphism (theorem II.1.3 and proposition III.3.2 of [CC99]).

**Proposition 2.5.9.** If $F = \mathbb{Q}_p$ and $\pi = p$, then the map

$$D(\varphi_1 = 1) \rightarrow D_{rig}^\dagger(V)^{\psi_1 = 1} \rightarrow \lim_{\rightarrow \leftarrow n} H_{an}^1(K_n,V) \rightarrow \lim_{\leftarrow n} H^1(K_n,V)$$

coincides with the map $\text{Log}^\dagger_{V^*_1} : D(\varphi_1 = 1) \rightarrow H_{w}^1(K,V) \subset \lim_{\leftarrow n} H^1(K_n,V)$. **Proof.** The map $\text{Log}^\dagger_{V^*_1}$ is constructed by mapping $x \in D(\varphi_1 = 1)$ to the sequence $(\ldots, \ell_{\psi,n}(x), \ldots) \in \lim_{\leftarrow n} H^1(K_n,V)$ (see theorem II.1.3 in [CC99] and the paragraph preceding it), where

$$\ell_{\psi,n}(x) = \left[ \sigma \mapsto \ell_{K_n}(\gamma_n) \left( \frac{\sigma - 1}{\gamma_n - 1} x - (\sigma - 1)b \right) \right]$$
on $G_{K_n}$ and where (see proposition I.4.1, lemma I.5.2 and lemma I.5.5 of ibid.)

1. $\gamma_n = \gamma_{[K_n : K_1]}$ and $\gamma_1$ is a fixed generator of $\Gamma_{K_1}$.

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2. \( \ell_{K_n}(\gamma_n) = \frac{\log \chi(\gamma_n)}{p^{\nu(K_n)}} \) where \( r(K_n) \) is the integer such that \( \log \chi(\Gamma_{K_n}) = p^{\nu(K_n)} \mathbb{Z}_p \).

3. \( b \in \mathcal{B}^1 \otimes_{\mathbb{Q}_p} V \) is such that \((\varphi - 1)b = a \) and \( a \in D^1(V)^{\psi = 1} \) is such that \((\gamma_n - 1)a = (\varphi - 1)x \) (using the fact that \( \gamma_n - 1 \) is bijective on \( D^1(V)^{\psi = 0} \)).

The theorem follows from comparing this with the explicit formula of theorem 2.5.8.

3 Explicit formulas for crystalline representations

In this chapter, we explain how the constructions of the previous chapter are related to \( p \)-adic Hodge theory, via Bloch and Kato’s exponential maps. Let \( B_{3R} \) be Fontaine’s ring of periods [Fon94] and let \( B_{max,F}^+ \) be the subring of \( B_{3R}^+ \) that is constructed in §8.5 of [Col02] (recall that \( B_{max,F} = F \otimes_{\mathbb{F}_p} B_{max} \) where \( F_0 = F \cap \mathbb{Q}_p \) and \( B_{max}^+ \) is a ring that is similar to Fontaine’s \( B_{cris} \)).

We assume throughout this chapter that \( K = F \) and that the representation \( V \) is crystalline and \( F \)-analytic.

3.1 Crystalline \( F \)-analytic representations

If \( V \) is an \( F \)-analytic crystalline representation of \( G_F \), let \( D_{cris}(V) = (B_{max,F}^+ \otimes_F V)^{G_F} \) (this is the “component at identity” of the usual \( D_{cris} \)).

By corollary 3.3.8 of [KR09], \( F \)-analytic crystalline representations of \( G_F \) are overconvergent. Moreover, if \( \mathcal{M}(D) \subset B^+_{rig,F}[1/t_z] \otimes_F D \) is the object constructed in §2.2 of ibid., then by §2.4 of ibid., \( \mathcal{M}(D_{cris}(V)) \) contains a basis of \( D^1(V) \) and \( D^1_{rig}(V) = B^+_{rig,F} \otimes B^+_{rig,F} \mathcal{M}(D_{cris}(V)) \). This implies that \( D^1_{rig}(V) \subset B^+_{rig,F}[1/t_z] \otimes_F D_{cris}(V) \).

**Theorem 3.1.1.** We have \( D^1_{rig}(V)^{\psi = 1} \subset B^+_{rig,F}[1/t_z] \otimes_F D_{cris}(V) \).

**Proof.** Take \( h \geq 0 \) such that the slopes of \( \pi^{-h}\varphi_q \) on \( D_{cris}(V) \) are \( \leq -d \). Let \( E \) be an extension of \( F \) such that \( E \) contains the eigenvalues of \( \varphi_q \) on \( D_{cris}(V) \). We show that \( D^1_{rig}(V)^{\psi = 1} \subset t_z^{-h}E \otimes_F B^+_{rig,F} \otimes_F D_{cris}(V) \).

Let \( e_1, \ldots, e_n \) be a basis of \( t_z^{-h}E \otimes_F D_{cris}(V) \) in which the matrix \( (p_{i,j}) \) of \( \varphi_q \) is upper triangular. If \( y = \sum_{i=1}^d y_i \otimes \varphi_q(e_i) \) with \( y_i \in E \otimes_F B^+_{rig,F} \), then \( \psi_q(y) = y \) if and only if \( \psi_q(y_k) = p_k y_k + \sum_{j > k} p_j y_j \) for all \( k \). The theorem follows from applying lemma 3.1.2 below to \( k = n, n-1, \ldots, 1 \).

**Lemma 3.1.2.** Take \( y \in E \otimes_F B^+_{rig,F} \) and \( \alpha \in F \) such that \( \text{val}_\pi(\alpha) \leq -d \). If \( \psi_q(y) - \alpha y \in E \otimes_F B^+_{rig,F} \), then \( y \in E \otimes_F B^+_{rig,F} \).

**Proof.** This is lemma 5.4 of [FX13].
3.2 Bloch-Kato’s exponentials for analytic representations

We now recall the definition of Bloch-Kato’s exponential map and its dual, and give a similar definition for $F$-analytic representations.

**Lemma 3.2.1.** We have an exact sequence

$$0 \to F \to \left( B_{max,F}^+[1/t]\right)^{\phi_{\max}^v = 1} \to B_{dR}/B_{dR}^+ \to 0.$$  

**Proof.** This is lemma 9.25 of [Col02].

If $V$ is a de Rham $F$-linear representation of $G_K$, we can $\otimes_F$ the above sequence with $V$ and we get a connecting homomorphism $\exp_{K,V} : (B_{dR} \otimes_F V)^{G_K} \to \mathbb{H}^1(K,V)$. Recall that if $W$ is an $F$-vector space, there is a natural injective map $W \otimes_F V \to W \otimes_{\mathbb{Q}_p} V$.

**Lemma 3.2.2.** If $V$ is $F$-analytic, the map $\exp_{K,V} : (B_{dR} \otimes_F V)^{G_K} \to \mathbb{H}^1(K,V)$ defined above coincides with Bloch-Kato’s exponential via the inclusion $(B_{dR} \otimes_F V)^{G_K} \subset (B_{dR} \otimes_{\mathbb{Q}_p} V)^{G_K}$, and its image is in $\mathbb{H}^1_{an}(K,V)$.

**Proof.** Bloch and Kato’s exponential is defined as follows (definition 3.10 of [BK90]): if $\phi$ denotes the Frobenius map that lifts $x \mapsto x^p$ and if $x \in (B_{dR} \otimes_{\mathbb{Q}_p} V)^{G_K}$, there exists $\tilde{x} \in B_{max,Q_p}^+ \otimes_{Q_p} V$ such that $\tilde{x} - x \in B_{dR}^+ \otimes_{Q_p} V$, and $\exp(x)$ is represented by the cocycle $g \mapsto (g - 1)\tilde{x}$.

Lemma 3.2.1 says that we can lift $x \in (B_{dR} \otimes_F V)^{G_K}$ to some $\tilde{x} \in (B_{max,F}^+[1/t\pi])^{\phi_{\max}^v = 1} \otimes_F V$ such that $\tilde{x} - x \in B_{dR}^+ \otimes_F V \subset B_{dR}^+ \otimes_{Q_p} V$. In addition, $B_{max,Q_p}^{\phi_{\max}^v = 1} = B_{dR}^+ \otimes_{Q_p} V$ (see lemma 1.1.11 of [Ber08]) so that $(B_{max,F}^+[1/t\pi])^{\phi_{\max}^v = 1} \subset F \otimes_{Q_p} B_{max,Q_p}^{\phi_{\max}^v = 1}$. We can therefore view $\tilde{x}$ as an element of $B_{max,Q_p}^+ \otimes_{Q_p} V$, and $\exp_{K,V}(x) = [g \mapsto (g - 1)\tilde{x}] = \exp(x)$.

The construction of $\exp_{K,V}(x)$ shows that the cocycle $\exp_{K,V}(x)$ is de Rham. At each embedding $\tau \neq \text{Id}$ of $F$, the extension of $F$ by $V$ given by $\exp_{K,V}(x)$ is therefore Hodge-Tate with weights 0. This finishes the proof of the lemma. 

Recall the following theorem of Kato (see §II.1 of [Kat93]).

**Theorem 3.2.3.** If $V$ is a de Rham representation, the map from $(B_{dR} \otimes_{Q_p} V)^{G_K}$ to $\mathbb{H}^1(K,B_{dR} \otimes_{Q_p} V)$ defined by $x \mapsto [g \mapsto \log(\chi_{cy}(\overline{g}))x]$ is an isomorphism, and the dual exponential map $\exp_{K,V}^{\ast\ast}(1) : H^1(K,V) \to (B_{dR} \otimes_{Q_p} V)^{G_K}$ is equal to the composition of the map $\mathbb{H}^1(K,V) \to H^1(K,B_{dR} \otimes_{Q_p} V)$ with the inverse of this isomorphism.

Concretely, if $c \in Z^1(K,B_{dR} \otimes_{Q_p} V)$ is some cocycle, there exists $w \in B_{dR} \otimes_{Q_p} V$ such that $c(g) = \log(\chi_{cy}(\overline{g})) \cdot \exp_{K,V}^{\ast\ast}(1)(c) + (g - 1)(w)$.

**Corollary 3.2.4.** If $c \in Z^1(K,B_{dR} \otimes_{F} V)$, and if there exist $x \in (B_{dR} \otimes_{F} V)^{G_K}$ and $w \in B_{dR} \otimes_{F} V$ such that $c(g) = f(\overline{g}) \cdot x + (g - 1)(w)$, then $\exp_{K,V}^{\ast\ast}(1)(c) = x$. 

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Proof. This follows from theorem 3.2.3 and from the fact that $g \mapsto \log(\chi(\mathfrak{f})/\chi_{\text{cyv}}(\mathfrak{f}))$ is $\mathcal{B}_{\text{dR}}$-admissible, since $t_{\pi}/t \in (\mathcal{B}_{\text{dR}}^+)^{\times}$ so that $\log(t_{\pi}/t) \in \mathcal{B}_{\text{dR}}^+$ is well-defined. ☐

3.3 Interpolating exponentials and their duals

Let $V$ be an $F$-analytic crystalline representation. By theorem 3.1.1, we have $D_{\text{rig}}(V)^{\psi_{\phi}=1} \subset \mathcal{B}_{\text{rig},F}^+[1/t_{\pi}] \otimes_F \mathcal{D}_{\text{cris}}(V)$. Let $\partial_V$ denote the map $\partial_D$ of §2.4 for $D = D_{\text{cris}}$.  

Theorem 3.3.1. If $y \in D_{\text{rig}}(V)^{\psi_{\phi}=1}$, then

$$\exp_{F_n,V}^{*}(1)(h_{F_n,V}(y)) = \begin{cases} q^{-n} \partial_V(\varphi_q^{-n}(y)) & \text{if } n \geq 1 \\ (1 - q^{-1}\varphi_q^{-1}) \partial_V(y) & \text{if } n = 0. \end{cases}$$

Proof. Since the diagram

$$\begin{array}{ccc}
H^1(F_{n+1}, V) & \xrightarrow{\exp_{F_{n+1},V}^{*}(1)} & F_{n+1} \otimes_F \mathcal{D}_{\text{cris}}(V) \\
\text{cor}_{F_{n+1}/F_n} & & \text{Tr}_{F_{n+1}/F_n} \\
H^1(F_n, V) & \xrightarrow{\exp_{F_n,V}^{*}(1)} & F_n \otimes_F \mathcal{D}_{\text{cris}}(V)
\end{array}$$

is commutative, we only need to prove the theorem when $n \geq n(F)$ by lemma 2.4.3 and proposition 2.5.6. By theorem 2.5.8, we have

$$h_{F_n,V}^{*}(1)(b_j^k) = \ell^{*}(b) \cdot \frac{b_j^k - 1}{b_j - 1} \cdot \frac{\nabla^{d-1}_{\partial_V(\varphi_q^{-m}(y))} - (b_j^k - 1)z}{\prod_{i \neq j} (b_i - 1)},$$

with $z \in \mathcal{B}_{\text{rig}}^+ \otimes_F V$ so that if $m \gg 0$, then $\varphi_q^{-m}(z) \in \mathcal{B}_{\text{dR}}^+ \otimes_F V$ (see §3 of [Ber16] and §2.2 of [Ber02]). Moreover, $\varphi_q^{-m}(y) \in F_m((t_{\pi})) \otimes_F \mathcal{D}_{\text{cris}}(V)$. Let $W = \{w \in F_m((t_{\pi})) \otimes_F \mathcal{D}_{\text{cris}}(V) \text{ such that } \partial_V(w) = 0\}$. The operator $\nabla$ is bijective on $W$, and $F_m((t_{\pi})) \otimes_F \mathcal{D}_{\text{cris}}(V)$ injects into $\mathcal{B}_{\text{dR}}^+ \otimes_F V$, hence there exists $u \in \mathcal{B}_{\text{dR}}^+ \otimes_F V$ such that

$$h_{F_n,V}^{*}(1)(b_j^k) = \ell^{*}(b) \cdot \frac{b_j^k - 1}{b_j - 1} \cdot \frac{\nabla^{d-1}_{\partial_V(\varphi_q^{-m}(y))} - (b_j^k - 1)u}{\prod_{i \neq j} (b_i - 1)} = \ell(b_j^k) \cdot \Theta(b)(\varphi_q^{-m}(y)) - (b_j^k - 1)u$$

by lemmas 2.4.1 and 2.4.3. This proves the theorem by corollary 3.2.4. ☐

We now give explicit formulas for $\exp_{F_n,V}$. Take $h \geq 0$ such that $\text{Fil}^{-h} \mathcal{D}_{\text{cris}}(V) = \mathcal{D}_{\text{cris}}(V)$, so that $t_{\pi}^h(\mathcal{B}_{\text{rig},F}^+ \otimes_F \mathcal{D}_{\text{cris}}(V)) \subset D_{\text{rig}}(V)$ (in the notation of §2.2 of [KR09], we have $t_{\pi}^h(\mathcal{B}_{\text{rig},F}^+ \otimes_F \mathcal{D}_{\text{cris}}(V)) \subset \mathcal{M}(\mathcal{D}_{\text{cris}}(V))$). In particular, if $y \in (\mathcal{B}_{\text{rig},F}^+ \otimes_F \mathcal{D}_{\text{cris}}(V))^{\psi_{\phi}=1}$, then $\nabla_{h-1} \circ \cdots \circ \nabla_0(y) \in D_{\text{rig}}(V)^{\psi_{\phi}=1}$. 

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Theorem 3.3.2. If $y \in (\mathcal{B}_{\text{rig}}^+ \otimes_F D_{\text{cris}}(V))^\psi=1$, then

$$h_{F_n,V}^1(\nabla_{h-1} \circ \cdots \circ \nabla_0(y)) =$$

$$(-1)^{h-1}(h-1)! \begin{cases} 
\exp_{F_n,V}(q^{-n} \partial_V((\varphi_q^{-n}(y)))) & \text{if } n \geq 1 \\
\exp_{F,V}((1-q^{-1} \varphi_q^{-1}) \partial_V(y)) & \text{if } n = 0
\end{cases}$$

Proof. Since the diagram

$$\begin{array}{ccc}
F_{n+1} \otimes_F D_{\text{cris}}(V) & \xrightarrow{\exp_{F_{n+1}/F}} & H^1(F_{n+1}, V) \\
\mathbf{Tr}_{F_{n+1}/F_n} & & \mathbf{cor}_{F_{n+1}/F_n} \\
F_n \otimes_F D_{\text{cris}}(V) & \xrightarrow{\exp_{F_n,V}} & H^1(F_n, V)
\end{array}$$

is commutative, we only need to prove the theorem when $n \geq n(F)$ by lemma 2.4.3 and proposition 2.5.6. By theorem 2.5.8, we have

$$h_{F_n,V}^1(\nabla_{h-1} \circ \cdots \circ \nabla_0(y))(b_j) =$$

$$= \ell^*(b) \frac{b_j - 1}{b_j - 1} \cdot \frac{\nabla_d \cdots (\nabla_{h-1} \circ \cdots \circ \nabla_0(y)) - (b_j - 1)z}{\prod_{i \neq j} (b_i - 1)}$$

so that $h_{F_n,V}^1(\nabla_{h-1} \circ \cdots \circ \nabla_0(y))(g) = (g-1)(\nabla_{h-1} \circ \cdots \circ \nabla_0 \circ \Theta_b(y)) = (g-1)z$.

Moreover, we have $z \in (\widetilde{\mathcal{B}}_{\text{rig}}^+ \otimes_F V)$ so that $m \gg 0$, then $\varphi_q^{-m}(z) \in (\mathcal{B}_{\text{rig}}^+ \otimes_F V)$. In addition, $\varphi_q^{-m}(y)$ belongs to $F_m[t_x] \otimes_F D_{\text{cris}}(V)$, so that $\varphi_q^{-m}(y) - \partial_V(\varphi_q^{-m}(y))$ belongs to $t_x F_m[t_x] \otimes_F D_{\text{cris}}(V)$ and therefore

$$(\nabla_{h-1} \circ \cdots \circ \nabla_0 \circ \Theta_b) \left( \varphi_q^{-m}(y) - \partial_V(\varphi_q^{-m}(y)) \right) \in t_x^2 F_m[t_x] \otimes_F D_{\text{cris}}(V) \subset \mathcal{B}_{\text{rig}}^+ \otimes_F V.$$
We can hence write
\[ h_{F_n,V}(\nabla_{n-1} \circ \nabla_1(y))(g) = (g-1)(\nabla_{n-1} \circ \nabla_1 \circ \Theta_b \circ \partial_V(f^{-m}(y))) - (g-1)u, \]
with \( u \in \mathcal{B}^+_{\text{rig}} \otimes_F V \). The theorem now follows from the fact that
\[ \Theta_b \circ \partial_V(f^{-m}(y)) = q^{-n} \partial_V(f^{-n}(y)) \in F_n \otimes_F \mathcal{D}_{\text{cris}}(V) \]
by lemmas 2.4.2 and 2.4.3, that \( \nabla_{n-1} \circ \cdots \circ \nabla_1 = (-1)^{h-1}(h-1)! \) on \( F_n \otimes_F \mathcal{D}_{\text{cris}}(V) \), and from the reminders given in §3.2, in particular the fact that \( \exp_{K,V} \) is the connecting homomorphism when tensoring the exact sequence of lemma 3.2.1 with \( V \) and taking Galois invariants.

### 3.4 Kummer theory and the representation \( F(\chi) \)

Throughout this section, \( V = F(\chi) \). Let \( L \subset \overline{\mathbb{Q}}_p \) be an extension of \( K \). The Kummer map \( \delta : \text{LT}(\mathfrak{m}_L) \to H^1(L,V) \) is defined as follows. Choose a generator \( u = (u_k)_{k \geq 0} \) of \( \text{LT}(\mathfrak{m}_L) = \lim_{\leftarrow} L/T \). If \( x \in \text{LT}(\mathfrak{m}_L) \), let \( x_k \in \text{LT}(\mathfrak{m}_{q^k}) \) be such that \( \pi^k(x_k) = x \). If \( g \in G_L \), then \( g(x_k) - x_k \in L/\pi^k \) so that we can write \( g(x_k) - x_k = [c_k(g)](u_k) \) for some \( c_k(g) \in \mathcal{O}_F/\pi^k \). If \( c(g) = (c_k(g))_{k \geq 0} \in \mathcal{O}_F \) then \( \delta(x) = [g \mapsto c(g)] \in H^1(L,V) \).

If \( x \in \text{LT}(\mathfrak{m}_L) \), and \( L/K \) is finite Galois, let \( \text{Tr}^{LT}_{L/K}(x) = \sum_{g \in \text{Gal}(L/K)} g(x) \) where the superscript \( \text{LT} \) means that the summation is carried out using the Lubin-Tate addition. If \( F = \mathbb{Q}_p \) and \( \text{LT} = \mathcal{G}_m \), we recover the classical Kummer map, and \( \text{Tr}^{LT}_{L/K}(x) = N_{L/K}(1+x) - 1 \).

**Lemma 3.4.1.** We have the following commutative diagram:

\[
\begin{array}{ccc}
\text{LT}(\mathfrak{m}_{K_{n+1}}) & \longrightarrow & H^1(K_{n+1},V) \\
\text{Tr}^{LT}_{K_{n+1}/K_n} & & \downarrow \text{cor}_{K_{n+1}/K_n} \\
\text{LT}(\mathfrak{m}_{K_n}) & \longrightarrow & H^1(K_n,V).
\end{array}
\]

**Proof.** This is a straightforward consequence of the explicit description of the corestriction map. \( \square \)

Recall that \( \varphi_q \circ \psi_q(f) = \frac{1}{q} \sum_{\omega \in \text{LT}[\pi]} f(T \cdot \omega) \), so that for \( n \geq 1 \):
\[
\psi_q(f)(u_n) = \frac{1}{q} \sum_{\omega \in \text{LT}[\pi]} f(u_{n+1} \otimes \omega) = \frac{1}{q} \text{Tr}^{F_{n+1}/F_n}(u_{n+1}).
\]

In particular, if \( f(T) \in \mathcal{B}^+_{\text{rig},F} \) is such that \( \psi_q(f(T)) = 1/\pi \cdot f(T) \) and \( y_n = f(u_n) \), then \( \text{Tr}^{F_{n+1}/F_n}(y_{n+1}) = q/\pi \cdot y_n \).

**Proposition 3.4.2.** Assume that \( F \neq \mathbb{Q}_p \). If \( \{y_n\}_{n \geq 1} \) is a sequence with \( y_n \in F_n \) and \( \text{Tr}^{F_{n+1}/F_n}(y_{n+1}) = q/\pi \cdot y_n \), there exists \( f(T) \in \mathcal{B}^+_{\text{rig},F} \) such that \( \psi_q(f(T)) = 1/\pi \cdot f(T) \) and \( y_n = f(u_n) \) for all \( n \geq 1 \).
Proof. By [Laz62], there exists a power series \( g(T) \in B_{\text{rig}, F}^+ \) such that \( g(u_n) = y_n \) for all \( n \geq 1 \). We also have
\[
\psi_q g(0) = \frac{1}{q} g(0) + \frac{1}{q} \text{Tr}_{F/F_n} g(u_1),
\]
and since \( q \neq \pi \) (because \( F \neq \mathbb{Q}_p \)), we can choose \( g(0) \) such that
\[
\frac{1}{\pi} g(0) = \frac{1}{q} g(0) + \frac{1}{q} \text{Tr}_{F/F_n} y_1.
\]
This implies that \( (\psi_q(g) - 1/\pi \cdot g)(u_n) = 0 \) for all \( n \geq 0 \), so that \( \psi_q(g) - 1/\pi \cdot g \in \ell_{\pi} B_{\text{rig}, F}^+ \). It is therefore enough to prove that \( \psi_q - 1/\pi : \ell_{\pi} B_{\text{rig}, F}^+ \to \ell_{\pi} B_{\text{rig}, F}^+ \) is onto. Since \( \psi_q(t_\pi f) = t_\pi \cdot t_\pi \psi_q(f) \), this amounts to proving that \( \psi_q - 1 : B_{\text{rig}, F}^+ \to B_{\text{rig}, F}^+ \) is onto, which follows from corollary 2.3.4.

**Definition 3.4.3.** Let \( S \) denote the set of sequences \( \{x_n\}_{n \geq 1} \) with \( x_n \in m_{F_n} \) and \( \text{Tr}_{F_{n+1}/F_n}(x_{n+1}) = [q/\pi](x_n) \) for \( n \geq 1 \).

The following proposition says that if \( F \neq \mathbb{Q}_p \), then \( S \) is quite large: for any \( k \geq 1 \), the “\( k \)-th component” map \( F \otimes_{\mathbb{Q}_p} S \to F_k \) is surjective (if \( F = \mathbb{Q}_p \), there are restrictions on “universal norms”).

**Proposition 3.4.4.** Assume that \( F \neq \mathbb{Q}_p \). If \( z \in m_{F_k} \), there exists \( \ell \geq 0 \) and \( x \in S \) such that \( x_\ell = [\pi^\ell](z) \).

**Proof.** We claim that \( \text{Tr}_{F_{n+1}/F_n}(\mathcal{O}_{F_{n+1}}) = \pi \mathcal{O}_{F_n} \). Indeed, let \( D \) denote the different. We have (see for instance proposition 7.11 of [Iwa86])
\[
\text{val}_p(D_{F_{n+1}/F_n}) = \frac{1}{e} \left( n + 1 - \frac{1}{q-1} \right) - \frac{1}{e} \left( n - \frac{1}{q-1} \right) = \text{val}_p(\pi).
\]
This implies that \( \text{Tr}_{F_{n+1}/F_n}(\mathcal{O}_{F_{n+1}}) = \pi \mathcal{O}_{F_n} \) by proposition 7 of Chapter III of [Ser68].

Since \( \pi \) divides \( q/\pi \), this shows that given \( y \in \mathcal{O}_{F_k} \), there exists a sequence \( \{y_n\}_{n \geq 1} \) with \( x_n \in \mathcal{O}_{F_n} \) such that \( y_n = y \), and \( \text{Tr}_{F_{n+1}/F_n}(y_{n+1}) = q/\pi \cdot y_n \) for \( n \geq 1 \). Take \( \ell_1, \ell_2 \geq 0 \) such that \( \pi^{\ell_1} \mathcal{O}_{F_{\ell_1}} \) is in the domain of \( \exp_{\text{LT}} \) and such that \( \pi^{\ell_2} \mathcal{O}_{\text{LT}}(z) \in \mathcal{O}_{F_{\ell_1}} \). Let \( y = \pi^{\ell_2} \mathcal{O}_{\text{LT}}(z) \). Let \( \{y_n\}_{n \geq 1} \) be a sequence as above, let \( x_n = \exp_{\text{LT}}(\pi^{\ell_1} y_n) \) and \( \ell = \ell_1 + \ell_2 \). The elements \( x_n \otimes [\pi^\ell](z) \), as well as \( \text{Tr}_{F_{n+1}/F_n}(x_{n+1}) \otimes [q/\pi](x_n) \) for all \( n \), have their \( \log_{\text{LT}} \) equal to zero and are in a domain in which \( \log_{\text{LT}} \) is injective. This proves the proposition. \( \Box \)

If \( x \in S \) and \( y_n = \log_{\text{LT}}(x_n) \), then \( y_n \in F_n \) and \( \text{Tr}_{F_{n+1}/F_n}(y_{n+1}) = q/\pi \cdot y_n \), so that by proposition 3.4.2, there exists \( f(T) \in B_{\text{rig}, F}^+ \) such that \( \psi_q(f(T)) = \pi^{-\ell} \cdot f(T) \) and \( y_n = f(u_n) \) for all \( n \geq 1 \). If \( f(T) \in B_{\text{rig}, F}^+ \) is such that \( \psi_q(f(T)) = \pi^{-\ell} \cdot f(T) \), then \( \partial f \in (B_{\text{rig}, F}^+)^{\psi_q=1} \) and \( \partial f \cdot u \) can be seen as an element of \( D_{\text{rig}}(V)^{\psi_q=1} \).
If \( x \in S \), and if \( f(T) \in B^{+}_{\text{rig}, F} \) is such that \( f(u_n) = \log_{LT}(x_n) \) and \( \psi_q(f(T)) = \pi^{-1} \cdot f(T) \), then \( h^{1}_{F_{n}, V}(\partial f(T) \cdot u) = (q/\pi)^{-n} \cdot \delta(x_n) \) for all \( n \geq 1 \).

**Proof.** Let \( y = f(T) \otimes t_{\pi}^{-1} u \), so that \( y \in (B^{+}_{\text{rig}, F} \otimes_{F} D_{\text{cris}}(V))^\psi_q=1 \). By theorem 3.3.2 applied to \( y \) with \( h = 1 \), we have \( h^{1}_{F_{n}, V}(\nabla(y)) = \exp_{F_{n}, V}(q^{-n} \partial_{V}(\varphi^{-n} q(y))) \) if \( n \geq 1 \). Since \( \varphi^{-n} \circ \partial = \pi^{n-1} \circ \varphi^{-n} \), this implies that

\[
\exp_{F_{n}, V}(q^{-n} \partial_{V}(\varphi^{-n} q(y))) = (q/\pi)^{-n} \cdot \exp_{F_{n}, V}(\log_{LT}(x_n) \cdot u).
\]

By example 3.10.1 of [BK90] and lemma 3.2.2, we have \( \delta(x_n) = \exp_{F_{n}, V}(\log_{LT}(x_n) \cdot u) \). This proves the theorem.

Remark 3.4.6. If \( F = \mathbb{Q}_{p} \) and \( \pi = q = p \) and \( x = \{x_n\}_{n \geq 1} \), this theorem says that \( \text{Exp}^*_{\mathbb{Q}_{p}}(\delta(x)) = \partial \log \text{Col}_{x}(T) \), which is (iii) of proposition V.3.2 of [CC99] (see theorem II.1.3 of ibid for the definition of the map \( \text{Exp}^*_{\mathbb{Q}_{p}} : H_{\text{cris}}^{1}(F, \mathbb{Q}_{p}(1)) \to D_{\text{rig}}(\mathbb{Q}_{p}(1))^{\psi_q=1} \)).

Remark 3.4.7. If \( x \in S \), then by proposition 3.4.2, there is a power series \( f \) such that \( f(u_n) = \log_{LT}(x_n) \) for \( n \geq 1 \). Is there a power series \( g(T) \in \mathcal{O}_{F}[T] \) such that \( g(u_n) = x_n \), so that \( f(T) = \log g(T)^2 \)? If \( F = \mathbb{Q}_{p} \), such a power series is the classical Coleman power series [Col79]. If \( F \neq \mathbb{Q}_{p} \) and \( x \in S \) and \( z \) is a \([q/\pi]-\text{torsion point} \), and \( n \geq d - 1 \) so that \( z \in F_{k} \), then the sequence \( x' = \{x'_n\}_{n \geq 1} \) defined by \( x'_n = x_n \) if \( n \neq k \) and \( x'_k = x_k \otimes z \) also belongs to \( S \). This means that we cannot naïvely interpolate \( x \).

### 3.5 Perrin-Riou’s big exponential map

In this last section, we explain how the explicit formulas of the previous sections can be used to give a Lubin-Tate analogue of Perrin-Riou’s “big exponential map” [PR94]. Take \( h \geq 1 \) such that \( \text{Fil}^{-h} D_{\text{cris}}(V) = D_{\text{cris}}(V) \). If \( f \in B^{+}_{\text{rig}, F} \otimes_{F} D_{\text{cris}}(V) \), let \( \Delta(f) \) be the image of \( \bigoplus_{h=0}^{h} \partial(f)^{0} \) in \( \bigoplus_{h=0}^{h} D_{\text{cris}}(V)/(1 - \pi^{h} \varphi_{q}) \).

**Lemma 3.5.1.** There is an exact sequence:

\[
0 \to \bigoplus_{k=0}^{h} t_{k} \Delta_{\text{cris}}(V)^{\psi_{q} = \pi^{-k}} \to B^{+}_{\text{rig}, F} \otimes_{F} D_{\text{cris}}(V)_{\psi_{q}=1} \xrightarrow{1 - \varphi_{q}} (B^{+}_{\text{rig}, F})_{\psi_{q}=0} \otimes_{F} D_{\text{cris}}(V) \Delta_{k=0}^{h} D_{\text{cris}}(V)_{1 - \pi^{k} \varphi_{q}} \to 0.
\]

**Proof.** Note that the map \( \varphi_{q} \) acts diagonally on tensor products. It is easy to see that ker\((1 - \varphi_{q}) = \bigoplus_{k=0}^{h} t_{k} D_{\text{cris}}(V)^{\psi_{q} = \pi^{-k}} \), that \( \Delta \) is surjective, and that \( \text{im}(1 - \varphi_{q}) \subset \text{ker} \Delta \), so we now prove that \( \text{im}(1 - \varphi_{q}) = \text{ker} \Delta \).

If \( f, g \in B^{+}_{\text{rig}, F} \otimes_{F} D_{\text{cris}}(V) \) and \( f = (1 - \varphi_{q}) g \), then \( \psi_{q}(f) = 0 \) if and only if \( \psi_{q}(g) = 0 \). It is therefore enough to show that if \( f \in B^{+}_{\text{rig}, F} \otimes_{F} D_{\text{cris}}(V) \) is such that \( \Delta(f) = 0 \), then \( f = (1 - \varphi_{q}) g \) for some \( g \in B^{+}_{\text{rig}, F} \otimes_{F} D_{\text{cris}}(V) \).
The map $1 - \varphi : T^{h+1}B_{\text{rig},F}^+ \otimes_F D_{\text{cris}}(V) \to T^{h+1}B_{\text{rig},F}^+ \otimes_F D_{\text{cris}}(V)$ is bijective because the slopes of $\varphi_q$ on $T^{h+1}B_{\text{rig},F}^+ \otimes_F D$ are $> 0$. This implies that $1 - \varphi_q$ induces a sequence

$$0 \to \oplus_{k=0}^h t_k D_{\text{cris}}(V)^{\varphi_q = \pi^{-k}} \to \frac{B_{\text{rig},F}^+ \otimes_F D_{\text{cris}}(V)}{T^{h+1}B_{\text{rig},F}^+ \otimes_F D_{\text{cris}}(V)} \overset{1 - \varphi_q}{\to} \frac{B_{\text{rig},F}^+ \otimes_F D_{\text{cris}}(V)}{T^{h+1}B_{\text{rig},F}^+ \otimes_F D_{\text{cris}}(V)} \overset{\Delta}{\to} \oplus_{k=0}^h D_{\text{cris}}(V)^{1 - \pi^k \varphi_q}$$

We have $\ker(1 - \varphi_q) = \oplus_{k=0}^h t_k D_{\text{cris}}(V)^{\varphi_q = \pi^{-k}}$ and by comparing dimensions, we see that $\coker(1 - \varphi_q) = \oplus_{k=0}^h D_{\text{cris}}(V)/(1 - \pi^k \varphi_q)$. This and the bijectivity of $1 - \varphi_q$ on $T^{h+1}B_{\text{rig},F}^+ \otimes_F D_{\text{cris}}(V)$ imply the claim. □

If $f \in ((B_{\text{rig},F}^+)_{\psi_q = 0} \otimes_F D_{\text{cris}}(V))^{\Delta = 0}$, then by lemma 3.5.1 there exists $y \in (B_{\text{rig},F}^+ \otimes_F D_{\text{cris}}(V))^{\psi_q = 1}$ such that $f = (1 - \varphi_q)y$. Since $\nabla_{h-1} \circ \cdots \circ \nabla_0$ kills $\oplus_{k=0}^h t_k D_{\text{cris}}(V)^{\varphi_q = \pi^{-k}}$ we see that $\nabla_{h-1} \circ \cdots \circ \nabla_0(y)$ does not depend upon the choice of such a $y$ (unless $D_{\text{cris}}(V)^{\varphi_q = \pi^{-n}} \neq 0$).

**Definition 3.5.2.** Let $h \geq 1$ be such that Fil$^{-h}D_{\text{cris}}(V) = D_{\text{cris}}(V)$ and such that $D_{\text{cris}}(V)^{\varphi_q = \pi^{-h}} = 0$. We deduce from the above construction a well-defined map:

$$\Omega_{V,h} : ((B_{\text{rig},F}^+)_{\psi_q = 0} \otimes_F D_{\text{cris}}(V))^{\Delta = 0} \to D_{\text{rig}}(V)^{\psi_q = 1},$$

given by $\Omega_{V,h}(f) = \nabla_{h-1} \circ \cdots \circ \nabla_0(y)$ where the element $y \in (B_{\text{rig},F}^+ \otimes_F D_{\text{cris}}(V))^{\psi_q = 1}$ is such that $f = (1 - \varphi_q)y$ and is provided by lemma 3.5.1.

If $D_{\text{cris}}(V)^{\varphi_q = \pi^{-h}} \neq 0$, we get a map

$$\Omega_{V,h} : ((B_{\text{rig},F}^+)_{\psi_q = 0} \otimes_F D_{\text{cris}}(V))^{\Delta = 0} \to D_{\text{rig}}(V)^{\psi_q = 1}/V^{G_F = \chi^h}.$$

Let $u$ be a basis of $F(\chi_\sigma)$ as above, and let $e_j = u^j$ if $j \in {\mathbb Z}$.

**Theorem 3.5.3.** Take $y \in (B_{\text{rig},F}^+ \otimes_F D_{\text{cris}}(V))^{\psi_q = 1}$ and let $h \geq 1$ be such that Fil$^{-h}D_{\text{cris}}(V) = D_{\text{cris}}(V)$. Let $f = (1 - \varphi_q)y$ so that $f \in ((B_{\text{rig},F}^+)_{\psi_q = 0} \otimes_F D_{\text{cris}}(V))^{\Delta = 0}$.

If $j \in {\mathbb Z}$ and $h + j \geq 1$, then

$$h_{F_{\omega},V(\chi^h)}^j(\Omega_{V,h}(f) \otimes e_j) = (-1)^{h+j-1}(h + j - 1)! x$$

$$\exp_{F_{\omega},V(\chi^h)}((q^{-n} \partial_{V(\chi^h)}(\varphi_q^{-n}(\partial^{-j} y \otimes t_\pi^{j} e_j)))) \quad \text{if } n \geq 1$$

$$\exp_{F,V(\chi^h)}((1 - q^{-1} \varphi_q^{-1}) \partial_{V(\chi^h)}(\partial^{-j} y \otimes t_\pi^{j} e_j)) \quad \text{if } n = 0.$$
If \( j \in \mathbb{Z} \) and \( h + j \leq 0 \), then

\[
\exp_{F_n,V}^*(1-j)(h^1_{F_n,V}(\chi_{j}^h) (\Omega_{V,h}(\ell) \otimes e_j)) = \frac{1}{(-h-j)!} \left( q^{-n} \partial_{V,\chi_{j}^h} (\varphi_{e_j}^{-n} (\partial^{-j} y \otimes t_{\pi^{-j}} e_j)) \right) \quad \text{if} \ n \geq 1
\]
\[
\left( 1 - q^{-1} \varphi_{e_j}^{-1} \right) \partial_{V,\chi_{j}^h} (\partial^{-j} y \otimes t_{\pi^{-j}} e_j) \quad \text{if} \ n = 0.
\]

**Proof.** If \( h + j \geq 1 \), the following diagram is commutative:

\[
\begin{array}{ccc}
\mathbb{B}^+_{\rig,F} \otimes_F D_{\cris}(V) & \xrightarrow{\partial_{j} \otimes \ell \otimes e_j} & \mathbb{B}^+_{\rig,F} \otimes_F D_{\cris}(V(\chi_{j}^h))
\end{array}
\]

and the theorem is a straightforward consequence of theorem 3.3.2 applied to \( \partial^{-j} y \otimes t^{-j} e_j \), \( h + j \) and \( V(\chi_{j}^h) \) (which are the \( j \)-th twists of \( y, h \) and \( V \)).

If \( h + j \leq 0 \), and \( \Gamma_{F_n} \) is torsion free, then theorem 3.3.1 shows that

\[
\exp_{F_n,V}^*(1-j)(h^1_{F_n,V}(\chi_{j}^h) (\nabla_{h-1} \circ \cdots \circ \nabla_0 (y) \otimes e_j)) = q^{-n} \partial_{V,\chi_{j}^h} (\varphi_{e_j}^{-n} (\nabla_{h-1} \circ \cdots \circ \nabla_0 (y) \otimes e_j))
\]

in \( D_{\cris}(V(\chi_{j}^h)) \), and a short computation involving Taylor series shows that

\[
\partial_{V,\chi_{j}^h} (\varphi_{e_j}^{-n} (\nabla_{h-1} \circ \cdots \circ \nabla_0 (y) \otimes e_j)) = (-h-j)!^{-1} \cdot \partial_{V,\chi_{j}^h} (\varphi_{e_j}^{-n} (\partial^{-j} y \otimes t_{\pi^{-j}} e_j)).
\]

To get the other \( n \), we corestrict. \( \square \)

**Corollary 3.5.4.** We have \( \Omega_{V,h}(x) \otimes e_j = \Omega_{V(h)}(\partial^{-j} x \otimes t_{\pi^{-j}} e_j) \) and \( \nabla_h \circ \Omega_{V,h}(x) = \Omega_{V,h+1}(x) \).

**Remark 3.5.5.** The notation \( \partial^{-j} \) is somewhat abusive if \( j \geq 1 \) as \( \partial \) is not injective on \( \mathbb{B}^+_{\rig,F} \) (it is surjective as can be seen by “integrating” directly a power series) but the reader can check that this leads to no ambiguity in the formulas of theorem 3.5.3 above.

If \( F = \mathbb{Q}_p \) and \( \pi = p \), definition 3.5.2 and theorem 3.5.3 are given in §II.5 of [Ber03]. They imply that \( \Omega_{V,h} \) coincides with Perrin-Riou’s exponential map (see theorem 3.2.3 of [PR94]) after making suitable identifications (theorem II.13 of [Ber03]).

Our definition therefore generalizes Perrin-Riou’s exponential map to the \( F \)-analytic setting. We hope to use the results of [Fou05] and [Fou08] to relate our constructions to suitable Iwasawa algebras as in the cyclotomic case.

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References


Iwahori Theory and Lubin-Tate $(\varphi, \Gamma)$-modules


