$p$-Adic Monodromy of the Universal Deformation of a HW-Cyclic Barsotti-Tate Group

Yichao Tian

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Abstract. Let $k$ be an algebraically closed field of characteristic $p > 0$, and $G$ be a Barsotti-Tate over $k$. We denote by $S$ the “algebraic” local moduli in characteristic $p$ of $G$, by $G$ the universal deformation of $G$ over $S$, and by $U \subseteq S$ the ordinary locus of $G$. The étale part of $G$ over $U$ gives rise to a monodromy representation $\rho_G$ of the fundamental group of $U$ on the Tate module of $G$. Motivated by a famous theorem of Igusa, we prove in this article that $\rho_G$ is surjective if $G$ is connected and HW-cyclic. This latter condition is equivalent to saying that Oort’s $a$-number of $G$ equals 1, and it is satisfied by all connected one-dimensional Barsotti-Tate groups over $k$.

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1. Introduction

1.1. A classical theorem of Igusa says that the monodromy representation associated with a versal family of ordinary elliptic curves in characteristic $p > 0$ is surjective [Igu, Ka2]. This important result has deep consequences in the theory of $p$-adic modular forms, and inspired various generalizations. Faltings and Chai [Ch2, FC] extended it to the universal family over the moduli space of higher dimensional principally polarized ordinary abelian varieties in characteristic $p$, and Ekedahl [Eke] generalized it to the jacobian of the universal $n$-pointed curve in characteristic $p$, equipped with a symplectic level structure. Recently, Chai and Oort [CO] proved the maximality of the $p$-adic monodromy over each “central leaf” in the moduli space of abelian varieties which is not contained in the supersingular locus. We refer to Deligne-Ribet [DR] and Hida [Hid] for other generalizations to some moduli spaces of PEL-type and their
arithmetic applications. Though it has been formulated in a global setting, the proof of Igusa's theorem is purely local, and it has got also local generalizations. Gross [Gro] generalized it to one-dimensional formal \( \Theta \)-modules over a complete discrete valuation ring of characteristic \( p \), where \( \Theta \) is the integral closure of \( \mathbb{Z}_p \) in a finite extension of \( \mathbb{Q}_p \). We refer to Chai [Ch2] and Achter-Norman [AN] for more results on local monodromy of Barsotti-Tate groups. Motivated by these results, it has been longly expected/conjectured that the monodromy of a \textit{versal} family of ordinary Barsotti-Tate groups in characteristic \( p > 0 \) is maximal. The aim of this paper is to prove the surjectivity of the monodromy representation associated with the universal deformation in characteristic \( p \) of a certain class of Barsotti-Tate groups.

1.2. To describe our main result, we introduce first the notion of HW-cyclic Barsotti-Tate groups. Let \( k \) be an algebraically closed field of characteristic \( p > 0 \), and \( G \) be a Barsotti-Tate group over \( k \). We denote by \( G^\vee \) the Serre dual of \( G \), and by \( \text{Lie}(G^\vee) \) its Lie algebra. The Frobenius homomorphism of \( G \) (or dually the Verschiebung of \( G^\vee \)) induces a semi-linear endomorphism \( \varphi_G \) on \( \text{Lie}(G^\vee) \), called the Hasse-Witt map of \( G \) (2.6.1). We say that \( G \) is HW-cyclic if \( c = \dim(\text{Lie}(G^\vee)) \geq 1 \) and there is a \( v \in \text{Lie}(G^\vee) \) such that \( v, \varphi_G(v), \ldots, \varphi_G^{c-1}(v) \) form a basis of \( \text{Lie}(G^\vee) \) over \( k \) (4.1). We prove in 4.7 that \( G \) is HW-cyclic and non-ordinary if and only if the \( \omega \)-number of \( G \), defined previously by Oort, equals 1. Basic examples of HW-cyclic Barsotti-Tate groups are given as follows. Let \( r, s \) be relatively prime integers such that \( 0 \leq s \leq r \) and \( r \neq 0 \), \( \lambda = s/r \), \( G^\lambda \) the Barsotti-Tate group over \( k \) whose (contravariant) Dieudonné module is generated by an element \( e \) over the non-commutative Dieudonné ring with the relation \((F^{r-s} - V^s) \cdot e = 0 \) (4.10). It is easy to see that \( G^\lambda \) is HW-cyclic for any \( 0 < \lambda < 1 \). Any connected Barsotti-Tate group over \( k \) of dimension 1 and height 0 is isomorphic to \( G^{1/0} \) [Dem, Chap.IV §8].

Let \( G \) be a Barsotti-Tate group of dimension \( d \) and height \( c + d \) over \( k \); assume \( c \geq 1 \). We denote by \( S \) the "algebraic" local moduli of \( G \) in characteristic \( p \), and by \( G \) the universal deformation of \( G \) over \( S \) (cf. 3.8). The scheme \( S \) is affine of ring \( R \simeq \mathbb{E}[[t_{ij}]]_{1 \leq i \leq c, 1 \leq j \leq d}] \), and the Barsotti-Tate group \( G \) is obtained by algebraizing the formal universal deformation of \( G \) over \( \text{Spf}(R) \) (3.7). Let \( U \) be the ordinary locus of \( G \) (i.e. the open subscheme of \( S \) parametrizing the ordinary fibers of \( G \)), and \( \mathfrak{f} \) a geometric point over the generic point of \( U \). For any integer \( n \geq 1 \), we denote by \( G(n) \) the kernel of the multiplication by \( p^n \) on \( G \), and by

\[
T_p(G, \mathfrak{f}) = \lim_{n \to \infty} G(n)(\mathfrak{f})
\]

the Tate module of \( G \) at \( \mathfrak{f} \). This is a free \( \mathbb{Z}_p \)-module of rank \( c \). We consider the monodromy representation attached to the \( \acute{e}tale \) part of \( G \) over \( U \)

\[
(1.2.1) \quad \rho_G : \pi_1(U, \mathfrak{f}) \to \text{Aut}_{\mathbb{Z}_p}(T_p(G, \mathfrak{f})) \simeq \text{GL}_c(\mathbb{Z}_p).
\]

The aim of this paper is to prove the following:

\[
\text{Documenta Mathematica 14 (2009) 397–440}
\]
Theorem 1.3. If $G$ is connected and HW-cyclic, then the monodromy representation $\rho_G$ is surjective.

Igusa's theorem mentioned above corresponds to Theorem 1.3 for $G = G^{1/2}$ (cf. 5.7). My interest in the $p$-adic monodromy problem started with the second part of my PhD thesis [Ti1], where I guessed 1.3 for $G = G^\lambda$ with $0 < \lambda < 1$ and proved it for $G^{1/2}$. After I posted the manuscript on ArXiv [Ti2], Strauch proved the one-dimensional case of 1.3 by using Drinfeld's level structures [Str, Theorem 2.1]. Later on, Lau [Lau] proved 1.3 without the assumption that $G$ is HW-cyclic. By using the Newton stratification of the universal deformation space of $G$ due to Oort, Lau reduced the higher dimensional case to the one-dimensional case treated by Strauch. In fact, Strauch and Lau considered more generally the monodromy representation over each $p$-rank stratum of the universal deformation space. In this paper, we provide first a different proof of the one-dimensional case of 1.3. Our approach is purely characteristic $p$, while Strauch used Drinfeld's level structure in characteristic 0. Then by following Lau's strategy, we give a new (and easier) argument to reduce the general case of 1.3 to the one-dimensional case for HW-cyclic groups. The essential part of our argument is a versality criterion by Hasse-Witt maps of deformations of a connected one-dimensional Barsotti-Tate group (Prop. 4.11). This criterion can be considered as a generalization of another theorem of Igusa which claims that the Hasse invariant of a versal family of elliptic curves in characteristic $p$ has simple zeros. Compared with Strauch's approach, our characteristic $p$ approach has the advantage of giving also results on the monodromy of Barsotti-Tate groups over a discrete valuation ring of characteristic $p$.

1.4. Let $A = k[[\pi]]$ be the ring of formal power series over $k$ in the variable $\pi$, $K$ its fraction field, and $v$ the valuation on $K$ normalized by $v(\pi) = 1$. We fix an algebraic closure $\overline{K}$ of $K$, and let $K^{\text{sep}}$ be the separable closure of $K$ contained in $\overline{K}$. $I$ be the Galois group of $K^{\text{sep}}$ over $K$, $I_p \subset I$ be the wild inertia subgroup, and $I_t = I/I_p$ the tame inertia group. For every integer $n \geq 1$, there is a canonical surjective character $\theta_p^{n-1} : I_t \to \mathbb{F}_p^\times$ (5.2), where $\mathbb{F}_p^\times$ is the finite subfield of $k$ with $p^n$ elements.

We put $S = \text{Spec}(A)$. Let $G$ be a Barsotti-Tate group over $S$, $G^\vee$ be its Serre dual, $\text{Lie}(G^\vee)$ the Lie algebra of $G^\vee$, and $\varphi_G$ the Hasse-Witt map of $G$, i.e. the semi-linear endomorphism of $\text{Lie}(G^\vee)$ induced by the Frobenius of $G$. We define $h(G)$ to be the valuation of the determinant of a matrix of $\varphi_G$, and call it the Hasse invariant of $G$ (5.4). We see easily that $h(G) = 0$ if and only if $G$ is ordinary over $S$, and $h(G) < \infty$ if and only if $G$ is generically ordinary. If $G$ is connected of height 2 and dimension 1, then $h(G) = 1$ is equivalent to that $G$ is versal (5.7).

Proposition 1.5. Let $S = \text{Spec}(A)$ be as above, $G$ be a connected HW-cyclic Barsotti-Tate group with Hasse invariant $h(G) = 1$, and $G(1)$ the kernel of the multiplication by $p$ on $G$. Then the action of $I$ on $G(1)(\overline{K})$ is tame; moreover,
$G(1)(\overline{K})$ is an $\mathbb{F}_p'$-vector space of dimension 1 on which the induced action of $I_t$ is given by the surjective character $\theta_{\mathbb{F}_p'} : I_t \to \mathbb{F}_p'$.

This proposition is an analog in characteristic $p$ of Serre’s result [Se3, Prop. 9] on the tameness of the monodromy associated with one-dimensional formal groups over a trait of mixed characteristic. We refer to 5.8 for the proof of this proposition and more results on the $p$-adic monodromy of HW-cyclic Barsotti-Tate groups over a trait in characteristic $p$.

1.6. This paper is organized as follows. In Section 2, we review some well known facts on ordinary Barsotti-Tate groups. Section 3 contains some preliminaries on the Dieudonné theory and the deformation theory of Barsotti-Tate groups. In Section 4, after establishing some basic properties of HW-cyclic groups, we give the fundamental relation between the versality of a Barsotti-Tate group and the coefficients of its Hasse-Witt matrix (Prop. 4.11). Section 5 is devoted to the study of the monodromy of a HW-cyclic Barsotti-Tate group over a complete trait of characteristic $p$. Section 6 is totally elementary, and contains a criterion (6.3) for the surjectivity of a homomorphism from a profinite group to $GL_n(\mathbb{Z}_p)$. Section 7 is the heart of this work, and it contains a proof of Theorem 1.3 in the one-dimensional case. Finally in Section 8, we follow Lau’s strategy and complete the proof of 1.3 by reducing the general case to the one-dimensional case treated in Section 7.

The proof in Section 7 of 1.3 in the one-dimensional case is based on an induction on the height $n+1 \geq 2$ of $G$. The case $n=1$ is just the classical Igusa’s theorem (5.7). For $n \geq 2$, by lemmas 6.3 and 6.5, it suffices to prove the following two statements: (a) the image of reduction modulo $p$ of $\rho_G$ contains a non-split Cartan subgroup; (b) under a suitable basis, the image of $\rho_G$ contains all matrix of the form \[
\begin{pmatrix}
B & b \\
0 & 1
\end{pmatrix}
\] with $B \in GL_{n-1}(\mathbb{Z}_p)$ and $b \in M_{(n-1) \times 1}(\mathbb{Z}_p)$.

The first statement follows easily from 1.5 by considering a certain base change of $G$ to a complete discrete valuation ring. To prove (b), we consider the formal completion $\text{Spec}(R')$ of the localization of the local moduli $S = \text{Spec}(R)$ of $G$ at the generic point of the locus where the universal deformation $G$ has $p$-rank $\leq 1$ (7.4). The ring $R'$ is a complete regular ring of dimension $n-1$, and the Barsotti-Tate group $\mathcal{G} = G \otimes_R R'$ has a connected part of height $n$ and an étale part of height 1. Let $K_0$ be the residue field of $R'$, and $\overline{K}_0$ an algebraic closure of $K_0$. In order to apply the induction hypothesis, we consider the set of $k$-algebra homomorphisms $\sigma : R' \to \overline{R'} = \overline{K}_0[[t_1, \ldots, t_{n-1}]]$ lifting the natural inclusion $K_0 \to \overline{K}_0$. The key point is that, the natural map $\sigma \mapsto \mathcal{G}_{R', \sigma} = \mathcal{G} \otimes_{R', \sigma} \overline{R'}$ gives a bijection between the set of such $\sigma$’s and the set of deformations of $\mathcal{G}_{R_0} = \mathcal{G} \otimes_R K_0$ to $\overline{R'}$; moreover, we can compute explicitly the Hasse-Witt map of the connected component $\mathcal{G}_{R', \sigma}$ of $\mathcal{G}_{R', \sigma}$ (Lemma 7.8).

From the versality criterion for one-dimensional Barsotti-Tate groups in terms of the Hasse-Witt map established in Section 4 (Prop. 4.11), it follows immediately that there exists a $\sigma$ such that the Barsotti-Tate group $\mathcal{G}_{R', \sigma}'$, which
is connected and one-dimensional of height $n$, is the universal deformation of its closed fiber. We fix such a $\sigma$. Then the set of all $\sigma'$ with $\mathcal{G}^p_{R,\sigma'} \simeq \mathcal{G}^p_{R,\sigma}$ as deformations of their common closed fiber is actually a group isomorphic to $\text{Ext}^1_{R}(\mathbb{Q}_p/\mathbb{Z}_p,\mathcal{G}^p_{R,\sigma})$ (Prop. 3.10). Let $\sigma_1$ be the element corresponding to neutral element in $\text{Ext}^1_{R}(\mathbb{Q}_p/\mathbb{Z}_p,\mathcal{G}^p_{R,\sigma})$. Applying the induction hypothesis to $\mathcal{G}^p_{R,\sigma_1}$, we see that the monodromy group of $\mathcal{G}^p_{R,\sigma_1}$, hence that of $\mathcal{G}$, contains the subgroup \[ \begin{pmatrix} \text{GL}_{n-1}(\mathbb{Z}_p) & 0 \\ 0 & 1 \end{pmatrix} \] under a suitable basis of the Tate module (7.5.3). In order to conclude the proof, we need another $\sigma_2$ such that $\mathcal{G}^p_{R,\sigma_2}$ has the same connected component as $\mathcal{G}^p_{R,\sigma_1}$, and that the induced extension between the Tate module of the étale part of $\mathcal{G}^p_{R,\sigma_2}$ and that of $\mathcal{G}^p_{R,\sigma_2}$ is non-trivial after reduction modulo $p$ (see 7.5 and 7.5.4). To verify the existence of such a $\sigma_2$, we reduce the problem to a similar situation over a complete trait of characteristic $p$ (see 7.9), and we use a criterion of non-triviality of extensions by Hasse-Witt maps (5.12).

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1.8. Notations. Let $S$ be a scheme of characteristic $p > 0$. A BT-group over $S$ stands for a Barsotti-Tate group over $S$. Let $G$ be a commutative finite group scheme (resp. a BT-group) over $S$. We denote by $G^\vee$ its Cartier dual (resp. its Serre dual), by $\omega_G$ the sheaf of invariant differentials of $G$ over $S$, and by $\text{Lie}(G)$ the sheaf of Lie algebras of $G$. If $S = \text{Spec}(A)$ is affine and there is no risk of confusions, we also use $\omega_G$ and $\text{Lie}(G)$ to denote the corresponding $A$-modules of global sections. We put $G^{(p)}$ the pull-back of $G$ by the absolute Frobenius of $S$, $F_G : G \to G^{(p)}$ the Frobenius homomorphism and $V_G : G^{(p)} \to G$ the Verschiebung homomorphism. If $G$ is a BT-group and $n$ an integer $\geq 1$, we denote by $G(n)$ the kernel of the multiplication by $p^n$ on $G$; we have $G^{\vee}(n) = (G^{\vee})(n)$ by definition. For an $\mathcal{O}_S$-module $M$, we denote by $M^{(p)} = \mathcal{O}_S \otimes_{\mathcal{O}_S} M$ the scalar extension of $M$ by the absolute Frobenius of $\mathcal{O}_S$. If $\varphi : M \to N$ be a semi-linear homomorphism of $\mathcal{O}_S$-modules, we denote by $\widehat{\varphi} : M^{(p)} \to N$ the linearization of $\varphi$, i.e., we have $\widehat{\varphi}(\lambda \otimes x) = \lambda \cdot \varphi(x)$, where $\lambda$ (resp. $x$) is a local section of $\mathcal{O}_S$ (resp. of $M$).

Starting from Section 5, $k$ will denote an algebraically closed field of characteristic $p > 0$.

2. Review of ordinary Barsotti-Tate groups

In this section, $S$ denotes a scheme of characteristic $p > 0$. 

Documenta Mathematica 14 (2009) 397–440
2.1. Let \( G \) be a commutative group scheme, locally free of finite type over \( S \). We have a canonical isomorphism of coherent \( \mathcal{O}_S \)-modules [Ill, 2.1]

\[
\text{Lie}(G^\vee) \simeq \mathcal{Homs}_{\mathcal{O}_S}(G, \mathbb{G}_a),
\]

where \( \mathcal{Homs}_{\mathcal{O}_S} \) is the sheaf of homomorphisms in the category of abelian fppf-sheaves over \( S \), and \( \mathbb{G}_a \) is the additive group scheme. Since \( \mathbb{G}_a^{(p)} \simeq \mathbb{G}_a \), the Frobenius homomorphism of \( \mathbb{G}_a \) induces an endomorphism

\[
\varphi_G : \text{Lie}(G^\vee) \to \text{Lie}(G^\vee),
\]

semi-linear with respect to the absolute Frobenius map \( F_S : \mathcal{O}_S \to \mathcal{O}_S \); we call it the Hasse-Witt map of \( G \). By the functoriality of Frobenius, \( \varphi_G \) is also the canonical map induced by the Frobenius of \( G \), or dually by the Verschiebung of \( G^\vee \).

2.2. By a commutative \( p \)-Lie algebra over \( S \), we mean a pair \( (L, \varphi) \), where \( L \) is an \( \mathcal{O}_S \)-module locally free of finite type, and \( \varphi : L \to L \) is a semi-linear endomorphism with respect to the absolute Frobenius \( F_S : \mathcal{O}_S \to \mathcal{O}_S \). When there is no risk of confusions, we omit \( \varphi \) from the notation. We denote by \( p\text{-Lie}_S \) the category of commutative \( p \)-Lie algebras over \( S \).

Let \( (L, \varphi) \) be an object of \( p\text{-Lie}_S \). We denote by

\[
\mathcal{U}(L) = \text{Sym}(L) = \oplus_{n \geq 0} \text{Sym}^n(L),
\]

the symmetric algebra of \( L \) over \( \mathcal{O}_S \). Let \( \mathcal{J}_p(L) \) be the ideal sheaf of \( \mathcal{U}(L) \) defined, for an open subset \( V \subset S \), by

\[
\Gamma(V, \mathcal{J}_p(L)) = \{ x^{(p)} - \varphi(x) : x \in \Gamma(V, \mathcal{U}(L)) \},
\]

where \( x^{(p)} = x \otimes x \otimes \cdots \otimes x \in \Gamma(V, \text{Sym}^p(L)) \). We put \( \mathcal{U}_p(L) = \mathcal{U}(L)/\mathcal{J}_p(L) \), and call it the \emph{\( p \)-enveloping algebra} of \( (L, \varphi) \). We endow \( \mathcal{U}_p(L) \) with the structure of a Hopf-algebra with the comultiplication given by \( \Delta(x) = 1 \otimes x + x \otimes 1 \) and the counverse given by \( i(x) = -x \).

Let \( G \) be a commutative group scheme, locally free of finite type over \( S \). We say that \( G \) is of \emph{coheight one} if the Verschiebung \( V_G : G^{(p)} \to G \) is the zero homomorphism. We denote by \( \mathfrak{G} V_S \) the category of such objects. For an object \( G \) of \( \mathfrak{G} V_S \), the Frobenius \( F_{G^\vee} \) of \( G^\vee \) is zero, so the Lie algebra \( \text{Lie}(G^\vee) \) is locally free of finite type over \( \mathcal{O}_S \) ([DG] VII.A Théo. 7.4(iii)). The Hasse-Witt map of \( G \) (2.1.2) endows \( \text{Lie}(G^\vee) \) with a commutative \( p \)-Lie algebra structure over \( S \).

**Proposition 2.3** ([DG] VII.A, Théo. 7.2 et 7.4). The functor \( \mathfrak{G} V_S \to p\text{-Lie}_S \) defined by \( G \mapsto \text{Lie}(G^\vee) \) is an anti-equivalence of categories; a quasi-inverse is given by \( (L, \varphi) \mapsto \text{Spec}(\mathcal{U}_p(L)) \).

2.4. Assume \( S = \text{Spec}(A) \) affine. Let \( (L, \varphi) \) be an object of \( p\text{-Lie}_S \) such that \( L \) is free of rank \( n \) over \( \mathcal{O}_S \), \( (e_1, \cdots, e_n) \) be a basis of \( L \) over \( \mathcal{O}_S \), \( (h_{ij})_{1 \leq i,j \leq n} \) be the matrix of \( \varphi \) under the basis \( (e_1, \cdots, e_n) \), i.e. \( \varphi(e_j) = \sum_{i=1}^n h_{ij}e_i \) for
1 \leq j \leq n. Then the group scheme attached to \((L, \varphi)\) is explicitly given by

\[ \text{Spec}(\mathcal{U}_p(L)) = \text{Spec} \left( \mathbb{A}(X_1, \cdots, X_n)/(X_p^{N} - \sum_{i=1}^{n} h_{ij}X_j)_{1 \leq j \leq n} \right), \]

with the commultiplication \(\Delta(X_j) = 1 \otimes X_j + X_j \otimes 1\). By the Jacobian criterion of étaleness [EGA, IV_0 22.6.7], the finite group scheme \(\text{Spec}(\mathcal{U}_p(L))\) is étale over \(S\) if and only if the matrix \((h_{ij})_{1 \leq i, j \leq n}\) is invertible. This condition is equivalent to that the linearization of \(\varphi\) is an isomorphism.

**Corollary 2.5.** An object \(G\) of \(\mathfrak{G}V_S\) is étale over \(S\), if and only if the linearization of its Hasse-Witt map (2.1.2) is an isomorphism.

**Proof.** The problem being local over \(S\), we may assume \(S\) affine and \(L = \text{Lie}(G^\vee)\) free over \(\mathcal{O}_S\). By Theorem 2.3, \(G\) is isomorphic to \(\text{Spec}(\mathcal{U}_p(L))\), and we conclude by the last remark of 2.4. \(\square\)

2.6. Let \(G\) be a \(\mathbb{B}T\)-group over \(S\) of height \(c + d\) and dimension \(d\). The Lie algebra \(\text{Lie}(G^\vee)\) is an \(\mathcal{O}_S\)-module locally free of rank \(c\), and canonically identified with \(\text{Lie}(G^\vee(1))(\mathbf{BBM} 3.3.2)\). We define the Hasse-Witt map of \(G\)

\[
(2.6.1) \quad \varphi_G : \text{Lie}(G^\vee) \to \text{Lie}(G^\vee)
\]

to be that of \(G(1)\) (2.1.2).

2.7. Let \(k\) be a field of characteristic \(p > 0\), \(G\) be a \(\mathbb{B}T\)-group over \(k\). Recall that we have a canonical exact sequence of \(\mathbb{B}T\)-groups over \(k\)

\[
(2.7.1) \quad 0 \to G^\circ \to G \to G^\text{ét} \to 0
\]

with \(G^\circ\) connected and \(G^\text{ét}\) étale ([Dem] Chap.II, § 7). This induces an exact sequence of Lie algebras

\[
(2.7.2) \quad 0 \to \text{Lie}(G^\text{ét\text{\text{}v}}) \to \text{Lie}(G^\vee) \to \text{Lie}(G^\text{ét\text{\text{}v}}) \to 0,
\]

compatible with Hasse-Witt maps.

**Proposition 2.8.** Let \(k\) be a field of characteristic \(p > 0\), \(G\) be a \(\mathbb{B}T\)-group over \(k\). Then \(\text{Lie}(G^\text{ét\text{\text{}v}})\) is the unique maximal \(k\)-subspace \(V\) of \(\text{Lie}(G^\vee)\) with the following properties:

(a) \(V\) is stable under \(\varphi_G\);

(b) the restriction of \(\varphi_G\) to \(V\) is injective.

**Proof.** It is clear that \(\text{Lie}(G^\text{ét\text{\text{}v}})\) satisfies property (a). We note that the Verschiebung of \(G^\text{ét}(1)\) vanishes; so \(G^\text{ét}(1)\) is in the category \(\mathfrak{G}V_{\text{Spec}(k)}\). Since \(k\) is a field, 2.5 implies that the restriction of \(\varphi_G\) to \(\text{Lie}(G^\text{ét\text{\text{}v}})\), which coincides with \(\varphi_{G^{	ext{ét}}_g}\), is injective. This proves that \(\text{Lie}(G^\text{ét\text{\text{}v}})\) verifies (b). Conversely, let \(V\) be an arbitrary \(k\)-subspace of \(\text{Lie}(G^\vee)\) with properties (a) and (b). We have to show that \(V \subset \text{Lie}(G^\text{ét\text{\text{}v}})\). Let \(\sigma\) be the Frobenius endomorphism of \(k\). If \(M\) is a \(k\)-vector space, for each integer \(n \geq 1\), we put \(M^{(n)} = k \otimes_{\sigma^n} M\), i.e. we have \(1 \otimes ax = \sigma^n(a) \otimes x\) in \(k \otimes_{\sigma^n} M\) for \(a \in k, x \in M\). Since \(\varphi_G|_{V} : V \to V\) is injective by assumption, the linearization \(\varphi_G|_{V^{(n)}} : V^{(n)} \to V\) of \(\varphi_G|_{V}\)
is injective (hence bijective) for any \( n \geq 1 \). We have \( V = \bar{\varphi}_G^o(V(p^n)) \). Since \( G^o \) is connected, there is an integer \( n \geq 1 \) such that the \( n \)-th iterated Frobenius \( F_{G^o(1)}^n : G^o(1) \to G^o(1)(p^n) \) vanishes. Hence by definition, the linearized \( n \)-iterated Hasse-Witt map \( \bar{\varphi}_G^o : \text{Lie}(G^o(1)(p^n)) \to \text{Lie}(G^o(1)) \) is zero. By the compatibility of Hasse-Witt maps, we have \( \bar{\varphi}_G^o(\text{Lie}(G^o(1)(p^n))) \subset \text{Lie}(G^{\text{ét}}(1)) \); in particular, we have \( V = \bar{\varphi}_G^o(V(p^n)) \subset \text{Lie}(G^{\text{ét}}) \). This completes the proof. \( \square \)

**Corollary 2.9.** Let \( k \) be a field of characteristic \( p > 0 \), \( G \) be a BT-group over \( k \). Then \( G \) is connected if and only if \( \varphi_G \) is nilpotent.

**Proof.** In the proof of the proposition, we have seen that the Hasse-Witt map of the connected part of \( G \) is nilpotent. So the “only if” part is verified. Conversely, if \( \varphi_G \) is nilpotent, \( \text{Lie}(G^{\text{ét}}) \) is zero by the proposition. Therefore \( G \) is connected. \( \square \)

**Definition 2.10.** Let \( S \) be a scheme of characteristic \( p > 0 \), \( G \) be a BT-group over \( S \). We say that \( G \) is *ordinary* if there exists an exact sequence of BT-groups over \( S \)

\[
0 \to G^{\text{mult}} \to G \to G^{\text{ét}} \to 0,
\]

such that \( G^{\text{mult}} \) is multiplicative and \( G^{\text{ét}} \) is étale.

We note that when it exists, the exact sequence (2.10.1) is unique up to a unique isomorphism, because there is no non-trivial homomorphisms between a multiplicative BT-group and an étale one in characteristic \( p > 0 \). The property of being ordinary is clearly stable under arbitrary base change and Serre duality. If \( S \) is the spectrum of a field of characteristic \( p > 0 \), \( G \) is ordinary if and only if its connected part \( G^o \) is of multiplicative type.

**Proposition 2.11.** Let \( G \) be a BT-group over \( S \). The following conditions are equivalent:

(a) \( G \) is ordinary over \( S \).

(b) For every \( x \in S \), the fiber \( G_x = G \otimes_S k(x) \) is ordinary over \( k(x) \).

(c) The finite group scheme \( \text{Ker} V_G \) is étale over \( S \).

(c') The finite group scheme \( \text{Ker} F_G \) is of multiplicative type over \( S \).

(d) The linearization of the Hasse-Witt map \( \varphi_G \) is an isomorphism.

First, we prove the following lemmas.

**Lemma 2.12.** Let \( T \) be a scheme, \( H \) be a commutative group scheme locally free of finite type over \( T \). Then \( H \) is étale (resp. of multiplicative type) over \( T \) if and only if, for every \( x \in T \), the fiber \( H \otimes_T k(x) \) is étale (resp. of multiplicative type) over \( k(x) \).

**Proof.** We will consider only the étale case; the multiplicative case follows by duality. Since \( H \) is \( T \)-flat, it is étale over \( T \) if and only if it is unramified over \( T \). By [EGA, IV 17.4.2], this condition is equivalent to that \( H \otimes_T k(x) \) is unramified over \( k(x) \) for every point \( x \in T \). Hence the conclusion follows. \( \square \)
Lemma 2.13. Let $G$ be a $BT$-group over $S$. Then $\text{Ker } V_G$ is an object of the category $\mathfrak{S} V_S$, i.e. it is locally free of finite type over $S$, and its Verschiebung is zero. Moreover, we have a canonical isomorphism $(\text{Ker } V_G)^{\vee} \simeq \text{Ker } F_{G^\vee}$, which induces an isomorphism of Lie algebras $\text{Lie}(\text{Ker } V_G)^{\vee} \simeq \text{Lie}(\text{Ker } F_{G^\vee}) = \text{Lie}(G^\vee)$, and the Hasse-Witt map (2.1.2) of $\text{Ker } V_G$ is identified with $\varphi_G$ (2.6.1).

Proof. The group scheme $\text{Ker } V_G$ is locally free of finite type over $S$ ([III] 1.3(b)), and we have a commutative diagram

$$
\begin{array}{ccc}
(\text{Ker } V_G)^{(p)} & \xrightarrow{V_{\text{Ker } V_G}} & \text{Ker } V_G \\
\downarrow & & \downarrow \\
(G^{(p)})^{(p)} & \xrightarrow{V_{G^{(p)}}} & G^{(p)}
\end{array}
$$

By the functoriality of Verschiebung, we have $V_{G^{(p)}} = (V_G)^{(p)}$ and $V_{(\text{Ker } V_G)^{(p)}} = (\text{Ker } V_G)^{(p)}$. Hence the composition of the left vertical arrow with $V_{G^{(p)}}$ vanishes, and the Verschiebung of $\text{Ker } V_G$ is zero.

By Cartier duality, we have $(\text{Ker } V_G)^{\vee} = \text{Coker}(F_{G^\vee(1)})$. Moreover, the exact sequence

$$
\cdots \to G^\vee(1) \xrightarrow{F_{G^\vee(1)}} (G^\vee(1))^{(p)} \xrightarrow{V_{G^\vee(1)}} G^\vee(1) \to \cdots ,
$$

induces a canonical isomorphism

$$
(\text{Ker } V_G)^{\vee} \simeq \text{Coker}(F_{G^\vee(1)}) \simeq \text{Im}(V_{G^\vee(1)}) = \text{Ker } F_{G^\vee(1)} = \text{Ker } F_{G^\vee}.
$$

Hence, we deduce that

$$
(\text{Ker } V_G)^{\vee} \simeq \text{Coker}(F_{G^\vee(1)}) \simeq \text{Ker } F_{G^\vee} \hookrightarrow G^\vee(1).
$$

Since the natural injection $\text{Ker } F_{G^\vee} \to G^\vee(1)$ induces an isomorphism of Lie algebras, we get

$$
\text{Lie}(\text{Ker } V_G)^{\vee} \simeq \text{Lie}(\text{Ker } F_{G^\vee}) = \text{Lie}(G^\vee(1)) = \text{Lie}(G^\vee).
$$

It remains to prove the compatibility of the Hasse-Witt maps with (2.13.3). We note that the dual of the morphism (2.13.2) is the canonical map $F : G(1) \to \text{Ker } V_G = \text{Im}(F_{G(1)})$ induced by $F_{G(1)}$. Hence by (2.1.1), the isomorphism (2.13.3) is identified with the functorial map

$$
\mathcal{H}om_{\text{S_{bad}}}(\text{Ker } V_G, \mathbb{G}_a) \to \mathcal{H}om_{\text{S_{bad}}}(G(1), \mathbb{G}_a)
$$

induced by $F$, and its compatibility with the Hasse-Witt maps follows easily from the definition (2.1.2). \hfill $\square$

Proof of 2.11. (a)$\Rightarrow$(b). Indeed, the ordinarity of $G$ is stable by base change. (b)$\Rightarrow$(c). By Lemma 2.12, it suffices to verify that for every point $x \in S$, the fiber $(\text{Ker } V_G) \otimes_S \kappa(x)$ is $\text{Ker } V_{G_x}$ is flat over $\kappa(x)$. Since $G_x$ is assumed to be ordinary, its connected part $(G_x)^{\circ}$ is multiplicative. Hence, the Verschiebung of
$(G_x)^\circ$ is an isomorphism, and Ker $V_G$ is canonically isomorphic to Ker \( V_G^{(p)} \subseteq (G_x^\circ)^{(p)} \simeq (G_x)^{d\circ}$, so our assertion follows.

(c) $\Leftrightarrow$ (d). It follows immediately from Lemma 2.13 and Corollary 2.5.

(c) $\Leftrightarrow$ (c'). By 2.12, we may assume that $S$ is the spectrum of a field. So the category of commutative finite group schemes over $S$ is abelian. We will just prove (c) $\Rightarrow$ (c'); the converse can be proved by duality. We have a fundamental short exact sequence of finite group schemes

\[(2.13.4) \quad 0 \to \text{Ker } F_G \to G(1) \to \text{Ker } V_G \to 0,\]

where $F$ is induced by $F_{G(1)}$. That induces a commutative diagram

\[
\begin{array}{c}
0 \quad \longrightarrow \quad (\text{Ker } F_G)^{(p)} \quad \longrightarrow \quad (G(1))^{(p)} \quad \longrightarrow \quad (\text{Ker } V_G)^{(p)} \quad \longrightarrow \quad 0 \\
\downarrow V' \quad \quad \quad \downarrow V_{G(1)} \quad \quad \quad \downarrow V'' \quad \quad \quad \downarrow 0 \\
0 \quad \longrightarrow \quad \text{Ker } F_G \quad \longrightarrow \quad G(1) \quad \longrightarrow \quad F \quad \longrightarrow \quad \text{Ker } V_G \quad \longrightarrow \quad 0
\end{array}
\]

where vertical arrows are the Verschiebung homomorphisms. We have seen that $V'' = 0$ (2.13). Therefore, by the snake lemma, we have a long exact sequence

\[(2.13.5) \quad 0 \to \text{Ker } V' \to \text{Ker } V_{G(1)} \overset{\alpha}{\longrightarrow} (\text{Ker } V_G)^{(p)} \to \to Coker V' \to \text{Coker } V_{G(1)} \overset{\beta}{\longrightarrow} \text{Ker } V_G \to 0,\]

where the map $\alpha$ is the Frobenius of $\text{Ker } V_G$ and $\beta$ is the composed isomorphism $Coker(V_{G(1)}) \simeq G(1)/\text{Ker } F_{G(1)} \simeq \text{Im}(F_{G(1)}) \simeq \text{Ker } V_G$.

Then condition (c) is equivalent to that $\alpha$ is an isomorphism; it implies that $\text{Ker } V' = \text{Coker } V' = 0$, i.e. the Verschiebung of $\text{Ker } F_G$ is an isomorphism, and hence (c').

(c) $\Rightarrow$ (a). For every integer $n > 0$, we denote by $F^n_G$ the composed homomorphism

\[
G \xrightarrow{F_G} G^{(p)} \xrightarrow{F_G^{(p)}} \cdots \xrightarrow{F_G^{(p^{n-1})}} G^{(p^n)},
\]

and by $V^n_G$ the composed homomorphism

\[
G^{(p^n)} \xrightarrow{V_G^{(p^{n-1})}} G^{(p^{n-1})} \xrightarrow{V_G^{(p^{n-2})}} \cdots \xrightarrow{V_G} G;
\]

$F^n_G$ and $V^n_G$ are isogenies of BT-groups. From the relation $V^n_G \circ F^n_G = p^n$, we deduce an exact sequence

\[(2.13.6) \quad 0 \to \text{Ker } F^n_G \to G(n) \xrightarrow{F^n} \text{Ker } V^n_G \to 0,\]
where $F^n$ is induced by $F^n_G$. For $1 \leq j < n$, we have a commutative diagram

$$
\begin{array}{ccc}
G(p^n) & \stackrel{V_{G(p^n)}^{n-j}}{\longrightarrow} & G(p') \\
\downarrow V^n_G & & \downarrow V^n_G \\
G & = & G.
\end{array}
$$

One notices that $\text{Ker} V_{G(p^n)}^{n-j} = (\text{Ker} V_{G}^{n-j}(p^n))^j$ by the functoriality of Verschiebung. Since all maps in (2.13.7) are isogenies, we have an exact sequence

$$0 \rightarrow (\text{Ker} V_{G(p^n)}^{n-j})^{j,n} \overset{\lambda_{n-j,n}}{\longrightarrow} \text{Ker} V^n_G \overset{p^{n-j}}{\longrightarrow} \text{Ker} V^n_{p} \rightarrow 0.
$$

Therefore, condition (c) implies by induction that $\text{Ker} V^n_G$ is an étale group scheme over $S$. Hence the $j$-th iteration of the Frobenius $\text{Ker} V^{n-j}_G \rightarrow (\text{Ker} V^{n-j}_G(p^n))^{j}$ is an isomorphism, and $\text{Ker} V^{n-j}_G$ is identified with a closed subgroup scheme of $\text{Ker} V^n_G$ by the composed map

$$i_{n-j,n} : \text{Ker} V^{n-j}_G \cong (\text{Ker} V^{n-j}_G(p^n))^{j,n} \overset{\lambda_{n-j,n}}{\longrightarrow} \text{Ker} V^n_G.
$$

We claim that the kernel of the multiplication by $p^{n-j}$ on $\text{Ker} V^n_G$ is $\text{Ker} V^{n-j}_G$. Indeed, from the relation $p^{n-j} \cdot \text{Id}_{G(p^n)} = F_{G(p^n)}^{n-j} \circ V^{n-j}_G(p^n)$, we deduce a commutative diagram (without dotted arrows)

$$
\begin{array}{ccc}
\text{Ker} V^n_G & \overset{p^{n-j}}{\longrightarrow} & \text{Ker} V^n_{p} \\
\downarrow \text{Ker} V^{n-j}_G & & \downarrow \text{Ker} V^{n-j}_{p} \\
\text{Ker} V^n_{G(p^n)} & \overset{\lambda_{n-j,n}}{\longrightarrow} & \text{Ker} V^n_{p^{n-j}} \overset{\lambda_{n-j,n}}{\longrightarrow} \text{Ker} V^n_{p^{n-j}} \\
\downarrow \text{Ker} V^{n-j}_{G(p^n)} & & \downarrow \text{Ker} V^{n-j}_{p^{n-j}} \\
\text{Ker} V^{n-j}_{G(p^n)} & \overset{\lambda_{n-j,n}}{\longrightarrow} & \text{Ker} V^n_{p^{n-j}} \overset{\lambda_{n-j,n}}{\longrightarrow} \text{Ker} V^n_{p^{n-j}}.
\end{array}
$$

It follows from (2.13.8) that the subgroup $\text{Ker} V^n_G$ of $G(p^n)$ is sent by $V^{n-j}_G$ onto $\text{Ker} V^n_{p}$. Therefore diagram (2.13.9) remains commutative when completed by the dotted arrows, hence our claim. It follows from the claim that $(\text{Ker} V^n_G)_{n \geq 1}$ constitutes an étale BT-group over $S$, denoted by $G^e$. By duality, we have an exact sequence

$$0 \rightarrow \text{Ker} F^n_G \rightarrow \text{Ker} F^n_G \rightarrow (\text{Ker} F^{n-j}_G(p^n)) \rightarrow 0.
$$

Condition (c') implies by induction that $\text{Ker} F^n_G$ is of multiplicative type. Hence the $j$-th iteration of Verschiebung $(\text{Ker} F^{n-j}_G(p^n)) \rightarrow \text{Ker} F^{n-j}_G$ is an isomorphism. We deduce from (2.13.10) that $(\text{Ker} F^n_G)_{n \geq 1}$ form a multiplicative BT-group over $S$ that we denote by $G^{\text{mult}}$. Then the exact sequences (2.13.6) give a decomposition of $G$ of the form \eqref{Gmult}.
Corollary 2.14. Let $G$ be a $B$T-group over $S$, and $S^{\text{ord}}$ be the locus in $S$ of the points $x \in S$ such that $G_x = G \otimes_S \kappa(x)$ is ordinary over $\kappa(x)$. Then $S^{\text{ord}}$ is open in $S$, and the canonical inclusion $S^{\text{ord}} \to S$ is affine.

The open subscheme $S^{\text{ord}}$ of $S$ is called the ordinary locus of $G$.

3. Preliminaries on Dieudonné Theory and Deformation Theory

3.1. We will use freely the conventions of 1.8. Let $S$ be a scheme of characteristic $p > 0$, $G$ be a Barsotti-Tate group over $S$, and $M(G) = D(G)_{(S,S)}$ be the coherent $\mathcal{O}_S$-module obtained by evaluating the (contravariant) Dieudonné crystal of $G$ at the trivial divided power immersion $S \hookrightarrow S$ [BBM, 3.3.6]. Recall that $M(G)$ is an $\mathcal{O}_S$-module locally free of finite type satisfying the following properties:

(i) Let $F_M : M(G)^{(p)} \to M(G)$ and $V_M : M(G) \to M(G)^{(p)}$ be the $\mathcal{O}_S$-linear maps induced respectively by the Frobenius and the Verschlebung of $G$. We have the following exact sequence:

$$
\cdots \to M(G)^{(p)} \xrightarrow{F_M} M(G) \xrightarrow{V_M} M(G)^{(p)} \to \cdots.
$$

(ii) There is a connection $\nabla : M(G) \to M(G) \otimes_{\mathcal{O}_S} \Omega^1_{S/F_p}$ for which $F_M$ and $V_M$ are horizontal morphisms.

(iii) We have two canonical filtrations on $M(G)$ by $\mathcal{O}_S$-modules locally free of finite type:

(3.1.1) $0 \to \omega_G \to M(G) \to \text{Lie}(G^\vee) \to 0$,

called the Hodge filtration on $M(G)$ [BBM, 3.3.5], and the conjugate filtration on $M(G)$

(3.1.2) $0 \to \text{Lie}(G^\vee)^{(p)} \xrightarrow{\delta_G} M(G) \to \omega_G^{(p)} \to 0$,

which is obtained by applying the Dieudonné functor to the exact sequence of finite group schemes $0 \to \text{Ker} F_G \to G(1) \to \text{Ker} V_G \to 0$ [BBM, 4.3.1, 4.3.6, 4.3.11]. Moreover, we have the following commutative diagram (cf. [Ka1, 2.3.2].

Documenta Mathematica 14 (2009) 397–440
and 2.3.4)]

\[ \begin{array}{c}
0 \\
\omega_G^{(p)} \\
\longrightarrow M(G)^{(p)} \\
\phi_G \\
\longrightarrow \text{Lie}(G^\vee)^{(p)}
\end{array} \] 

\[ \begin{array}{c}
0 \\
\omega_G \\
\longrightarrow M(G) \\
\psi_G \\
\longrightarrow \text{Lie}(G^\vee)
\end{array} \] 

\[ \begin{array}{c}
0 \\
\omega_G^{(p)} \\
\longrightarrow M(G)^{(p)} \\
\phi_G \\
\longrightarrow \text{Lie}(G^\vee)^{(p)}
\end{array} \] 

where the columns are the Hodge filtrations and the anti-diagonal is the conjugate filtration. By functoriality, we see easily that \( \varphi_G \) above is nothing but the linearization of the Hasse-Witt map \( \varphi_G \) (2.6.1), and the morphism \( \psi_G^* : \text{Lie}(G)^{(p)} \rightarrow \text{Lie}(G) \), which is obtained by applying the functor \( \mathcal{H}om_{\mathcal{O}_S}(\_ , \mathcal{O}_S) \) to \( \psi_G \), is identified with the linearization \( \varphi_G^{\vee^\vee} \) of \( \varphi_G^\vee \).

The formation of these structures on \( M(G) \) commutes with arbitrary base changes of \( S \). In the sequel, we will use \( (M(G), F_M, \nabla) \) to emphasize these structures on \( M(G) \).

3.2. In the reminder of this section, \( k \) will denote an algebraically closed field of characteristic \( p > 0 \). Let \( S \) be a scheme formally smooth over \( k \) such that \( \Omega^1_{S/k} = \Omega^1_{S/k} \) is an \( \mathcal{O}_S \)-module locally free of finite type, e.g. \( S = \text{Spec}(A) \) with \( A \) a formally smooth \( k \)-algebra with a finite \( p \)-basis over \( k \). Let \( G \) be a BT-group over \( S \). We put KS to be the composed morphism

\[ \text{KS} : \omega_G \rightarrow M(G) \xrightarrow{\nabla} M(G) \otimes_{\mathcal{O}_S} \Omega^1_{S/k} \xrightarrow{\text{pr}} \text{Lie}(G^\vee) \otimes_{\mathcal{O}_S} \Omega^1_{S/k} \]

which is \( \mathcal{O}_S \)-linear. We put \( \mathcal{T}_{S/k} = \mathcal{H}om_{\mathcal{O}_S}(\Omega^1_{S/k}, \mathcal{O}_S) \), and define the Kodaira-Spencer map of \( G \)

\[ \text{Kod} : \mathcal{T}_{S/k} \rightarrow \mathcal{H}om_{\mathcal{O}_S}(\omega_G, \text{Lie}(G^\vee)) \]

to be the morphism induced by KS. We say that \( G \) is versal if Kod is surjective.

3.3. Let \( r \) be an integer \( \geq 1 \), \( R = k[[t_1, \ldots, t_r]] \), \( m \) be the maximal ideal of \( R \). We put \( \mathcal{I} = \text{Spf}(R) \), \( S = \text{Spec}(R) \), and for each integer \( n \geq 0 \), \( S_n = \text{Spec}(R/m^{n+1}) \). By a BT-group \( \mathcal{G} \) over the formal scheme \( \mathcal{I} \), we mean a sequence of BT-groups \( (G_n)_{n \geq 0} \) over \( (S_n)_{n \geq 0} \) equipped with isomorphisms \( G_{n+1} \times_{S_{n+1}} S_n \simeq G_n \).
According to [deJ, 2.4.4], the functor $G \mapsto (G \times S \rightarrow S)_{n \geq 0}$ induces an equivalence of categories between the category of BT-groups over $S$ and the category of BT-groups over $\mathcal{J}$. For a BT-group $\mathcal{J}$ over $\mathcal{S}$, the corresponding BT-group $G$ over $S$ is called the algebraization of $\mathcal{J}$. We say that $\mathcal{J}$ is versal over $\mathcal{S}$, if its algebraization $G$ is versal over $S$. Since $S$ is local, by Nakayama’s Lemma, $\mathcal{J}$ or $G$ is versal if and only if the reduction of Kod modulo the maximal ideal is surjective.

(3.3.1) $\text{Kod}_0 : \mathcal{S}_{j/k} \otimes_{\kappa} k \longrightarrow \text{Hom}_k(\omega_{G_0}, \text{Lie}(G_0^0))$

is surjective.

3.4. We recall briefly the deformation theory of a BT-group. Let $\mathfrak{A}_k$ be the category of local artinian $k$-algebras with residue field $k$. We notice that all morphisms of $\mathfrak{A}_k$ are local. A morphism $A' \rightarrow A$ in $\mathfrak{A}_k$ is called a small extension, if it is surjective and its kernel $I$ satisfies $I \cdot m_{A'} = 0$, where $m_{A'}$ is the maximal ideal of $A'$.

Let $G_0$ be a BT-group over $k$, and $A$ an object of $\mathfrak{A}_k$. A deformation of $G_0$ over $A$ is a pair $(G, \phi)$, where $G$ is a BT-group over Spec$(A)$ and $\phi$ is an isomorphism $\phi : G \otimes_A k \iso G_0$. When there is no risk of confusion, we will denote a deformation $(G, \phi)$ simply by $G$. Two deformations $(G, \phi)$ and $(G', \phi')$ over $A$ are isomorphic if there exists an isomorphism of BT-groups $\psi : G \iso G'$ over $A$ such that $\phi = \phi' \circ (\psi \otimes_A k)$. Let’s denote by $D$ the functor which associates with each object $A$ of $\mathfrak{A}_k$ the set of isomorphism classes of deformations of $G_0$ over $A$. If $f : A \rightarrow B$ is a morphism of $\mathfrak{A}_k$, then the map $D(f) : D(A) \rightarrow D(B)$ is given by extension of scalars. We call $D$ the deformation functor of $G_0$ over $\mathfrak{A}_k$.

**Proposition 3.5 ([III], 4.8).** Let $G_0$ be a BT-group over $k$ of dimension $d$ and height $c + d$. $D$ be the deformation functor of $G_0$ over $\mathfrak{A}_k$.

(i) Let $A' \rightarrow A$ be a small extension in $\mathfrak{A}_k$ with ideal $I$, $x = (G, \phi)$ be an element in $D(A)$, $D_x(A')$ be the subset of $D(A')$ with image $x$ in $D(A)$. Then the set $D_x(A')$ is a nonempty homogeneous space under the group $\text{Hom}_k(\omega_{G_0}, \text{Lie}(G_0^0)) \otimes_k I$.

(ii) The functor $D$ is pro-representable by a formally smooth formal scheme $\mathcal{F}$ over $k$ of relative dimension $cd$, i.e. $\mathcal{F} = \text{Spf}(R)$ with $R \simeq k[[t_{ij} : 1 \leq i, j \leq d]]$, and there exists a unique deformation $(\mathcal{F}, \psi)$ of $G_0$ over $\mathcal{F}$ such that, for any object $A$ of $\mathfrak{A}_k$ and any deformation $(G, \phi)$ of $G_0$ over $A$, there is a unique homomorphism of local $k$-algebras $\varphi : R \rightarrow A$ with $(G, \phi) = D(\varphi)(\mathcal{F}, \psi)$.

(iii) Let $\mathcal{F}_{j/k}(0) = \mathcal{F}_{j/k} \otimes_{\kappa} k$ be the tangent space of $\mathcal{F}$ at its unique closed point.

$\text{Kod}_0 : \mathcal{F}_{j/k}(0) \longrightarrow \text{Hom}_k(\omega_{G_0}, \text{Lie}(G_0^0))$

be the Kodaira-Spencer map of $\mathcal{F}$ evaluated at the closed point of $\mathcal{F}$. Then $\text{Kod}_0$ is bijective, and it can be described as follows. For an element $f \in \mathcal{F}_{j/k}(0)$, i.e. a homomorphism of local $k$-algebras $f : R \rightarrow k[\epsilon]/\epsilon^2$, $\text{Kod}_0(f)$ is the difference of deformations

$[\mathcal{F} \otimes_R (k[\epsilon]/\epsilon^2)] - [G_0 \otimes_k (k[\epsilon]/\epsilon^2)]$,

which is a well-defined element in $\text{Hom}_k(\omega_{G_0}, \text{Lie}(G_0^0))$ by (i).
Remark 3.6. Let \((e_j)_{1 \leq j \leq d}\) be a basis of \(\omega_{G_0}\), \((f_i)_{1 \leq i \leq n}\) be a basis of \(\text{Lie}(G_0)\). In view of 3.5(iii), we can choose a system of parameters \((t_{ij})_{1 \leq i \leq n, 1 \leq j \leq d}\) of \(\mathcal{F}\) such that

\[
\text{Kod}_0 \left( \frac{\partial}{\partial t_{ij}} \right) = e_j^* \otimes f_i,
\]

where \((e_j^*)_{1 \leq j \leq d}\) is the dual basis of \((e_j)_{1 \leq j \leq d}\). Moreover, if \(m\) is the maximal ideal of \(R\), the parameters \(t_{ij}\) are determined uniquely modulo \(m^2\).

Corollary 3.7 (Algebraization of the universal deformation). The assumptions being those of (3.5), we put moreover \(S = \text{Spec}(R)\) and \(G\) the algebraization of the universal formal deformation \(\mathcal{F}\). Then the BT-group \(G\) is versal over \(S\), and satisfies the following universal property: Let \(A\) be a noetherian complete local \(k\)-algebra with residue field \(k\), \(G\) be a BT-group over \(A\) endowed with an isomorphism \(G \otimes_A k \simeq G_0\). Then there exists a unique continuous homomorphism of local \(k\)-algebras \(\varphi : R \to A\) such that \(G \simeq G \otimes_R A\).

Proof. By the last remark of 3.3, \(G\) is clearly versal. It remains to prove that it satisfies the universal property in the corollary. Let \(G\) be a deformation of \(G_0\) over a noetherian complete local \(k\)-algebra \(A\) with residue field \(k\). We denote by \(m_A\) the maximal ideal of \(A\), and put \(A_n = A/m_A^{n+1}\) for each integer \(n \geq 0\). Then by 3.5(b), there exists a unique local homomorphism \(\varphi_n : R \to A_n\) such that \(G \otimes A_n \simeq G \otimes R A_n\). The \(\varphi_n\)’s form a projective system \((\varphi_n)_{n \geq 0}\), whose projective limit \(\varphi : R \to A\) answers the question. \(\square\)

Definition 3.8. The notations are those of (3.7). We call \(S\) the local moduli in characteristic \(p\) of \(G_0\), and \(G\) the universal deformation of \(G_0\) in characteristic \(p\).

If there is no confusions, we will omit “in characteristic \(p\)” for short.

3.9. Let \(G\) be a BT-group over \(k\), \(G^o\) be its connected part, and \(G^{et}\) be its étale part. Let \(r\) be the height of \(G^{et}\). Then we have \(G^{et} \simeq (\mathbb{Q}_p/\mathbb{Z}_p)^r\), since \(k\) is algebraically closed. Let \(D_G\) (resp. \(D_G^o\)) be the deformation functor of \(G\) (resp. \(G^o\)) over \(\mathcal{A}\). If \(A\) is an object in \(\mathcal{A}\) and \(\mathcal{G}\) is a deformation of \(G\) (resp. \(G^o\)) over \(A\), we denote by \([\mathcal{G}]\) its isomorphism class in \(D_G(A)\) (resp. in \(D_G^o(A)\)).

Proposition 3.10. The assumptions are as above, let \(\Theta : D_G \to D_G^o\) be the morphism of functors that maps a deformation of \(G\) to its connected component.

(i) The morphism \(\Theta\) is formally smooth of relative dimension \(r\).

(ii) Let \(A\) be an object of \(\mathcal{A}\), and \(\mathcal{G}\) be a deformation of \(G^o\) over \(A\). Then the subset \(\Theta_A^o([\mathcal{G}])\) of \(D_G(A)\) is canonically identified with \(\text{Ext}_A^1(\mathbb{Q}_p/\mathbb{Z}_p, \mathcal{G})\), where \(\text{Ext}_A^1\) means the group of extensions in the category of abelian \(p\)-fppf-sheaves on \(\text{Spec}(A)\).

Proof. (i) Since \(D_G\) and \(D_G^o\) are both pro-representable by a noetherian local complete \(k\)-algebra and formally smooth over \(k\) (3.5), by a formal completion version of [EGA, IV 17.11.1(d)], we only need to check that the tangent map

\[
\Theta_{k[e]/e^2} : D_G(k[e]/e^2) \to D_G^o(k[e]/e^2)
\]

Documenta Mathematica 14 (2009) 397–440
is surjective with kernel of dimension $r$ over $k$. By 3.5(iii), $D_G(k[e]/e^2)$ (resp. $D_G(k[e]/e^2)$) is isomorphic to $\text{Hom}_k(\omega_G, \text{Lie}(G^\vee))$ (resp. $\text{Hom}_k(\omega_G, \text{Lie}(G^{\text{et}}))$) by the Kodaira-Spencer morphism. In view of the canonical isomorphism $\omega_G \cong \omega_{G^{\text{et}}}$, $\Theta_{k[e]/e^2}$ corresponds to the map
$$\Theta_{k[e]/e^2} : \text{Hom}_k(\omega_G, \text{Lie}(G^\vee)) \to \text{Hom}_k(\omega_G, \text{Lie}(G^{\text{et}}))$$
induced by the canonical surjection $\text{Lie}(G^\vee) \to \text{Lie}(G^{\text{et}})$. It is clear that $\Theta_{k[e]/e^2}$ is surjective of kernel $\text{Hom}_k(\omega_G, \text{Lie}(G^{\text{et}}))$, which has dimension $r$ over $k$.

(ii) Since $G^{\text{et}}$ is isomorphic to $(\mathbb{Q}_p/\mathbb{Z}_p)^r$, every element in $\text{Ext}_A^1(\mathbb{Q}_p/\mathbb{Z}_p, \mathfrak{g}^p)^r$ defines clearly an element of $D_G(A)$ with image $[\mathfrak{g}^p]$ in $D_G(A)$. Conversely, for any $\mathfrak{g} \in D_G(A)$ with connected component isomorphic to $\mathfrak{g}^p$, the isomorphism $G^{\text{et}} \cong (\mathbb{Q}_p/\mathbb{Z}_p)^r$ lifts uniquely to an isomorphism $\mathfrak{g}^{\text{et}} \cong (\mathbb{Q}_p/\mathbb{Z}_p)^r$ because $A$ is henselian. The canonical exact sequence $0 \to \mathfrak{g}^p \to \mathfrak{g} \to \mathfrak{g}^{\text{et}} \to 0$ shows that $\mathfrak{g}$ comes from an element of $\text{Ext}_A^1(\mathbb{Q}_p/\mathbb{Z}_p, \mathfrak{g}^p)^r$.

\section{4. HW-cyclic Barsotti-Tate groups}

\begin{definition}
Let $S$ be a scheme of characteristic $p > 0$, $G$ be a BT-group over $S$ such that $c = \dim(G^\vee)$ is constant. We say that $G$ is HW-cyclic, if $c \geq 1$ and there exists an element $v \in \Gamma(S, \text{Lie}(G^\vee))$ such that 
$$v, \varphi_G(v), \ldots, \varphi_G^{-1}(v)$$
generate $\text{Lie}(G^\vee)$ as an $\mathcal{O}_S$-module, where $\varphi_G$ is the Hasse-Witt map (2.6.1) of $G$.
\end{definition}

\begin{remark}
It is clear that a BT-group $G$ over $S$ is HW-cyclic, if and only if $\text{Lie}(G^\vee)$ is free over $\mathcal{O}_S$ and there exists a basis of $\text{Lie}(G^\vee)$ over $\mathcal{O}_S$ under which $\varphi_G$ is expressed by a matrix of the form
$$\begin{pmatrix}
0 & 0 & \cdots & 0 & -a_1 \\
1 & 0 & \cdots & 0 & -a_2 \\
0 & 1 & \cdots & 0 & -a_3 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \cdots & 1 & -a_c
\end{pmatrix},
$$
where $a_i \in \Gamma(S, \mathcal{O}_S)$ for $1 \leq i \leq c$.
\end{remark}

\begin{lemma}
Let $R$ be a local ring of characteristic $p > 0$, $k$ be its residue field.
(i) A BT-group $G$ over $R$ is HW-cyclic if and only if $G \otimes k$.
(ii) Let $0 \to G' \to G \to G'' \to 0$ be an exact sequence of BT-groups over $R$. If $G$ is HW-cyclic, then so is $G'$. In particular, if $R$ is henselian, the connected part of a HW-cyclic BT-group over $R$ is HW-cyclic.
\end{lemma}

\begin{proof}
(i) The property of being HW-cyclic is clearly stable under arbitrary base changes, so the "only if" part is clear. Assume that $G_0 = G \otimes k$ is HW-cyclic. Let $\varpi$ be an element of $\text{Lie}(G_0^{\vee}) = \text{Lie}(G^\vee) \otimes k$ such that

\document{Documenta Mathematica 14 (2009) 397–440}
\((\tau, \varphi_G(\tau), \ldots, \varphi_G^{-1}(\tau))\) is a basis of \(\text{Lie}(G_V^\lor)\). Let \(v\) be any lift of \(\tau\) in \(\text{Lie}(G_V^\lor)\). Then by Nakayama’s lemma, \((v, \varphi_G(v), \ldots, \varphi_G^{-1}(v))\) is a basis of \(\text{Lie}(G_V^\lor)\).

(ii) By statement (i), we may assume \(R = k\). The exact sequence of BT-groups induces an exact sequence of Lie algebras

\[(4.3.1) \quad 0 \to \text{Lie}(G''^\lor) \to \text{Lie}(G^\lor) \to \text{Lie}(G'^\lor) \to 0,\]

and the Hasse-Witt map \(\varphi_G\) is induced by \(\varphi_G^\lor\) by functoriality. Assume that \(G\) is HW-cyclic and \(G^\lor\) has dimension \(c\). Let \(u\) be an element of \(\text{Lie}(G^\lor)\) such that

\[u, \varphi_G(u), \ldots, \varphi_G^{-1}(u)\]

form a basis of \(\text{Lie}(G^\lor)\) over \(k\). We denote by \(u'\) the image of \(u\) in \(\text{Lie}(G'^\lor)\). Let \(r \leq c\) be the maximal integer such that the vectors

\[u', \varphi_G(u'), \ldots, \varphi_G^{-1}(u')\]

are linearly independent over \(k\). It is easy to see that they form a basis of the \(k\)-vector space \(\text{Lie}(G'^\lor)\). Hence \(G'^\lor\) is HW-cyclic. \(\Box\)

**Lemma 4.4.** Let \(S = \text{Spec}(R)\) be an affine scheme of characteristic \(p > 0\), \(G\) be a HW-cyclic BT-group over \(R\) with \(c = \dim(G^\lor)\) constant, and

\[
\begin{pmatrix}
0 & 0 & \cdots & 0 & -a_1 \\
1 & 0 & \cdots & 0 & -a_2 \\
0 & 1 & \cdots & 0 & -a_3 \\
\vdots & \ddots & \ddots & \ddots & \vdots \\
0 & 0 & \cdots & 1 & -a_c
\end{pmatrix} \in M_{c\times c}(R),
\]

be a matrix of \(\varphi_G\). Put \(a_{c+1} = 1\), and \(P(X) = \sum_{i=0}^{c} a_{i+1}X^i \in R[X]\).

(i) Let \(V_G : G^{(p)} \to G\) be the Verschiebung homomorphism of \(G\). Then \(\text{Ker} V_G\) is isomorphic to the group scheme \(\text{Spec}(R[X]/P(X))\) with comultiplication given by \(X \mapsto 1 \otimes X + X \otimes 1\).

(ii) Let \(x \in S\), and \(G_x\) be the fibre of \(G\) at \(x\). Put

\[(4.4.1) \quad i_0(x) = \min_{0 \leq i \leq c} \{i; a_{i+1}(x) \neq 0\},
\]

where \(a_i(x)\) denotes the image of \(a_i\) in the residue field of \(x\). Then the étale part of \(G_x\) has height \(c - i_0(x)\), and the connected part of \(G_x\) has height \(d + i_0(x)\).

In particular, \(G_x\) is connected if and only if \(a_i(x) = 0\) for \(1 \leq i \leq c\).

**Proof.** (i) By 2.3 and 2.13, \(\text{Ker} V_G\) is isomorphic to the group scheme

\[
\text{Spec} \left( R[X_1, \ldots, X_c]/(X_1^p - X_2, \ldots, X_{c-1}^p - X_c, X_c^p + a_1X_1 + \cdots + a_cX_c) \right)
\]

with comultiplication \(\Delta(X_i) = 1 \otimes X_i + X_i \otimes 1\) for \(1 \leq i \leq c\). By sending \((X_1, X_2, \ldots, X_c) \mapsto (X, X^p, \ldots, X^{p^{c-1}})\), we see that the above group scheme is isomorphic to \(\text{Spec}(R[X]/P(X))\) with comultiplication \(\Delta(X) = 1 \otimes X + X \otimes 1\).
(ii) By base change, we may assume that $S = x = \text{Spec}(k)$ and hence $G = G_x$. Let $G(1)$ be the kernel of the multiplication by $p$ on $G$. Then we have an exact sequence

$$0 \to \text{Ker } F_G \to G(1) \to \text{Ker } V_G \to 0.$$ 

Since $\text{Ker } F_G$ is an infinitesimal group scheme over $k$, we have $G(1)(\overline{k}) = (\text{Ker } V_G)(\overline{k})$, where $\overline{k}$ is an algebraic closure of $k$. By the definition of $i_0(x)$, we have $P(X) = Q(X^{p^{\alpha_0(x)}})$, where $Q(X)$ is an additive separable polynomial in $k[X]$ with $\deg(Q) = p^{\alpha_0(x)} - \alpha_0(x)$. Hence the roots of $P(X)$ in $\overline{k}$ form an $\mathbb{F}_p$-vector space of dimension $c - \alpha_0(x)$. By (i), $(\text{Ker } V_G)(\overline{k})$ can be identified with the additive group consisting of the roots of $P(X)$ in $\overline{k}$. Therefore, the étale part of $G$ has height $c - \alpha_0(x)$, and the connected part of $G$ has height $d + \alpha_0(x)$. \hfill $\square$

4.5. Let $k$ be a perfect field of characteristic $p > 0$, and $\alpha_p = \text{Spec}(k[X]/X^p)$ be the finite group scheme over $k$ with comultiplication map $\Delta(X) = 1 \otimes X + X \otimes 1$. Let $G$ be a BT-group over $k$. Following Oort, we call

$$a(G) = \dim_k \text{Hom}_{k_{\text{fppf}}} (\alpha_p, G)$$

the $a$-number of $G$, where $\text{Hom}_{k_{\text{fppf}}}$ means the homomorphisms in the category of abelian fppf-sheaves over $k$. Since the Frobenius of $\alpha_p$ vanishes, any morphism of $\alpha_p$ in $G$ factorize through $\text{Ker } (F_G)$. Therefore we have

$$\text{Hom}_{k_{\text{fppf}}} (\alpha_p, G) = \text{Hom}_{k_{\text{fppf}}} (\alpha_p, \text{Ker } (F_G)) = \text{Hom}_{k_{\text{fppf}}} (\text{Ker } (F_G), \alpha_p) = \text{Hom}_{k_{\text{fppf}}} (\text{Lie}(\alpha_p), \text{Lie}(\text{Ker } (F_G))),$$

where $\text{Hom}_{k_{\text{fppf}}}$ denotes the homomorphisms in the category of commutative group schemes over $k$, and the last equality uses Proposition 2.3. Since we have a canonical isomorphism $\text{Lie}(\text{Ker } (F_G)) \simeq \text{Lie}(G)$ and $\text{Lie}(\alpha_p)$ has dimension one over $k$ with $\varphi_{\alpha_p} = 0$, we get

$$a(G) = \dim_k \{ x \in \text{Lie}(G) | \varphi_G \gamma(x) = 0 \} = \dim_k \text{Ker } (\varphi_G \gamma).$$

Due to the perfectness of $k$, we have also $a(G) = \dim_k \text{Ker } (\varphi_G \gamma)$, where $\varphi_G \gamma$ is the linearization of $\varphi_G \gamma$. By Proposition 2.11, we see that $a(G) = 0$ if and only if $G$ is ordinary.

**Lemma 4.6.** Let $G$ be a BT-group over $k$, and $G^\vee$ its Serre dual. Then we have $a(G) = a(G^\vee)$.

**Proof.** Let $\psi_G : \omega_G \to \omega_G^{(p)}$ be the $k$-linear map induced by the Verschiebung of $G$. Then $\psi_G^\vee$, the morphism obtained by applying the functor $\text{Hom}_{k}(\_ , k)$ to $\psi_G$, is identified with $\varphi_G \gamma$. By (4.5.1) and the exactitude of the functor $\text{Hom}_{k}(\_ , k)$, we have $a(G) = \dim_k \text{Ker } (\psi_G^\vee) = \dim_k \text{Coker } (\psi_G)$. Using the additivity of $\dim_k$, we get finally $a(G) = \dim_k \text{Ker } (\psi_G)$. By considering the commutative diagram (3.1.3), we have

$$a(G) = \dim_k \left( \omega_G \cap \phi_G (\text{Lie}(G^\vee)^{(p)}) \right).$$

**Documenta Mathematica** 14 (2009) 397–440
On the other hand, it follows also from (3.1.3) that
\[ a(G^\nu) = \dim_k \ker(\tilde{\varphi}_G) = \dim_k \left( \varphi_G(Lie(G^\nu)^{(p)}) \cap \omega_G \right). \]

The lemma now follows immediately.

\[ \square \]

**Proposition 4.7.** Let \( k \) be a perfect field of characteristic \( p > 0 \), \( G \) a BT-group over \( k \). Consider the following conditions:

(i) \( G \) is HW-cyclic and non-ordinary;
(ii) the connected part \( G^o \) of \( G \) is HW-cyclic and not of multiplicative type;
(iii) \( a(G^\nu) = a(G) = 1 \).

We have (i) \( \Rightarrow \) (ii) \( \Leftrightarrow \) (iii). If \( k \) is algebraically closed, we have moreover (ii) \( \Rightarrow \) (i).

**Remark 4.8.** In [Ool, Lemma 2.2], Oort proved the following assertion, which is a generalization of (iii) \( \Rightarrow \) (ii): Let \( k \) be an algebraically closed field of characteristic \( p > 0 \), and \( G \) be a connected BT-group with \( a(G) = 1 \). Then there exists a basis of the Dieudonné module \( M \) of \( G \) over \( W(k) \), such that the action of Frobenius on \( M \) is given by a display-matrix of "normal form" in the sense of [Ool, 2.1].

**Proof.** (i) \( \Rightarrow \) (ii) follows from 4.3(ii).

(ii) \( \Rightarrow \) (iii). First, we note that \( a(G) = a(G^o) \), so we may assume \( G \) connected. Since \( G \) is not of multiplicative type, we have \( c = \dim(G^\nu) \geq 1 \). By Lemma 4.4(ii), there exists a basis of \( Lie(G^\nu) \) over \( k \) under which \( \varphi_G \) is expressed by

\[
\begin{pmatrix}
0 & 0 & \cdots & 0 & 0 \\
1 & 0 & \cdots & 0 & 0 \\
0 & 1 & \cdots & 0 & 0 \\
\vdots & \ddots & \ddots & \ddots & \vdots \\
0 & 0 & \cdots & 1 & 0
\end{pmatrix} \in M_{c \times c}(k).
\]

According to (4.5.1), \( a(G^\nu) \) equals to \( \dim_k \ker(\varphi_G) \), i.e. the \( k \)-dimension of the solutions of the equation system in \( (x_1, \ldots, x_c) \)

\[
\begin{pmatrix}
0 & 0 & \cdots & 0 & x_1^p \\
1 & 0 & \cdots & 0 & x_2^p \\
\vdots & \ddots & \ddots & \ddots & \vdots \\
0 & 0 & \cdots & 1 & x_c^p
\end{pmatrix} = 0
\]

The solutions \( (x_1, \ldots, x_c) \) form clearly a vector space over \( k \) of dimension 1, i.e. we have \( a(G^\nu) = 1 \).

(iii) \( \Rightarrow \) (ii). Let \( G^e \) be the étale part of \( G \). Since \( k \) is perfect, the exact sequence (2.7.1) splits [Dem, Chap. II §7]; so we have \( G \simeq G^e \times G^o \). We put \( M = Lie(G^\nu) \), \( M_1 = Lie(G^{e\nu}) \) and \( M_2 = Lie(G^{o\nu}) \) for short. By 2.8 and 2.9, we have a decomposition \( M = M_1 \oplus M_2 \), such that \( M_1, M_2 \) are stable under \( \varphi_G \), and the action of \( \varphi_G \) is nilpotent on \( M_1 \) and bijective on \( M_2 \). We note
that \( a(G^0) = a(G^\circ) = a(G) = 1 \). By the last remark of 4.5, \( G^\circ \) is not of multiplicative type, hence \( \text{dim}_k M_1 = \text{dim}(G^0) \geq 1 \). It remains to prove that \( G^\circ \) is HW-cyclic. Let \( n \) be the minimal integer such that \( \varphi^n_G(M_1) = 0 \). We have a strictly increasing filtration

\[
0 \subset \text{Ker}(\varphi_G) \subset \cdots \subset \text{Ker}(\varphi^n_G) = M_1.
\]

If \( n = 1 \), then \( M_1 \) is one-dimensional, hence \( G^\circ \) is clearly HW-cyclic. Assume \( n \geq 2 \). For \( 2 \leq m \leq n \), \( \varphi^{m-1}_G \) induces an injective map

\[
\varphi^{m-1}_G : \text{Ker}(\varphi^m_G)/\text{Ker}(\varphi^{m-1}_G) \rightarrow \text{Ker}(\varphi_G).
\]

Since \( \text{dim}_k \text{Ker}(\varphi_G) = a(G^0) = 1 \), \( \varphi^{m-1}_G \) is necessarily bijective. So we have \( \text{dim}_k \text{Ker}(\varphi^n_G) = m \) for \( 1 \leq m \leq n \). Let \( v \) be an element of \( M_1 \) but not in \( \text{Ker}(\varphi^{n-1}_G) \). Then \( v, \varphi_G(v), \ldots, \varphi^{n-1}_G(v) \) are linearly independent, hence they form a basis of \( M_1 \) over \( k \). This proves that \( G^\circ \) is HW-cyclic.

Assume \( k \) algebraically closed. We prove that (ii) \( \Rightarrow \) (i). Noting that \( G \) is ordinary if and only if \( G^\circ \) is of multiplicative type, we only need to check that \( G \) is HW-cyclic. We conserve the notations above. Since \( \varphi_G \) is bijective on \( M_2 \) and \( k \) algebraically closed, there exists a basis \( (e_1, \ldots, e_m) \) of \( M_2 \) such that \( \varphi_G(e_i) = e_i \) for \( 1 \leq i \leq m \). Let \( v \in M_1 \) but not in \( \text{Ker}(\varphi^{n-1}_G) \) as above, and put \( u = v + \lambda_1 e_1 + \cdots + \lambda_m e_m \), where \( \lambda_i (1 \leq i \leq m) \) are some elements in \( k \) to be determined later. Then we have

\[
\begin{pmatrix}
\varphi^n_G(u) \\
\vdots \\
\varphi^{n+m-1}_G(u)
\end{pmatrix} =
\begin{pmatrix}
\lambda_1^n & \cdots & \lambda_m^n \\
\vdots & \ddots & \vdots \\
\lambda_1^{n+m-1} & \cdots & \lambda_m^{n+m-1}
\end{pmatrix}
\begin{pmatrix}
e_1 \\
\vdots \\
e_m
\end{pmatrix}.
\]

Let \( L(\lambda_1, \ldots, \lambda_m) \in k[\lambda_1, \ldots, \lambda_m] \) be the determinant polynomial of the matrix on the right side. An elementary computation shows that the polynomial \( L(\lambda_1, \ldots, \lambda_m) \) is not null. We can choose \( \lambda_1, \ldots, \lambda_m \in k \) such that \( L(\lambda_1, \ldots, \lambda_m) \neq 0 \) because \( k \) is algebraically closed. So \( \varphi^n_G(u), \varphi^{n+m-1}_G(u) \) form a basis of \( M_2 \) over \( k \). Since

\[
\varphi^n_G(u) \equiv \varphi^n_G(v) \mod M_2 \quad \text{for} \quad 0 \leq i \leq n,
\]

by the choice of \( u \), we see that \( \{ u, \varphi_G(u), \ldots, \varphi^{n+m-1}_G(u) \} \) form a basis of \( M = \text{Lie}(G^\circ) \) over \( k \).

By combining 4.6 and 4.7, we obtain the following

**Corollary 4.9.** Let \( k \) be an algebraically closed field of characteristic \( p > 0 \). Then a BT-group over \( k \) is HW-cyclic if and only if so is its Serre dual.

4.10. Examples. Let \( k \) be a perfect field, \( W(k) \) be the ring of Witt vectors with coefficients in \( k \), and \( \sigma \) be the Frobenius automorphism of \( W(k) \). Let \( s, r \) be relatively prime integers such that \( 0 \leq s \leq r \) and \( r \neq 0 \); put \( \lambda = \frac{s}{r} \).

We consider the Dieudonné module \( M^\lambda \simeq W(k)[F,V]/(F^{r-s} - V^s) \), where \( W(k)[F,V] \) is the non-commutative ring with relations \( VF = VF = p, Fa = \sigma(a)F \) and \( V\sigma(a) = aV \) for all \( a \in W(k) \). We note that \( M^\lambda \) is free of rank
r over $W(k)$ and $M^\lambda/VM^\lambda \simeq k[F]/F^{r-s}$. By the contravariant Dieudonné theory, $M^\lambda$ corresponds to a BT-group $G^\lambda$ over $k$ of height $r$ with $\text{Lie}(G^\lambda) = M^\lambda/VM^\lambda$. We see easily that $G^\lambda$ is HW-cyclic, and we call it the elementary BT-group of slope $\lambda$. We note that $G^0 \simeq \mathbb{Q}_p/\mathbb{Z}_p$, $G^1 \simeq \mu_{p^\infty}$, and $(G^\lambda)^\vee \simeq G^{1-\lambda}$ for $0 \leq \lambda \leq 1$.

Assume $k$ algebraically closed. Then by the Dieudonné-Manin's classification of isocrystals [Dem, Chap.IV §4], any BT-group over $k$ is isogenous to a finite product of $G^\lambda$'s; moreover, any connected one-dimensional BT-group over $k$ of height $r$ is necessarily isomorphic to $G^{1/r}$ [Dem, Chap.IV §8], hence in particular HW-cyclic.

**Proposition 4.11.** Let $k$ be an algebraically closed field of characteristic $p > 0$, $R$ be a noetherian complete regular local $k$-algebra with residue field $k$, and $S = \text{Spec}(R)$. Let $G$ be a connected HW-cyclic BT-group over $R$ of dimension $d \geq 1$ and height $c + d$.

$$\mathfrak{h} = \begin{pmatrix}
0 & 0 & \cdots & 0 & -a_1 \\
1 & 0 & \cdots & 0 & -a_2 \\
0 & 1 & \cdots & 0 & -a_3 \\
\vdots & \ddots & \ddots & \ddots & \ddots \\
0 & 0 & \cdots & 1 & -a_c
\end{pmatrix} \in M_{c \times c}(R)$$

be a matrix of $\varphi_G$.

(i) If $G$ is versal over $S$, then $\{a_1, \cdots, a_c\}$ is a subset of a regular system of parameters of $R$.

(ii) Assume that $d = 1$. The converse of (i) is also true, i.e. if $\{a_1, \cdots, a_c\}$ is a subset of a regular system of parameters of $R$ then $G$ is versal over $S$. Furthermore, $G$ is the universal deformation of its special fiber $\bar{G}$ if and only if $\{a_1, \cdots, a_c\}$ is a system of regular parameters of $R$.

**Proof.** Let $(M(G), F_M, \nabla)$ be the finite free $\mathcal{O}_S$-module equipped with a semilinear endomorphism $F_M$ and a connection $\nabla : M(G) \rightarrow M(G) \otimes_{\mathcal{O}_S} \Omega^1_{\mathcal{O}_S/k}$, obtained by evaluating the Dieudonné crystal of $G$ at the trivial immersion $S \rightarrow S$ (cf. 3.1). Recall that we have a commutative diagram

\begin{equation}
\begin{array}{ccc}
M(G)^{(p)} & \xrightarrow{F_M} & M(G) \\
\downarrow{p^r} & \swarrow{\varphi_G} & \downarrow{p^r} \\
\text{Lie}(G^\vee)^{(p)} & \xrightarrow{\tilde{\varphi}_G} & \text{Lie}(G^\vee),
\end{array}
\end{equation}

where $\varphi_G$ is universally injective (3.13). Let $\{v_1, \cdots, v_c\}$ be a basis of $\text{Lie}(G^\vee)$ over $\mathcal{O}_S$ under which $\varphi_G$ is expressed by $\mathfrak{h}$, i.e. we have $\varphi_G^{-1}(v_1) = v_i$ for $1 \leq i \leq c$ and $\varphi_G(v_1) = \varphi_G(v_c) = -\sum_{i=1}^c a_i v_i$. Let $f_1$ be a lift of $v_1$ to $\Gamma(S, M(G))$, and put $f_{i+1} = \varphi_G(v_1)^{(p)}$ for $1 \leq i \leq c - 1$, where $v_i^{(p)} = 1 \otimes v_i \in \Gamma(S, \text{Lie}(G^\vee)^{(p)})$. The image of $f_i$ in $\Gamma(S, \text{Lie}(G^\vee))$ is thus $v_i$ for $1 \leq i \leq c$ by
(4.11.1). We put

\[ e_1 = \phi_G (v_1^{(p)}) + a_1 f_1 + \cdots + a_c f_c \in \Gamma (S, M(G)). \]

The image of \( e_1 \) in \( \Gamma (S, \text{Lie}(G')) \) is \( \varphi_G (v_1) + \sum_{i=1}^c a_i v_i = 0 \); so we have \( e_1 \in \Gamma (S, \omega_G) \). By 4.4(ii), we notice that \( a_1, \cdots, a_c \) belong to the maximal ideal \( \mathfrak{m}_R \) of \( R \), as \( G \) is connected. Hence, we have \( \overline{e_1} = \phi_G (v_1^{(p)}) \), where for a \( R \)-module \( M \) and \( x \in M \), we denote by \( \overline{x} \) the canonical image of \( x \) in \( M \otimes k \).

Since \( \phi_G \) commutes with base change and is universally injective, we get \( \overline{e_1} = \phi_G (v_1^{(p)}) = \phi_G \otimes k (v_1^{(p)}) \neq 0 \). Therefore, we can choose \( e_2, \cdots, e_d \in \Gamma (S, \omega_G) \) such that \( (e_1, \cdots, e_d) \) becomes a basis of \( \omega_G \) over \( \mathcal{O}_S \), so \( (e_1, \cdots, e_d, f_1, \cdots, f_c) \) is a basis of \( M(G) \). Since \( F_M \) is horizontal for the connection \( \nabla \) (cf. 3.1(ii)), we have

\[ \nabla (\phi_G (v_1^{(p)})) = \nabla (F_M (f_c^{(p)})) = 0. \]

In view of (4.11.2), we get

\[ \nabla (e_1) = \sum_{i=1}^c f_i \otimes da_i + \sum_{i=1}^c a_i \nabla (f_i) \]

\[ \equiv \sum_{i=1}^c f_i \otimes da_i \pmod {\mathfrak{m}_R}. \]

Let \( \text{KS}_0 \) and \( \text{Kod}_0 \) be respectively the reductions modulo \( \mathfrak{m}_R \) of (3.2.1) and (3.2.2). Since \( (\overline{\pi}_i)_{1 \leq i \leq c} \) is a base of \( \text{Lie}(G') \otimes k \), we can write

\[ \text{KS}_0 (e_j) = \sum_{i=1}^c \overline{\pi}_i \otimes \theta_{i,j} \quad \text{for } 1 \leq j \leq d, \]

where \( \theta_{i,j} \in \Omega_{S/k} \otimes k \). From (4.11.3), we deduce that \( \theta_{i,1} = da_i \). By the definition of \( \text{Kod}_0 \), we have

\[ \text{Kod}_0 (\partial) = \sum_{j=1}^d \sum_{i=1}^c \langle \partial, \theta_{i,j} \rangle \overline{\pi}_j \otimes \overline{\pi}_i \]

where \( \partial \in \mathcal{I}_{S/k} \otimes k \), \( \langle \cdot, \cdot \rangle \) is the canonical pairing between \( \mathcal{I}_{S/k} \otimes k \) and \( \Omega_{S/k} \otimes k \), and \( \langle \overline{\pi}_i \rangle_{1 \leq i \leq d} \) denotes the dual basis of \( (\overline{\pi}_i)_{1 \leq i \leq d} \). Now assume that \( G \) is versal over \( S \), i.e., \( \text{Kod}_0 \) is surjective by definition (3.2). In particular, there are \( \partial_1, \cdots, \partial_c \in \mathcal{I}_{S/k} \otimes k \) such that \( \text{Kod}_0 (\partial_i) = \overline{\pi}_i \otimes v_i \) for \( 1 \leq i \leq c \), i.e., we have

\[ \langle \partial_i, da_j \rangle = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{if } i \neq j \end{cases} \quad \text{for } 1 \leq i, j \leq c, \]

and

\[ \langle \partial_i, \theta_{j,\ell} \rangle = \begin{cases} 0 & \text{for } 1 \leq i, j \leq c, 2 \leq \ell \leq d. \end{cases} \]

From (4.11.5), we see easily that \( da_1, \cdots, da_c \) are linearly independent in \( \Omega_{S/k} \otimes k \simeq \mathfrak{m}_R/\mathfrak{m}_R^2 \); therefore, \( (a_1, \cdots, a_c) \) is a part of a regular system of parameters of \( R \). Statement (i) is proved.
For statement (ii), we assume $d = 1$ and that $(a_1, \ldots, a_c)$ is a part of a regular system of parameters of $R$. Then the formula (4.11.4) is simplified as

$$Kod_0(\partial) = \sum_{i=1}^c < \partial, da_i > \overline{\epsilon_i^+} \otimes \overline{\epsilon_i^-}.$$ 

Since $da_1, \ldots, da_c$ are linearly independent in $\Omega^1_{S/k} \otimes k$, there exist $\partial_1, \ldots, \partial_c \in \mathcal{I}_{S/k} \otimes k$ such that (4.11.5) holds, i.e. $(\overline{\epsilon_i^+} \otimes \overline{\epsilon_i^-})_{1 \leq i \leq c}$ are in the image of $Kod_0$. But the elements $(\overline{\epsilon_i^+} \otimes \overline{\epsilon_i^-})_{1 \leq i \leq c}$ form already a basis of $\mathcal{H}om_{\sigma_\ast}(\omega_G, \text{Lie}(G')) \otimes k$. So $Kod_0$ is surjective, and hence $G$ is versal over $S$ by Nakayama’s lemma.

Let $G_0$ be the special fiber of $G$. It remains to prove that when $d = 1$, $G$ is the universal deformation of $G_0$ if and only if $\dim(S) = c$ and $G$ is versal over $S$. Let $S$ be the local moduli in characteristic $p$ of $G_0$. By the universal property of $G$ (3.7), there exists a unique morphism $f : S \to \mathbf{S}$ such that $G \simeq G \times_S S$. Since $S$ and $\mathbf{S}$ are local complete regular schemes over $k$ with residue field $k$ of the same dimension, $f$ is an isomorphism if and only if the tangent map of $f$ at the closed point of $S$, denoted by $T_f$, is an isomorphism. By the functoriality of Kodaira-Spencer maps (3.2.2), we have a commutative diagram

$$\begin{array}{ccc}
\mathcal{I}_{S/k} \otimes_{\sigma_\ast} k & \xrightarrow{Kod_0^S} & \text{Hom}_k(\omega_{G_0}, \text{Lie}(G_0')) \\
| T_f | & | & | \\
\mathcal{I}_{S/k} \otimes_{\sigma_\ast} k & \xrightarrow{Kod_0^S} & \text{Hom}_k(\omega_{G_0}, \text{Lie}(G_0'))
\end{array}$$

where horizontal arrows are the Kodaira-Spencer maps evaluated at the closed points (3.3.1). Since $Kod_0^S$ and $Kod_0^k$ are isomorphisms according to the first part of this proposition, we deduce that so is $T_f$. This completes the proof. \(\square\)

5. Monodromy of a HW-cyclic BT-group over a Complete Trait of Characteristic \(p > 0\)

5.1. Let $k$ be an algebraically closed field of characteristic $p > 0$, $A$ be a complete discrete valuation ring of characteristic $p$, with residue field $k$ and fraction field $K$. We put $S = \text{Spec}(A)$, and denote by $s$ its closed point, by $\eta$ its generic point. Let $\overline{K}$ be an algebraic closure of $K$. $K^{\text{sep}}$ be the maximal separable extension of $K$ contained in $\overline{K}$, $K^t$ be the maximal tamely ramified extension of $K$ contained in $K^{\text{sep}}$. We put $I = \text{Gal}(K^{\text{sep}}/K)$, $I_p = \text{Gal}(K^{\text{sep}}/K^t)$ and $I_1 = I/I_p = \text{Gal}(K^t/K)$.

Let $\pi$ be a uniformizer of $A$; so we have $A \simeq k[[\pi]]$. Let $\nu$ be the valuation on $K$ normalized on $K$ by $\nu(\pi) = 1$; we denote also by $\nu$ the unique extension of $\nu$ to $\overline{K}$. For every $\alpha \in \overline{Q}$, we denote by $m_\alpha$ (resp. by $m_\alpha^+$) the set of elements $x \in K^{\text{sep}}$ such that $\nu(x) \geq \alpha$ (resp. $\nu(x) > \alpha$). We put

$$V_\alpha = m_\alpha/m_\alpha^+,$$

which is a $k$-vector space of dimension $1$ equipped with a continuous action of the Galois group $I$. 

Documenta Mathematica 14 (2009) 397–440
5.2. First, we recall some properties of the inertia groups $I_p$ and $I_t$ [Se, Chap. IV]. The subgroup $I_p$, called the \textit{wild inertia subgroup}, is the unique maximal pro-$p$-group contained in $I$ and hence normal in $I$. The quotient $I_t = I/I_p$ is a commutative profinite group, called the \textit{tame inertia group}. We have a canonical isomorphism

\begin{equation}
\theta : I_t \cong \lim_{\langle d, p \rangle = 1} \mu_d,
\end{equation}

where the projective system is taken over positive integers prime to $p$, $\mu_d$ is the group of $d$-th roots of unity in $k$, and the transition maps $\mu_m \rightarrow \mu_d$ are given by $\zeta \mapsto \zeta^{m/d}$, whenever $d$ divides $m$. We denote by $\theta_d : I_t \rightarrow \mu_d$ the projection induced by (5.2.1). Let $q$ be a power of $p$, $\mathbb{F}_q$ be the finite field of $k$ with $q$ elements. Then $\mu_{q-1} = \mathbb{F}_q^\times$, and we can write $\theta_{q-1} : I_t \rightarrow \mathbb{F}_q^\times$. The character $\theta_d$ is characterized by the following property.

**Proposition 5.3** ([Se3 Prop.7]. Let $a, b$ be relatively prime positive integers with $d$ prime to $p$. Then the natural action of $I_p$ on the $k$-vector space $V_{a/d}$ (5.1.1) is trivial, and the induced action of $I_t$ on $V_{a/d}$ is given by the character $(\theta_a)^a : I_t \rightarrow \mu_d$. In particular, if $q$ is a power of $p$, the action of $I_t$ on $V_{1/(q-1)}$ is given by the character $\theta_{q-1} : I_t \rightarrow \mathbb{F}_q^\times$ and any $I$-equivariant $\mathbb{F}_q$-subspace of $V_{1/(q-1)}$ is an $\mathbb{F}_q$-vector space.

5.4. Let $G$ be a BT-group over $S$. We define $h(G)$ to be the valuation of the determinant of a matrix of $\varphi_G$ if $\dim(G^{\vee}) \geq 1$, and $h(G) = 0$ if $\dim(G^{\vee}) = 0$. We call $h(G)$ the \textit{Hasse invariant} of $G$.

(a) $h(G)$ does not depend on the choice of the matrix representing $\varphi_G$. Indeed, let $c$ be the rank of $\text{Lie}(G^{\vee})$ over $A$, $\mathfrak{g} \in M_{c \times c}(A)$ be a matrix of $\varphi_G$. Any other matrix representing $\varphi_G$ can be written in the form $U^{-1} \cdot \mathfrak{g} \cdot U^{(p)}$, where $U \in \text{GL}_c(A)$, $U^{-1}$ is the inverse of $U$, and $U^{(p)}$ is the matrix obtained by applying the Frobenius map of $A$ to the coefficients of $U$.

(b) By 2.11, the generic fiber $G_\eta$ is ordinary if and only if $h(G) < \infty$; $G$ is ordinary over $T$ if and only if $h(G) = 0$.

(c) Let $0 \rightarrow G' \rightarrow G \rightarrow G'' \rightarrow 0$ be a short exact sequence of BT-groups over $T$, then we have $h(G) = h(G') + h(G'')$. Indeed, the exact sequence of BT-groups induces a short exact sequence of Lie algebras (cf. [BBM] 3.3.2)

$0 \rightarrow \text{Lie}(G^{\vee}) \rightarrow \text{Lie}(G^{\vee}) \rightarrow \text{Lie}(G^{\vee}) \rightarrow 0$,

from which our assertion follows easily.

**Proposition 5.5.** Let $G$ be a BT-group over $S$. Then we have $h(G) = h(G^{\vee})$.

**Proof.** The proof is very similar to that of Lemma 4.6. First, we have

\[ h(G) = \text{leng}(\text{Lie}(G^{\vee})/\tilde{\varphi}_G(\text{Lie}(G^{\vee})^{(p)})), \]

where $\tilde{\varphi}_G$ is the linearization of $\varphi_G$, and “leng” means the length of a finite $A$-module (note that this formulae holds even if $\dim(G^{\vee}) = 0$). By the commutative diatgram (3.1.3), we have

\[ h(G) = \text{leng} \text{M}(G)/\phi_G(\text{Lie}(G^{\vee})^{(p)}) + \omega_G). \]
On the other hand, by applying the functor $\text{Hom}_A(\_, A)$ to the $A$-linear map $\varphi_{G'} : \text{Lie}(G)(p) \to \text{Lie}(G)$, we obtain a map $\psi_G : \omega_G \to \omega_G(p)$. If $U$ is a matrix of $\varphi_{G'}$, then the transpose of $U$, denoted by $U^t$, is a matrix of $\psi_G$. So we have

$$h(G') = v(\det(U)) = v(\det(U^t)) = \text{leng}(\omega_G(p)/\psi_G(\omega_G)).$$

By diagram 3.1.3, we get

$$h(G') = \text{leng} M(G)/(\phi_G(\text{Lie}(G')(p)) + \omega_G) = h(G).$$

\[\square\]

5.6. Let $G$ be a BT-group over $S$, $c = \dim(G')$. We put

\[T_p(G) = \text{lim}_{n} G(n)(\mathbb{K})\]

the Tate module of $G$, where $G(n)$ is the kernel of $p^n : G \to G$. It is a free $\mathbb{Z}_p$-module of rank $\leq c$, and the equality holds if and only if the generic fiber $G_\eta$ is ordinary. The Galois group $I$ acts continuously on $T_p(G)$. We are interested in the image of the monodromy representation

\[\rho : I = \text{Gal}(K^{\text{sep}}/K) \to \text{Aut}_{\mathbb{Z}_p}(T_p(G)).\]

We denote by

\[\overline{\rho} : I = \text{Gal}(K^{\text{sep}}/K) \to \text{Aut}_{\mathbb{F}_p}(G(1)(\overline{K}))\]

its reduction mod $p$.

**Theorem 5.7 (Reformulation of Igusa’s theorem).** Let $G$ be a connected BT-group over $S$ of height 2 and dimension 1. Then $G$ is versal (3.2) if and only if $h(G) = 1$; moreover, if this condition is satisfied, the monodromy representation $\rho : I = \text{Gal}_{\mathbb{Z}_p}(T_p(G)) \simeq \mathbb{Z}_p^\times$ is surjective.

**Proof.** Since $\text{Lie}(G')$ is an $\mathcal{O}_S$-module free of rank 1, the condition that $h(G) = 1$ is equivalent to that any matrix of $\varphi_{G'}$ is represented by a uniformizer of $A$. Hence the first part of this theorem follows from Proposition 4.11(ii).

We follow [Ka2, Thm. 4.3] to prove the surjectivity of $\rho$ under the assumption that $h(G) = 1$. For each integer $n \geq 1$, let

$$\rho_n : I \to \text{Aut}_{\mathbb{Z}_p^e}(G(n)(\overline{K})) \simeq \mathbb{Z}/p^n\mathbb{Z}^\times$$

be the reduction mod $p^n$ of $\rho$. $K_n$ be the subfield of $K^{\text{sep}}$ fixed by the kernel of $\rho_n$. Then $\rho_n$ induces an injective homomorphism $\text{Gal}(K_n/K) \to (\mathbb{Z}/p^n\mathbb{Z})^\times$. By taking projective limits, we are reduced to proving the surjectivity of $\rho_n$ for every $n \geq 1$. It suffices to verify that

$$| \text{Im}(\rho_n)| = |K_n : K| \geq p^{n-1}(p-1)$$

(then the equality holds automatically).
We regard $G$ as a formal group over $S$. Then by [Ka2, 3.6], there exists a
parameter $X$ of the formal group $G$ normalized by the condition that $[\xi](X) = \xi(X)$ for all $(p-1)$-th root of unity $\xi \in \mathbb{Z}_p$. For such a parameter, we have
\[
[p](X) = a_1X^p + \alpha X^{p^2} + \sum_{m \geq 2} c_m X^{p^m(1+m(p-1))} \in A[[X]],
\]
where we have $v(a_1) = h(G) = 1$ by [Ka2, 3.6.1 and 3.6.5], and $v(\alpha) = 0$, as $G$
is of height 2. For each integer $i \geq 0$, we put
\[
V^{(p^i)}(X) = a_1^{p^i}X + \alpha^{p^i}X^p + \sum_{m \geq 2} c_m^{(p^i)} X^{1+m(p-1)} \in A[[X]],
\]
then we have $[p^n](X) = V^{(p^{n-1})} \circ V^{(p^{n-2})} \circ \cdots \circ V(X^p)$. Hence each point of $G(n)(K)$ is given by a sequence $y_1, \ldots, y_n \in K_{\text{sep}}$ (or simply an element $y_n \in K_{\text{sep}}$) satisfying the equations
\[
\begin{align*}
V(y_1) &= a_1 y_1 + \alpha y_1^p + \cdots = 0; \\
V^{(p)}(y_2) &= a_1^{p} y_2 + \alpha^p y_2^p + \cdots = y_1; \\
& \vdots \\
V^{(p^{n-1})}(y_n) &= a_1^{p^{n-1}} y_n + \alpha^{p^{n-1}} y_n^p + \cdots = y_{n-1}.
\end{align*}
\]
Let $y_n \in K_{\text{sep}}$ be such that $y_1 \neq 0$. By considering the Newton polygons of
the equations above, we verify that
\[
v(y_i) = \frac{1}{p^{i-1}(p-1)} 
\text{ for } 1 \leq i \leq n.
\]
In particular, the ramification index $e(K_n/K)$ is at least $p^{n-1}(p-1)$. By the
definition of $K_n$, the Galois group $\text{Gal}(K_{\text{sep}}/K_n)$ must fix $y_n \in K_{\text{sep}}$, i.e. $K_n$
is an extension of $K(y_n)$. Therefore, we have $[K_n : K] \geq [K(y_n) : K] \geq e(K(y_n)/K) \geq p^{n-1}(p-1)$. \(\square\)

PROPOSITION 5.8. Let $G$ be a HW-cyclic BT-group over $S$ of height $c + d$ and
dimension $d$ such that $G \otimes K$ is ordinary,
\[
\mathfrak{b} = \begin{pmatrix}
0 & 0 & \cdots & 0 & -a_1 \\
1 & 0 & \cdots & 0 & -a_2 \\
0 & 1 & \cdots & 0 & -a_3 \\
\vdots & \ddots & \ddots & \vdots \\
0 & 0 & \cdots & 1 & -a_c
\end{pmatrix}
\]
be a matrix of $\varphi_G$. Put $q = p^c$, $a_{c+1} = 1$, and $P(X) = \sum_{i=0}^c a_{i+1}X^i \in A[X]$.
(i) Assume that $G$ is connected and the Hasse invariant $b(G) = 1$. Then the
representation $\varphi$ (5.6.3) is tame, $G(1)(\overline{K})$ is endowed with the structure of an
$\mathbb{F}_q$-vector space of dimension 1, and the induced action of $I_t$ is given by the
character $\theta_{q-1} : I_t \rightarrow \mathbb{F}_q^*$. (ii) Assume that $c > 1$, $v(a_i) \geq 2$ for $1 \leq i \leq c - 1$ and $v(a_c) = 1$. Then the
order of $\text{Im}(\varphi)$ is divisible by $p^{c-1}(p-1)$.
(iii) Put \( i_0 = \min_{0 \leq i \leq c}\{i; v(a_{i+1}) = 0\} \). Assume that there exists \( \alpha \in k \) such that \( v(P(\alpha)) = 1 \). Then we have \( i_0 \leq c - 1 \) and the order of \( \text{Im}(\overline{\rho}) \) is divisible by \( p^{i_0} \).

Proof. Since \( G \) is generically ordinary, we have \( a_1 \neq 0 \) by 2.11(d). Hence \( P(X) \in K[X] \) is a separable polynomial. By 4.4, \( G(1)(\overline{K}) \simeq (\text{Ker} \ V_G)(K^{\text{sep}}) \) is identified with the additive group consisting of the roots of \( P(X) \) in \( K^{\text{sep}} \).

(i) By definition of the Hasse invariant, we have \( v(a_1) = h(G) = 1 \). By 4.4(ii), the assumption that \( G \) is connected is equivalent to saying \( v(a_1) \geq 1 \) for \( 1 \leq i \leq c \). From the Newton polygon of \( P(X) \), we deduce that all the non-zero roots of \( P(X) \) in \( K^{\text{sep}} \) have the same valuation \( 1/(q - 1) \). We denote by

\[ \psi : G(1)(\overline{K}) \rightarrow V_{1/(q-1)} \]

the map which sends each root \( x \in K^{\text{sep}} \) of \( P(X) \) to the class of \( x \) in \( V_{1/(q-1)} = m_1/(q-1)/m_1^+(q-1) \). We remark that \( G(1)(\overline{K}) \) is an \( \mathbb{F}_q \)-vector space of dimension \( c \). Hence \( G(1)(\overline{K}) \) is automatically of dimension 1 over \( \mathbb{F}_q \) once we know it is an \( \mathbb{F}_q \)-vector space. By 5.3, it suffices to show that \( \psi \) is an injective \( I \)-equivariant homomorphism of groups. By 4.4(i), \( \psi \) is obviously an \( I \)-equivariant homomorphism of groups. Let \( x_0 \) be a root of \( P(X) \), and put \( Q(y) = P(x_0y) \). Then the polynomial \( Q(y) \) has the form \( Q(y) = x_0^n Q_1(y) \), where

\[ Q_1(y) = y^n + b_1 y^{n-1} + \cdots + b_{n-1} y + b_n \]

with \( b_i = a_i / x_0(q-1)^{i-1} \in K^{\text{sep}} \). We have \( v(b_i) > 0 \) for \( 2 \leq i \leq c \) and \( v(b_1) = 0 \). Let \( \overline{b}_1 \) be the class of \( b_1 \) in the residue field \( k = \mathbb{m}_0 / \mathbb{m}_0^+ \). Then the images of the roots of \( P(X) \) in \( V_{1/(q-1)} \) are \( x_0 \overline{b}_1^{1/(q-1)} \), where \( \zeta \) runs over the finite field \( \mathbb{F}_q \). Therefore, \( \psi \) is injective.

(ii) By computing the slopes of the Newton polygon of \( P(X) \), we see that \( P(X) \) has \( p^{e-1}(p-1) \) roots of valuation \( 1/(p^e - p^{e-1}) \). Let \( L \) be the sub-extension of \( K^{\text{sep}} \) obtained by adding to \( K \) all the roots of \( P(x) \). Then the ramification index \( e(L/K) \) is divisible by \( p^{e-1}(p-1) \). Let \( \overline{L} \) be the sub-extension of \( K^{\text{sep}} \) fixed by the kernel of \( \overline{\rho} \). The Galois group \( \text{Gal}(K^{\text{sep}}/\overline{L}) \) fixes the roots of \( P(x) \) by definition. Hence we have \( L \subset \overline{L} \), and \( [\text{Im}(\overline{\rho})] = [L : K] \) is divisible by \( [L : K] \); in particular, it is divisible by \( p^{e-1}(p-1) \).

(iii) Note that the relation \( i_0 \leq c - 1 \) is equivalent to saying that \( G \) is not connected by 4.4(ii). Assume conversely \( i_0 = c \), i.e. \( G \) is connected. Then we would have

\[ P(X) \equiv X^c \mod (\pi A[X]) \]

But \( v(P(\alpha)) = 1 \) implies that \( \alpha^p \in \pi A \), i.e. \( \alpha = 0 \); hence we would have \( P(\alpha) = 0 \), which contradicts the condition \( v(P(\alpha)) = 1 \). We put \( Q(X) = P(X + \alpha) = P(X) + P(\alpha) \). As \( v(P(\alpha)) = 1 \), then \( (0, 1) \) and \( (p^{i_0}, 0) \) are the first two break points of the Newton polygon of \( Q(X) \). Hence there exists \( p^{i_0} \) roots of \( Q(X) \) of valuation \( 1/p^{i_0} \). Let \( L \) be the sub-extension of \( K \) in \( K^{\text{sep}} \) generated by the roots of \( P(X) \). The ramification index \( e(L/K) \) is divisible by \( p^{i_0} \). As in the proof of (ii), if \( \overline{L} \) is the sub-extension of \( K^{\text{sep}} \)
5.9. Let $G$ be a BT-group over $S$ with connected part $G^0$, and étale part $G^{\text{ét}}$ of height $r$. We have a canonical exact sequence of $I$-modules
\begin{equation}
0 \to G^0(1)(\overline{K}) \to G(1)(\overline{K}) \to G^{\text{ét}}(1)(\overline{K}) \to 0
\end{equation}
giving rise to a class $\overline{C} \in \text{Ext}^1_{\mathbb{F}_p[I]}(G^{\text{ét}}(1)(\overline{K}), G^0(1)(\overline{K}))$, which vanishes if and only if (5.9.1) splits. Since $I$ acts trivially on $G^{\text{ét}}(1)(\overline{K})$, we have an isomorphism of $I$-modules $G^{\text{ét}}(1)(\overline{K}) \cong \mathbb{F}_p$. Recall that for any $\mathbb{F}_p[I]$-module $M$, we have a canonical isomorphism ([Sel] Chap.VII, §2)
\[ \text{Ext}^1_{\mathbb{F}_p[I]}(\mathbb{F}_p, M) \cong H^1(I, M). \]
Hence we deduce that
\begin{equation}
\overline{C} \in \text{Ext}^1_{\mathbb{F}_p[I]}(G^{\text{ét}}(1)(\overline{K}), G^0(1)(\overline{K})) 
\cong H^1(I, G^0(1)(\overline{K})).
\end{equation}

Proposition 5.10. Let $G$ be a HW-cyclic BT-group over $S$ such that $h(G) = 1$, \( \overline{\mathfrak{m}} \) (5.6.3) be the representation of $I$ on $G(1)(\overline{K})$. Then the cohomology class $\overline{C}$ does not vanish if and only if the order of the group $\text{Im}(\overline{\mathfrak{m}})$ is divisible by $p$.

First, we prove the following result on cohomology of groups.

Lemma 5.11. Let $F$ be a field, $\Gamma$ be a commutative group, and $\chi : \Gamma \to F^\times$ be a non-trivial character of $\Gamma$. We denote by $F(\chi)$ an $F$-vector space of dimension 1 endowed with an action of $\Gamma$ given by $\chi$. Then we have $H^1(\Gamma, F(\chi)) = 0$.

Proof. Let $C$ be a 1-cocycle of $\Gamma$ with values in $F(\chi)$. We prove that $C$ is a 1-coboundary. For any $g, h \in \Gamma$, we have
\begin{align*}
C(gh) &= C(g) + \chi(g)C(h), \\
C(hg) &= C(h) + \chi(h)C(g).
\end{align*}
Since $\Gamma$ is commutative, it follows from the relation $C(gh) = C(hg)$ that
\begin{equation}
(\chi(g) - 1)C(h) = (\chi(h) - 1)C(g).
\end{equation}
If $\chi(g) \neq 1$ and $\chi(h) \neq 1$, then
\[ \frac{1}{\chi(g) - 1}C(g) = \frac{1}{\chi(h) - 1}C(h). \]
Therefore, there exists $x \in F(\overline{\mathfrak{m}})$ such that $C(g) = (\chi(g) - 1)x$ for all $g \in \Gamma$ with $\chi(g) \neq 1$. If $\chi(g) = 1$, we have also $C(g) = 0 = (\chi(g) - 1)x$ by (5.11.1). This shows that $C$ is a 1-coboundary. \qed

Proof of 5.10. By 4.3(ii) and 5.4(c), the connected part $G^0$ of $G$ is HW-cyclic with $h(G^0) = h(G) = 1$. Assume that $T_p(G^0)$ has rank $\ell$ over $\mathbb{Z}_p$, and $T_p(G^{\text{ét}})$ has rank $r$. Then by 5.8(a), $G^0(1)(\overline{K})$ is an $\mathbb{F}_p$-vector space of dimension 1 with $q = p^\ell$, and the action of $I$ on $G^0(1)(\overline{K})$ factors through the character $\overline{\chi} : I \to I_q \xrightarrow{\theta_{q-1}} \mathbb{F}_q^\times$. We write $G^0(1)(\overline{K}) = \mathbb{F}_q(\overline{\chi})$ for short. If the cohomology class $\overline{C}$ is zero, then the exact sequence (5.9.1) splits, i.e. we have an isomorphism

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Yichao Tian

Documenta Mathematica 14 (2009) 397–440

fixed by the kernel of $\overline{\mathfrak{m}}$, then it is an extension of $L$. Therefore, we have $|\text{Im}(\overline{\mathfrak{m}})| = |L : K|$ is divisible by $|L : K|$, and in particular, divisible by $p^{\ell_0}$. \qed
of Galois modules $G(1)(\mathcal{K}) \simeq \mathbb{F}_q(\chi) \oplus \mathbb{F}_p$. It is clear that the group $\text{Im}(\overline{\rho})$ has order $q-1$. Conversely, if the cohomology class $\overline{C}$ is not zero, we will show that there exists an element in $\text{Im}(\overline{\rho})$ of order $p$. We choose a basis adapted to the exact sequence (5.9.1) such that the action of $g \in I$ is given by

\[(5.11.2) \quad \overline{\rho}(g) = \begin{pmatrix} \overline{\chi}(g) & \overline{C}(g) \\ 0 & 1_r \end{pmatrix},\]

where $1_r$ is the unit matrix of type $(r, r)$ with coefficients in $\mathbb{F}_p$, and the map $g \mapsto \overline{C}(g)$ gives rise to a 1-cocycle representing the cohomology class $\overline{C}$. Let $I_1$ be the kernel of $\overline{\chi} : I \to \mathbb{F}_q^\times$, $\Gamma$ be the quotient $I/I_1$, so $\overline{\chi}$ induces an isomorphism $\overline{\chi} : \Gamma \cong \mathbb{F}_q^\times$. We have an exact sequence

\[0 \to H^1(\Gamma, \mathbb{F}_q(\overline{\chi})) \xrightarrow{\text{Inf}} H^1(I, \mathbb{F}_q(\overline{\chi})) \xrightarrow{\text{Res}} H^1(I_1, \mathbb{F}_q(\overline{\chi})),\]

where “Inf” and “Res” are respectively the inflation and restriction homomorphisms in group cohomology. Since $H^1(\Gamma, \mathbb{F}_q(\overline{\chi})) = 0$ by 5.11, the restriction of the cohomology class $\overline{C}$ to $H^1(I_1, \mathbb{F}_q(\overline{\chi}))$ is non-zero. Hence there exists $h \in I_1$ such that $\overline{C}(h) \neq 0$. As we have $\overline{\chi}(h) = 1$, then

\[\overline{\rho}(h)^p = \begin{pmatrix} 1_r & p\overline{C}(h) \\ 0 & 1_r \end{pmatrix} = 1_{t+r}.\]

Thus the order of $\overline{\rho}(h)$ is $p$. \hfill $\square$

**Corollary 5.12.** Let $G$ be a $\mathbb{Q}$-cyclic B.T.-group over $S$,

\[h = \begin{pmatrix} 0 & 0 & \cdots & 0 & -a_1 \\ 1 & 0 & \cdots & 0 & -a_2 \\ 0 & 1 & \cdots & 0 & -a_3 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 & -a_c \end{pmatrix}\]

be a matrix of $\varphi_G$, $P(X) = X^{p^c} + a_c X^{p^{c-1}} + \cdots + a_1 X \in A[X]$. If $h(G) = 1$ and if there exists $a \in k \subseteq A$ such that $\varphi(P(a)) = 1$, then the cohomology class (5.9.2) is not zero, i.e. the extension of $I$-modules (5.9.1) does not split.

**Proof.** Since $\varphi(a_1) = h(G) = 1$, the integer $i_0$ defined in 5.8(iii) is at least 1. Then the corollary follows from 5.8(iii) and 5.10. \hfill $\square$

### 6. Lemmas in Group Theory

In this section, we fix a prime number $p \geq 2$ and an integer $n \geq 1$.

6.1. Recall that the general linear group $\text{GL}_n(\mathbb{Z}_p)$ admits a natural exhaustive decreasing filtration by normal subgroups

\[\text{GL}_n(\mathbb{Z}_p) \supset 1 + p M_n(\mathbb{Z}_p) \supset \cdots \supset 1 + p^m M_n(\mathbb{Z}_p) \supset \cdots,\]

where $M_n(\mathbb{Z}_p)$ denotes the ring of matrix of type $(n, n)$ with coefficients in $\mathbb{Z}_p$. We endow $\text{GL}_n(\mathbb{Z}_p)$ with the topology for which $(1 + p^m M_n(\mathbb{Z}_p))_{m \geq 1}$ form a
fundamental system of neighborhoods of $1$. Then $GL_n(\mathbb{Z}_p)$ is a complete and separated topological group.

6.2. Let $\mathcal{G}$ be a profinite group, $\rho : \mathcal{G} \to GL_n(\mathbb{Z}_p)$ be a continuous homomorphism of topological groups. By taking inverse images, we obtain a decreasing filtration $(F^m\mathcal{G}, m \in \mathbb{Z}_{\geq 0})$ on $\mathcal{G}$ by open normal subgroups:

$$F^0\mathcal{G} = \mathcal{G}, \quad \text{and} \quad F^m\mathcal{G} = \rho^{-1}(1 + p^mM_n(\mathbb{Z}_p)) \text{ for } m \geq 1.$$  

Furthermore, the homomorphism $\rho$ induces a sequence of injective homomorphisms of finite groups

$$(6.2.1) \quad \rho_0 : F^0\mathcal{G}/F^1\mathcal{G} \to GL_n(\mathbb{F}_p)$$

$$(6.2.2) \quad \rho_m : F^m\mathcal{G}/F^{m+1}\mathcal{G} \to M_n(\mathbb{F}_p), \quad \text{for } m \geq 1.$$  

**Lemma 6.3.** The homomorphism $\rho$ is surjective if and only if the following conditions are satisfied:

(i) The homomorphism $\rho_0$ is surjective.

(ii) For every integer $m \geq 1$, the subgroup $\text{Im}(\rho_m)$ of $M_n(\mathbb{F}_p)$ contains an element of the form

$$\begin{pmatrix} x & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix}$$

with $x \neq 0$; or equivalently, there exists, for every $m \geq 1$, an element $g_m \in \mathcal{G}$ such that $\rho(g_m)$ is of the form

$$\begin{pmatrix} 1 + p^m a_{1,1} & p^{m+1} a_{1,2} & \cdots & p^{m+1} a_{1,n} \\ p^{m+1} a_{2,1} & 1 + p^{m+1} a_{2,2} & \cdots & p^{m+1} a_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ p^{m+1} a_{n,1} & p^{m+1} a_{n,2} & \cdots & 1 + p^{m+1} a_{n,n} \end{pmatrix},$$

where $a_{i,j} \in \mathbb{Z}_p$ for $1 \leq i, j \leq n$ and $a_{1,1}$ is not divisible by $p$.

**Proof.** We notice first that $\rho$ is surjective if and only if $\rho_0$ is surjective for every $m \geq 0$, because $\mathcal{G}$ is complete and $GL_n(\mathbb{Z}_p)$ is separated [Bou, Chap. III §2 n°8 Cor.2 au Théo. 1]. The surjectivity of $\rho_0$ is condition (i). Condition (ii) is clearly necessary. We prove that it implies the surjectivity of $\rho_m$ for all $m \geq 1$, under the assumption of (i). First, we remark that under condition (i), if $A$ lies in $\text{Im}(\rho_m)$, then for any $U \in GL_n(\mathbb{F}_p)$ the conjugate matrix $U \cdot A \cdot U^{-1}$ lies also in $\text{Im}(\rho_m)$. In fact, let $\tilde{A}$ be a lift of $A$ in $M_n(\mathbb{Z}_p)$ and $\tilde{U} \in GL_n(\mathbb{Z}_p)$ a lift of $U$. By assumption, there exist $g, h \in \mathcal{G}$ such that $\rho(g) \equiv 1 + p^m \tilde{A} \mod (1 + p^{m+1}M_n(\mathbb{Z}_p))$ and $\rho(h) \equiv \tilde{U} \mod (1 + pM_n(\mathbb{Z}_p))$.

Therefore, we have $\rho(hgh^{-1}) \equiv (1 + p^m \tilde{U} \cdot \tilde{A} \cdot \tilde{U}^{-1}) \mod (1 + p^{m+1}M_n(\mathbb{Z}_p))$. Hence $hgh^{-1} \in F^m\mathcal{G}$ and $\rho_m(hgh^{-1}) \equiv U \cdot A \cdot U^{-1}$.

For $1 \leq i, j \leq n$, let $E_{i,j} \in M_n(\mathbb{F}_p)$ be the matrix whose $(i,j)$-th entry is 0 and the other entries are 0. The matrices $E_{i,j}(1 \leq i, j \leq n)$ form clearly...
a basis of $M_n(\mathbb{F}_p)$ over $\mathbb{F}_p$. To prove the surjectivity of $\rho_m$, we only need to verify that $E_{i,j} \in \text{Im}(\rho_m)$ for $1 \leq i, j \leq n$, because $\text{Im}(\rho_m)$ is an $\mathbb{F}_p$-subspace of $M_n(\mathbb{F}_p)$. By assumption, we have $E_{1,1} \in \text{Im}(\rho_m)$. For $2 \leq i \leq n$, we put $U_i = E_{1,i} - E_{i,1} + \sum_{j \neq 1, i} E_{j,j}$. Then we have $U_i \in GL_n(\mathbb{Z}_p)$ and $U_i \cdot E_{1,1} \cdot U_i^{-1} = E_{i,i} \in \text{Im}(\rho_m)$. For $1 \leq i < j \leq n$, we put $U_{i,j} = I + E_{i,j}$ where $I$ is the unit matrix. Then we have $U_{i,j} \cdot E_{i,i} \cdot U_{i,j}^{-1} = E_{i,i} + E_{i,j} \in \text{Im}(\rho_m)$, and hence $E_{i,j} \in \text{Im}(\rho_m)$. This completes the proof.

\[ \square \]

**Remark 6.4.** By using the arguments in [Se2, Chap. IV 3.4 Lemma 3], we can prove the following stronger form of Lemma 6.3: If $p = 2$, condition (i) and (ii) for $m = 1, 2$ are sufficient to guarantee the surjectivity of $\rho$; if $p \geq 3$, then (i) and (ii) just for $m = 1$ suffice already.

A subgroup $C$ of $GL_n(\mathbb{F}_p)$ is called a non-split Cartan subgroup, if the subset $C \cup \{0\}$ of the matrix algebra $M_n(\mathbb{F}_p)$ is a field isomorphic to $\mathbb{F}_{p^n}$; such a group is cyclic of order $p^n - 1$.

**Lemma 6.5.** Assume that $n \geq 2$. We denote by $H$ the subgroup of $GL_n(\mathbb{F}_p)$ consisting of all the elements of the form \( \begin{pmatrix} A & b \\ 0 & 1 \end{pmatrix} \), where $A \in GL_{n-1}(\mathbb{F}_p)$ and \( b \equiv \begin{pmatrix} b_1 \\ \vdots \\ b_{n-1} \end{pmatrix} \) with $b_i \in \mathbb{F}_p (1 \leq i \leq n - 1)$. Let $G$ be a subgroup of $GL_n(\mathbb{F}_p)$.

Then $G = GL_n(\mathbb{F}_p)$ if and only if $G$ contains $H$ and a non-split Cartan subgroup of $GL_n(\mathbb{F}_p)$.

**Proof.** The “only if” part is clear. For the “if” part, let $C$ be a non-split Cartan subgroup contained in $G$. For a finite group $A$, we denote by $|A|$ its order. An easy computation shows that $|GL_n(\mathbb{F}_p)| = |H| \cdot |C|$. So we just need to prove that $U \cap C = \{1\}$; since then we will have $|GL_n(\mathbb{F}_p)| = |G|$, hence $G = GL_n(\mathbb{F}_p)$. Let $g \in H \cap C$, and $P(T) \in \mathbb{F}_p[T]$ be its characteristic polynomial. We fix an isomorphism $C \simeq \mathbb{F}_{p^n}$, and let $\zeta \in \mathbb{F}_{p^n}$ be the element corresponding to $g$. We have $P(T) = \prod_{\sigma \in \text{Gal}(\mathbb{F}_{p^n}/\mathbb{F}_p)} (T - \sigma(\zeta))$ in $\mathbb{F}_{p^n}[T]$. On the other hand, the fact that $g \in H$ implies that $(T - 1)$ divides $P(T)$. Therefore, we get $\zeta = 1$, i.e. $g = 1$.

\[ \square \]

**Remark 6.6.** E. Lau point out the following strengthened version of 6.5: When $n \geq 3$, a subgroup $G \subset GL_n(\mathbb{F}_p)$ coincides with $GL_n(\mathbb{F}_p)$ if and only if $G$ contains a non-split Cartan subgroup and the subgroup \( \begin{pmatrix} GL_{n-1}(\mathbb{F}_p) & 0 \\ 0 & 1 \end{pmatrix} \). This can be used to simplify the induction process in the proof of Theorem 7.3 when $n \geq 3$. 

\[ \text{Documenta Mathematica 14} \ (2009) \ 397–440 \]
7. Proof of Theorem 1.3 in the One-dimensional Case

7.1. We start with a general remark on the monodromy of BT-groups. Let $X$ be a scheme, $G$ be an ordinary BT-group over a scheme $X$, $G^\text{et}$ be its étale part (2.10.1). If $\eta$ is a geometric point of $X$, we denote by

\[ T_\rho(G, \eta) = \lim_{n} G(n)(\eta) = \lim_{n} G^\text{et}(n)(\eta) \]

the Tate module of $G$ at $\eta$, and by $\rho(G)$ the monodromy representation of $\pi_1(X, \eta)$ on $T_\rho(G, \eta)$. Let $f : Y \to X$ be a morphism of schemes, $\xi$ be a geometric point of $Y$, $G_Y = G \times_X Y$. Then by the functoriality, we have a commutative diagram

\[
\begin{array}{ccc}
\pi_1(Y, \xi) & \xrightarrow{\pi_1(f)} & \pi_1(X, f(\xi)) \\
\rho(G_Y) & \downarrow & \rho(G) \\
\text{Aut}_{Z_p}(T_\rho(G_Y, \xi)) & = & \text{Aut}_{Z_p}(T_\rho(G, f(\xi)))
\end{array}
\]

In particular, the monodromy of $G_Y$ is a subgroup of the monodromy of $G$. In the sequel, diagram (7.1.1) will be refereed as the functoriality of monodromy for the BT-group $G$ and the morphism $f$.

7.2. Let $k$ be an algebraically closed field of characteristic $p > 0$, $G$ be the unique connected BT-group over $k$ of dimension 1 and height $n + 1 \geq 2$ (4.10). We denote by $S$ the algebraic local moduli of $G$ in characteristic $p$, by $G$ the universal deformation of $G$ over $S$, and by $U$ the ordinary locus of $G$ over $S$ (3.8). Recall that $S$ is affine of ring $R \simeq k[[t_1, \ldots, t_n]]$ (3.7), and that $G$ and $G$ are HW-cyclic (cf. 4.3(i) and 4.10). Let $\eta$ be a geometric point of $U$ over its generic point. We put

\[ T_\rho(G, \eta) = \lim_{m \in \mathbb{Z}_{\geq 1}} G(m)(\eta) \]

to be the Tate module of $G$ at the point $\eta$. This is a free $\mathbb{Z}_p$-module of rank $n$. We have the monodromy representation

\[ \rho_n : \pi_1(U, \eta) \to \text{Aut}_{\mathbb{Z}_p}(T_\rho(G, \eta)) \simeq \text{GL}_n(\mathbb{Z}_p). \]

The following is the one-dimensional case of Theorem 1.3.

**Theorem 7.3.** Under the above assumptions, the homomorphism $\rho_n$ is surjective for $n \geq 1$.

7.4. First, we assume $n \geq 2$. By Proposition 4.11(ii), we may assume that

\[
\mathfrak{b} = \begin{pmatrix}
0 & 0 & \cdots & 0 & -t_1 \\
1 & 0 & \cdots & 0 & -t_2 \\
0 & 1 & \cdots & 0 & -t_3 \\
& \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \cdots & 1 & -t_n
\end{pmatrix}
\]
is a matrix of the Hasse-Witt map \( \varphi_G \). Let \( p \) be the prime ideal of \( R \) generated by \( t_1, \ldots , t_{n-1} \). Then the closed subscheme of \( S \) defined by \( p \) is just the locus where the \( p \)-rank of \( G \) is \( \leq 1 \) by 4.4(ii). Let \( K_0 \simeq k((t_n)) \) be the fraction field of \( R/p \), \( R' = \overline{R}_p \) be the completion of the localization of \( R \) at \( p \), and \( \mathcal{G}_R = G \otimes_R R' \). Since the natural map \( R \to R' \) is injective, for any \( a \in R \), we will denote also by \( a \) its image in \( R' \). Since the Hasse-Witt map commutes with base change, the image of \( \eta \) in \( M_{n \times n}(R') \), denoted also by \( \eta \), is a matrix of \( \varphi_{\mathcal{G}_R} \). We see easily that the etale part of \( \mathcal{G}_R \) has height 1 and its connected part \( \mathcal{G}_{R_0}^\text{et} \) has height \( n \). We have an exact sequence of \( \text{BT} \)-groups over \( R' \)

\[
(7.4.2) \quad 0 \to \mathcal{G}_R^\circ \to \mathcal{G}_R^\text{et} \to \mathcal{G}_{R_0}^\text{et} \to 0.
\]

We fix an imbedding \( i : K_0 \to \overline{K}_0 \) of \( K_0 \) into an algebraically closed field. Put \( \mathcal{G}_{K_0}^* = \mathcal{G}_R^* \otimes_{\overline{K}_0} \overline{R}' \) for \( *= \emptyset, \text{et}, \circ \). We have \( \mathcal{G}_{K_0}^\circ \simeq \mathbb{Q}_p/\mathbb{Z}_p \), and \( \mathcal{G}_{K_0}^* \) is the unique connected one-dimensional \( \text{BT} \)-group over \( K_0 \) of height \( n \) (cf. 4.10). We put \( \overline{R} = \overline{K}_0[[x_1, \ldots , x_{n-1}]] \)

\[
(7.4.3) \quad \Sigma = \{ \text{ring homomorphisms } \sigma : R' \to \overline{R}' \text{ lifting } R' \to K_0 \to \overline{K}_0 \}
\]

Let \( \sigma \in \Sigma \). We deduce from (7.4.2) by base change an exact sequence of \( \text{BT} \)-groups over \( \overline{R}' \)

\[
(7.4.4) \quad 0 \to \mathcal{G}_{R_0}^\circ_{R', \sigma} \to \mathcal{G}_{R, \sigma}^\circ \to \mathcal{G}_{R, \sigma}^\text{et} \to 0,
\]

where we have put \( \mathcal{G}_{R, \sigma}^\circ = \mathcal{G}_R^* \otimes_{\overline{K}_0} \overline{R}' \) for \( *= \emptyset, \text{et}, \circ \). Due to the henselian property of \( \overline{R}' \), the isomorphism \( \mathcal{G}_{K_0}^\circ \simeq \mathbb{Q}_p/\mathbb{Z}_p \) lifts uniquely to an isomorphism \( \mathcal{G}_{K_0}^\circ \simeq \mathbb{Q}_p/\mathbb{Z}_p \). Assume that \( \mathcal{G}_{R, \sigma}^\circ \) is generically ordinary over \( \overline{S}' = \text{Spec}(\overline{R}') \)

Let \( \overline{U}'_\sigma \subset \overline{S}' \) be its ordinary locus, and \( \overline{\pi} \) be a geometric point over the generic point of \( \overline{U}'_\sigma \). The exact sequence (7.4.4) induces an exact sequence of Tate modules

\[
(7.4.5) \quad 0 \to T_p(\mathcal{G}_{R_0, \sigma}^\circ, \overline{\pi}) \to T_p(\mathcal{G}_{R, \sigma}^\circ, \overline{\pi}) \to T_p(\mathcal{G}_{R, \sigma}^\text{et}, \overline{\pi}) \to 0
\]

compatible with the actions of \( \pi_1(\overline{U}'_\sigma, \overline{\pi}) \). Since we have \( T_p(\mathcal{G}_{R, \sigma}^\text{et}, \overline{\pi}) \simeq T_p(\mathbb{Q}_p/\mathbb{Z}_p, \overline{\pi}) = \mathbb{Z}_p \), this determines a cohomology class

\[
(7.4.6) \quad C_\sigma \in \text{Ext}_1^{\text{et}}(\mathbb{Z}_p, T_p(\mathcal{G}_{U_0, \sigma}^\circ, \overline{\pi})) \approx H^1(\pi_1(\overline{U}'_\sigma, \overline{\pi}), \mathcal{G}_{R, \sigma}^\circ) \subset H^1(\pi_1(\overline{U}'_\sigma, \overline{\pi}), \mathcal{G}_{R, \sigma}^\circ(1))
\]

We consider also the “mod-\( p \) version” of (7.4.5)

\[
0 \to \mathcal{G}_{R, \sigma}^\circ(1)(\overline{\pi}) \to \mathcal{G}_{R, \sigma}^\circ(1)(\overline{\pi}) \to \mathbb{F}_p \to 0
\]

which determines a cohomology class

\[
(7.4.7) \quad \overline{C}_\sigma \in \text{Ext}_1^{\text{et}}(\mathbb{Z}_p, T_p(\mathcal{G}_{R_0, \sigma}^\circ(1), \overline{\pi})) \approx H^1(\pi_1(\overline{U}'_\sigma, \overline{\pi}), \mathcal{G}_{R, \sigma}^\circ(1))
\]

It is clear that \( \overline{C}_\sigma \) is the image of \( C_\sigma \) by the canonical reduction map

\[
H^1(\pi_1(\overline{U}'_\sigma, \overline{\pi}), T_p(\mathcal{G}_{R_0, \sigma}^\circ(1))) \to H^1(\pi_1(\overline{U}'_\sigma, \overline{\pi}), \mathcal{G}_{R, \sigma}^\circ(1))
\]
Lemma 7.5. Under the above assumptions, there exist \( \sigma_1, \sigma_2 \in \Sigma \) satisfying the following properties:
(i) We have \( \mathcal{G}_{\mathbb{R}, \sigma_1} = \mathcal{G}_{\mathbb{R}, \sigma_2} \), and it is the universal deformation of \( \mathcal{G}_{\mathbb{K}, \sigma_1} \).
(ii) We have \( C_{\sigma_1} = 0 \) and \( C_{\sigma_2} \neq 0 \).

Before proving this lemma, we prove first Theorem 7.3.

Proof of 7.3. First, we notice that the monodromy of a BT-group is independent of the base point. So we can change \( \overline{\boldsymbol{T}} \) to any geometric point of \( \mathcal{U} \) when discussing the monodromy of \( \mathcal{G} \). We make an induction on the codimension \( n = \dim(G^\sigma) \). The case of \( n = 1 \) is proved in Theorem 5.7. Assume that \( n \geq 2 \) and the theorem is proved for \( n - 1 \). We denote by
\[
\overline{\rho}_n : \pi_1(U, \overline{T}) \to \text{Aut}_{\mathbb{F}_p}(G(1)(\overline{T})) \simeq \text{GL}_n(\mathbb{F}_p)
\]
the reduction of \( \rho_n \) modulo \( p \). By Lemma 6.3 and 6.5, to prove the surjectivity of \( \rho_n \), we only need to verify the following conditions:
(a) \( \text{Im}(\overline{\rho}_n) \) contains a non-split Cartan subgroup of \( \text{GL}_n(\mathbb{F}_p) \);
(b) \( \text{Im}(\rho_n) \) contains the subgroup \( H \subset \text{GL}_n(\mathbb{Z}_p) \) consisting of all the elements of the form \( \begin{pmatrix} B & b \\ 0 & 1 \end{pmatrix} \) for \( B \in \text{GL}_{n-1}(\mathbb{Z}_p) \) and \( b \in M_{1 \times 1}(\mathbb{Z}_p) \).

For condition (a), let \( A = k[[\pi]] \), \( T = \text{Spec}(A) \). \( \xi \) be its generic point, \( \overline{\xi} \) be a geometric point over \( \xi \), and \( I = \text{Gal}(\overline{\xi}/\xi) \) be the absolute Galois group over \( \xi \). We keep the notations of 7.4. Let \( j^*: R \to A \) be the homomorphism of \( k \)-algebras such that \( j^*(t) = \pi \) and \( j^*(t_i) = 0 \) for \( 2 \leq i \leq n \). We denote by \( f : T \to S \) the corresponding morphism of schemes, and put \( G_T = G \times_A T \). By the functoriality of the size of the Hasse-Witt maps,
\[
\begin{pmatrix}
0 & 0 & \cdots & 0 & -\pi \\
1 & 0 & \cdots & 0 & 0 \\
\vdots & \ddots & \vdots & \vdots & \vdots \\
0 & 0 & \cdots & 1 & 0
\end{pmatrix}
\]
is a matrix of \( \varphi_{G_T} \). By definition 5.4, the Hasse invariant of \( G_T \) is \( h(G) = 1 \).

Hence \( G_T \) is generically ordinary; so \( f(\xi) \in U \). Let
\[
\overline{\overline{\rho}}_T : I = \text{Gal}(\overline{\xi}/\xi) \to \text{Aut}_{\mathbb{F}_p}(G_T(1)(\overline{\xi}))
\]
be the mod-\( p \) monodromy representation attached to \( G_T \). Proposition 5.8(i) implies that \( \text{Im}(\overline{\rho}_T) \) is a non-split Cartan subgroup of \( \text{GL}_n(\mathbb{F}_p) \). On the other hand, by the functoriality of monodromy, we get \( \text{Im}(\overline{\rho}_T) \subset \text{Im}(\overline{\rho}_n) \). This verifies condition (a).

To check condition (b), we consider the constructions in 7.4. Let \( \mathcal{S}' = \text{Spec}(R') \), \( f : \mathcal{S}' \to \mathcal{S} \) be the morphism of schemes corresponding to the natural ring homomorphism \( R \to R' \), \( U' \) be the ordinary locus of \( \mathcal{G}_{R'} \), and \( \overline{\xi} \) be a geometric point of \( U' \). From (7.4.2), we deduce an exact sequence of Tate modules
\[
0 \to T_p(\mathcal{G}_{\overline{\mathbb{R}}, \overline{\xi}}) \to T_p(\mathcal{G}_{\mathbb{R}, \overline{\xi}}) \to T_p(\mathcal{G}_{\mathbb{R}, \overline{\xi}}) \to 0.
\]
Let $\rho_{\mathfrak{g}} : \pi_1(U', \overline{\mathfrak{z}}) \to \text{Aut}_{\mathbb{Z}_p}(T_p(\mathfrak{g}_{R'} \otimes \overline{\mathfrak{z}})) \simeq \text{GL}_n(\mathbb{Z}_p)$ be the monodromy representation of $\mathfrak{g}_{R'}$. Under any basis of $T_p(\mathfrak{g}_{R'} \otimes \overline{\mathfrak{z}})$ adapted to (7.5.1), the action of $\pi_1(U', \overline{\mathfrak{z}})$ on $T_p(\mathfrak{g}_{R'} \otimes \overline{\mathfrak{z}})$ is given by

$$\rho_{\mathfrak{g}'} : g \in \pi_1(U', \overline{\mathfrak{z}}) \mapsto \left( \begin{array}{cc} \rho_{\mathfrak{g}'}(g) & * \\ 0 & \rho_{\mathfrak{g}'}(g) \end{array} \right)$$

where $g \mapsto \rho_{\mathfrak{g}'}(g) \in \text{GL}_{n-1}(\mathbb{Z}_p)$ (resp. $g \mapsto \rho_{\mathfrak{g}'}(g) \in \mathbb{Z}_p$) gives the action of $\pi_1(U', \overline{\mathfrak{z}})$ on $T_p(\mathfrak{g}_{R'} \otimes \overline{\mathfrak{z}})$ (resp. on $T_p(\mathfrak{g}_{R'} \otimes \overline{\mathfrak{z}})$). Note that $f(U') \subset \mathfrak{U}$. So by the functoriality of monodromy, we get $\text{Im}(\rho_{\mathfrak{g}'}) \subset \text{Im}(\rho_{\mathfrak{g}})$. To complete the proof of Theorem 7.3, it suffices to check condition (b) with $\rho_{\mathfrak{g}}$ replaced by $\rho_{\mathfrak{g}'}$, under the induction hypothesis that 7.3 is valid for $n - 1$. Let $\sigma_1, \sigma_2 : R' \to \overline{R}'$ be the homomorphisms given by 7.5. For $i = 1, 2$, we denote by $f_i : \overline{S}' = \text{Spec}(\overline{R}') \to S' = \text{Spec}(R')$ the morphism of schemes corresponding to $\sigma_i$, and put $\mathfrak{g}_i = \mathfrak{g}_{R', \sigma_i} = \mathfrak{g}_{R_{\times \sigma}, \overline{R}}$ to simply the notations. By condition 7.5(i), we can denote by $\mathfrak{g}^\circ$ the common connected component of $\mathfrak{g}_1$ and $\mathfrak{g}_2$. Let $\overline{U}' \subset \overline{S}'$ be the ordinary locus of $\mathfrak{g}^\circ$. Then we have $f_i(\overline{U}') \subset U'$ for $i = 1, 2$. Let $\overline{\mathfrak{m}}$ be a geometric point over the generic point of $\overline{U}'$. We have an exact sequence of Tate modules

$$(7.5.2) \quad 0 \to T_{\mathfrak{g}}(\overline{\mathfrak{m}}) \to T_{\mathfrak{g}_i}(\overline{\mathfrak{m}}) \to T_{\mathfrak{g}_i}(\mathbb{Q}_p / \mathbb{Z}_p, \overline{\mathfrak{m}}) \to 0$$

compatible with the actions of $\pi_1(U', \overline{\mathfrak{m}})$. We denote by $\rho_{\mathfrak{g}_i} : \pi_1(U', \overline{\mathfrak{m}}) \to \text{Aut}_{\mathbb{Z}_p}(T_{\mathfrak{g}_i}(\overline{\mathfrak{m}})) \simeq \text{GL}_n(\mathbb{Z}_p)$ the monodromy representation of $\mathfrak{g}_i$. In a basis adapted to (7.5.2), the action of $\pi_1(U', \overline{\mathfrak{m}})$ on $T_{\mathfrak{g}_i}(\overline{\mathfrak{m}})$ is given by

$$\rho_{\mathfrak{g}_i} : g \mapsto \left( \begin{array}{cc} \rho_{\mathfrak{g}_i}(g) & C_{\sigma_i}(g) \\ 0 & 1 \end{array} \right),$$

where $\rho_{\mathfrak{g}_i} : \pi_1(U', \overline{\mathfrak{m}}) \to \text{GL}_{n-1}(\mathbb{Z}_p)$ is the monodromy representation of $\mathfrak{g}^\circ$, and the cohomology class in $H^1(\pi_1(U', \overline{\mathfrak{m}}), T_{\mathfrak{g}_i}(\mathfrak{g}^\circ))$ given by $g \mapsto C_{\sigma_i}(g)$ is nothing but the class defined in (7.4.6). By 7.5(i) and the induction hypothesis, $\rho_{\mathfrak{g}_i}$ is surjective. Since the cohomology class $C_{\sigma_i} = 0$ by 7.5(ii), we may assume $C_{\sigma_i}(g) = 0$ for all $g \in \pi_1(U', \overline{\mathfrak{m}})$. Therefore $\text{Im}(\rho_{\mathfrak{g}_i})$ contains all the matrix of the form $\left( \begin{array}{cc} B & 0 \\ 0 & 1 \end{array} \right)$ with $B \in \text{GL}_{n-1}(\mathbb{Z}_p)$. By the functoriality of monodromy, $\text{Im}(\rho_{\mathfrak{g}_i})$ contains $\text{Im}(\rho_{\mathfrak{g}'_i})$. Hence we have

$$(7.5.3) \quad \left( \begin{array}{cc} \text{GL}_{n-1}(\mathbb{Z}_p) & 0 \\ 0 & 1 \end{array} \right) \subset \text{Im}(\rho_{\mathfrak{g}_i}) \subset \text{Im}(\rho_{\mathfrak{g}'_i}).$$

On the other hand, since the cohomology class $\overline{C}_{\sigma_2} \neq 0$, there exists a $g \in \pi_1(U', \overline{\mathfrak{m}})$ such that $b_2 = \overline{C}_{\sigma_2}(g) \neq 0$. Hence the matrix $\rho_{\mathfrak{g}_2}(g)$ has the form $\left( \begin{array}{cc} B_2 & b_2 \\ 0 & 1 \end{array} \right)$ such that $B_2 \in \text{GL}_{n-1}(\mathbb{Z}_p)$ and the image of $b_2 \in M_{1 \times n-1}(\mathbb{Z}_p)$.
in $M_{1 \times n-1}(\mathbb{F}_p)$ is non-zero. By the functoriality of monodromy, we have 
$\Im(\rho_{\tilde{\eta}^j}) \subset \Im(\rho_{\tilde{\eta}^{i_j}})$; in particular, we have $(B_2 \ b_2) \in \Im(\rho_{\tilde{\eta}^{i_j}})$. In view of (7.5.3), we get
\[(7.5.4) \quad \left( \begin{array}{cc}
GL_{n-1}(\mathbb{Z}_p) & 0 \\
0 & 1
\end{array} \right) \left( \begin{array}{cc}
B_2 & b_2 \\
0 & 1
\end{array} \right) \left( \begin{array}{cc}
GL_{n-1}(\mathbb{Z}_p) & 0 \\
0 & 1
\end{array} \right) \subset \Im(\rho_{\tilde{\eta}^{i_j}}).
\]
But the subset of $GL_n(\mathbb{Z}_p)$ on the left hand side is just the subgroup $H$ described in condition (b). Therefore, condition (b) is verified for $\rho_{\tilde{\eta}^{i_j}}$, and the proof of 7.3 is complete.

The rest of this section is dedicated to the proof of Lemma 7.5.

**Lemma 7.6.** Let $k$ be an algebraically closed field of characteristic $p > 0$, $A$ be a noetherian henselian local $k$-algebra with residue field $k$, $G$ be a $BT$-group over $A$, and $G^{\text{ét}}$ be its étale part. Put

$$\text{Lie}(G^\vee)^{\varphi = 1} = \{ x \in \text{Lie}(G^\vee) \text{ such that } \varphi_G(x) = x \}.$$ 

Then $\text{Lie}(G^\vee)^{\varphi = 1}$ is an $\mathbb{F}_p$-vector space of dimension equal to the rank of $\text{Lie}(G^{\text{ét}})$, and the $A$-submodule $\text{Lie}(G^{\text{ét}})$ of $\text{Lie}(G^\vee)$ is generated by $\text{Lie}(G^\vee)^{\varphi = 1}$.

**Proof.** Let $r$ be the rank of $\text{Lie}(G^{\text{ét}})$, $G^o$ be the connected part of $G$, and $s$ be the height of $\text{Lie}(G^{\text{ét}})$. We have an exact sequence of $A$-modules

$$0 \rightarrow \text{Lie}(G^{\text{ét}}) \rightarrow \text{Lie}(G^\vee) \rightarrow \text{Lie}(G^0) \rightarrow 0,$$

compatible with Hasse-Witt maps. We choose a basis of $\text{Lie}(G^\vee)$ adapted to this exact sequence, so that $\varphi_G$ is expressed by a matrix of the form

$$\left( \begin{array}{cc}
U & W \\
0 & V
\end{array} \right)$$

with $U \in M_{r \times r}(A)$, $V \in M_{s \times s}(A)$, and $W \in M_{r \times s}(A)$. An element of $\text{Lie}(G^\vee)^{\varphi = 1}$ is given by a vector $(x \ y)$, where $x = (x_1 \ldots x_r)$ and $y = (y_1 \ldots y_s)$ with $x_i, y_j \in A$, satisfying

$$(7.6.1) \quad \left( \begin{array}{cc}
U & W \\
0 & V
\end{array} \right) \cdot \left( \begin{array}{c}
x^{(p)} \\
y^{(p)}
\end{array} \right) = \left( \begin{array}{c}
x \\
y
\end{array} \right) \Leftrightarrow \left\{ \begin{array}{c}
U \cdot x^{(p)} + W \cdot y^{(p)} = x \\
V \cdot y^{(p)} = y
\end{array} \right.,$$

where $x^{(p)}$ (resp. $y^{(p)}$) is the vector obtained by applying $a \mapsto a^p$ to each $x_i (1 \leq i \leq r)$ (resp. $y_j (1 \leq j \leq s)$). By 2.9, the Hasse-Witt map of the special fiber of $G^o$ is nilpotent. So there exists an integer $N \geq 1$ such that $\varphi_G^N(\text{Lie}(G^{\text{ét}})) \subset m_A \cdot \text{Lie}(G^{\text{ét}})$, i.e. we have $V \cdot V^{(p)} \ldots V^{(p^{N-1})} \equiv 0 \pmod{m_A}$. From the equation $V \cdot y^{(p)} = y$, we deduce that

$$y = V \cdot V^{(p)} \ldots V^{(p^{N-1})} \cdot y^{(p^N)} \equiv 0 \pmod{m_A}.$$
But this implies that \( y^{(p^N)} \equiv 0 \pmod{\mathfrak{m}_A^{p^N}} \). Hence we get \( y = V \cdot y^{(p)} \equiv 0 \pmod{\mathfrak{m}_A^{p^N+1}} \). Repeting this argument, we get finally \( y \equiv 0 \pmod{\mathfrak{m}_A^k} \) for all integers \( k \geq 1 \), so \( y = 0 \). This implies that \( \text{Lie}(G^\nu)^{\nu = 1} \subset \text{Lie}(G^\nu) \), and the equation (7.6.1) is simplified as \( U \cdot x^{(p)} = x \). Since the linearization of \( \varphi_{G^\nu} \) is bijective by 2.11, we have \( U \in \text{GL}_r(A) \). Let \( \overline{U} \) be the image of \( U \) in \( \text{GL}_r(k) \), and \( \text{Sol} \) be the solutions of the equation \( U \cdot x^{(p)} = x \). As \( k \) is algebraically closed, \( \text{Sol} \) is an \( \mathbb{F}_p \)-space of dimension \( r \), and \( \text{Lie}(G^\nu) \otimes k \) is generated by \( \text{Sol} \) (cf. [Ka2, Prop. 4.1]). By the henselian property of \( A \), every elements in \( \text{Sol} \) lifts uniquely to a solution of \( U \cdot x^{(p)} = x \), i.e. the reduction map \( \text{Lie}(G^\nu)^{\nu = 1} \overset{\xi}{\to} \text{Sol} \) is bijective. By Nakayama's lemma, \( \text{Lie}(G^\nu)^{\nu = 1} \) generates the \( A \)-module \( \text{Lie}(G^\nu) \).

7.7. We keep the notations of 7.4. Let \( \text{Comp}_{R^\nu} \) be the category of noetherian complete local \( K_0 \)-algebras with residue field \( K_0 \), \( D_{\mathfrak{m}_{R^\nu}} \) (resp. \( D_{\mathfrak{m}_{R^\nu}^c} \)) be the functor which associates to every object \( A \) of \( \text{Comp}_{R^\nu} \) the set of isomorphism classes of deformations of \( \mathfrak{G}_{R^\nu} \) (resp. \( \mathfrak{G}_{R^\nu}^c \)) of \( \mathfrak{G}_{R^\nu} \). If \( A \) is an object in \( \text{Comp}_{R^\nu} \) and \( G \) is a deformation of \( \mathfrak{G}_{R^\nu} \) (resp. \( \mathfrak{G}_{R^\nu}^c \)) over \( A \), we denote by \( [G] \) its isomorphic class in \( D_{\mathfrak{m}_{R^\nu}}(A) \) (resp. in \( D_{\mathfrak{m}_{R^\nu}^c}(A) \)).

**Lemma 7.8.** Let \( S \) be the set defined in (7.4.3).

(i) The morphism of sets \( \Phi : S \to D_{\mathfrak{m}_{R^\nu}}(R^\nu) \) given by \( \sigma \mapsto [\mathfrak{G}_{R^\nu}^\nu, \sigma] \) is bijective.

(ii) Let \( \sigma \in S \). Then there exists a basis of \( \text{Lie}(\mathfrak{G}_{R^\nu}^\nu, \sigma) \) such that \( \varphi_{\mathfrak{G}_{R^\nu}^\nu, \sigma} \) is represented by a matrix of the form

\[
\begin{pmatrix}
0 & \cdots & 0 & a_1 \\
1 & \cdots & 0 & a_2 \\
\vdots & \ddots & \vdots & \vdots \\
0 & \cdots & 1 & a_{n-1}
\end{pmatrix}
\]

with \( a_i \equiv a \cdot \sigma(t_i) \pmod{\mathfrak{m}_R^{2^n}} \) for \( 1 \leq i \leq n-1 \), where \( a \in \overline{R}^\times \) and \( \mathfrak{m}_R \) is the maximal ideal of \( R^\nu \). In particular, \( \mathfrak{G}_{R^\nu}^\nu, \sigma \) is the universal deformation of \( \mathfrak{G}_{R^\nu}^\nu, \sigma \).

**Proof.** (i) We begin with a remark on the Kodaira-Spencer map of \( \mathfrak{G}_{R^\nu} \). Let \( \mathcal{E}_{S/k} = \text{Hom}_{\mathcal{O}_S}(\Omega_{S/k}^1, \mathcal{O}_S) \) be the tangent sheaf of \( S \). Since \( G \) is universal, the Kodaira-Spencer map (3.2.2)

\[
\text{Kod} : \mathcal{T}_{S/k} \to \text{Hom}_{\mathcal{O}_S}(\Omega_{S/k}^1, \text{Lie}(G^\nu))
\]

is an isomorphism. By functoriality, this induces an isomorphism of \( R^\nu \)-modules

\[
\text{Kod}_{R^\nu} : T_{R^\nu/k} \overset{\sim}{\to} \text{Hom}_{R^\nu}(\mathfrak{m}_{R^\nu}^{p^N+1}, \text{Lie}(G^\nu))
\]

where \( T_{R^\nu/k} = \text{Hom}_{R^\nu}(\Omega_{R^\nu/k}^1, R^\nu) = \Gamma(S, \mathcal{T}_{S/k}) \otimes_R R^\nu \).

For each integer \( \nu \geq 0 \), we put \( \overline{R}^\nu = \overline{R}/\mathfrak{m}_R^{2^{\nu+1}} \), \( \Sigma_\nu \) to be the set of liftings of \( R \to K_0 \to \overline{R}^\nu \) to \( R \to \overline{R}^\nu \), and \( \Phi_\nu : \Sigma_\nu \to D_{\mathfrak{m}_{R^\nu}}(R^\nu) \) to be the morphism of
sets \( \sigma_\nu \mapsto [\mathcal{G}_{R'} \otimes_{\sigma_{\nu}} \overline{R}_{\nu}^'] \). We prove by induction on \( \nu \) that \( \Phi_\nu \) is bijective for all \( \nu \geq 0 \). This will complete the proof of (i). For \( \nu = 0 \), the claim holds trivially. Assume that it holds for \( \nu - 1 \) with \( \nu \geq 1 \). We have a commutative diagram

\[
\begin{array}{ccc}
\Sigma_\nu & \xrightarrow{\Phi_\nu} & \mathcal{D}_{\varphi_{R_0}}(\overline{R}_{\nu}^') \\
\downarrow & & \downarrow \\
\Sigma_{\nu-1} & \xrightarrow{\Phi_{\nu-1}} & \mathcal{D}_{\varphi_{R_0}}(\overline{R}_{\nu-1}'),
\end{array}
\]

where the vertical arrows are the canonical reductions, and the lower arrow is an isomorphism by induction hypothesis. Let \( \tau \) be an arbitrary element of \( \Sigma_{\nu-1} \). We denote by \( \Sigma_{\nu,\tau} \subset \Sigma_\nu \) the preimage of \( \tau \), and by \( \mathcal{D}_{\Phi_{\nu-1}(\tau)}(\overline{R}_{\nu}^') \subset \mathcal{D}_{\varphi_{R_0}}(\overline{R}_{\nu}^') \) the preimage of \( \Phi_{\nu-1}(\tau) \). It suffices to prove that \( \Phi_\nu \) induces a bijection between \( \Sigma_{\nu,\tau} \) and \( \mathcal{D}_{\Phi_{\nu-1}(\tau)}(\overline{R}_{\nu}^') \). Let \( I_\nu = m_{R_\nu}^\nu / m_{R_\nu}^{\nu+1} \) be the ideal of the reduction map \( \overline{R}_{\nu}^' \rightarrow \overline{R}_{\nu-1}^' \). By \( \text{EGA}, \; \text{0}_{1V} \; \text{21.2.5 and 21.9.4} \), we have \( \Omega_{R_\nu/k}^1 \cong \Omega_{R_\nu/k}^1 \) and they are free over \( A \) of rank \( n \). By \( \text{EGA}, \; \text{0}_{1V} \; \text{20.1.3j} \), \( \Sigma_{\nu,\tau} \) is a (nonempty) homogenous space under the group

\[
\text{Hom}_{R_0}(\Omega_{R_\nu/k}^1 \otimes_{R'} K_0, I_\nu) = T_{R'/k} \otimes_{R'} I_\nu.
\]

On the other hand, according to 3.5(i), \( \mathcal{D}_{\Phi_{\nu-1}(\tau)}(\overline{R}_{\nu}^') \) is a homogenous space under the group

\[
\text{Hom}_{R_0}(\omega_{R_0}, \text{Lie}(\mathcal{G}_{R_0}^\lor)) \otimes_{R_0} I_\nu = \text{Hom}_{R'}(\omega_{R_\nu}, \text{Lie}(\mathcal{G}_{R_\nu}^\lor)) \otimes_{R'} I_\nu.
\]

Moreover, it is easy to check that the morphism of sets \( \Phi_\nu : \Sigma_{\nu,\tau} \rightarrow \mathcal{D}_{\Phi_{\nu-1}(\tau)}(\overline{R}_{\nu}^') \) is compatible with the homomorphism of groups

\[
\text{Kod}_{R'} \otimes_{R'} \text{Id} : T_{R'/k} \otimes_{R'} I_\nu \rightarrow \text{Hom}_{R'}(\omega_{R_\nu}, \text{Lie}(\mathcal{G}_{R_\nu}^\lor)) \otimes_{R'} I_\nu,
\]

where \( \text{Kod}_{R'} \) is the Kodaira-Spencer map (7.8.2) associated to \( \mathcal{G}_{R_\nu}^\lor \). The bijectivity of \( \Phi_\nu \) now follows from the fact that \( \text{Kod}_{R'} \) is an isomorphism.

(ii) The second part of the statement follows immediately from 4.11. It remains to compute the Hasse-Witt map of \( \mathcal{G}_{R_\nu,\sigma}^\lor \). We determine first the submodule \( \text{Lie}(\mathcal{G}_{R_\nu,\sigma}^\lor) \) of \( \text{Lie}(\mathcal{G}_{R_\nu,\sigma}^\lor) \). We choose a basis of \( \text{Lie}(G^\lor) \) over \( \mathcal{O}_R \) such that \( \varphi_G \) is expressed by the matrix \( h(7.4.1) \). As \( \mathcal{G}_{R_\nu,\sigma}^\lor \) derives from \( G \) by base change \( R \rightarrow R' \), there exists a basis \( (e_1, \ldots, e_n) \) of \( \text{Lie}(\mathcal{G}_{R_\nu,\sigma}^\lor) \) such that \( \varphi_{\mathcal{G}_{R_\nu,\sigma}^\lor} \) is expressed by

\[
h^\sigma = \begin{pmatrix}
0 & 0 & \cdots & 0 & -\sigma(t_1) \\
1 & 0 & \cdots & 0 & -\sigma(t_2) \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \cdots & 1 & -\sigma(t_n)
\end{pmatrix}.
\]

By Lemma 7.6, \( \text{Lie}(\mathcal{G}_{R_\nu,\sigma}^\lor) \) is generated by \( \text{Lie}(\mathcal{G}_{R_\nu,\sigma}^\lor)^{\varphi=1} \). If \( \sum_{i=1}^n x_i e_n \in \text{Lie}(\mathcal{G}_{R_\nu,\sigma}^\lor)^{\varphi=1} \) with \( x_i \in R' \) for \( 1 \leq i \leq n \), then \( (x_i)_{1 \leq i \leq n} \) must satisfy the
equation \( b^\sigma \cdot \begin{pmatrix} x_1^p \\ \vdots \\ x_n^p \end{pmatrix} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}; \) or equivalently,

\[
\begin{align*}
   x_1 &= -\sigma(t_1)x_n^p \\
   x_2 &= -\sigma(t_2)x_n^p - \sigma(t_1)^{p-2}x_n^{p-1} \\
   \cdots \\
   x_{n-1} &= -\sigma(t_{n-1})x_n^p - \cdots - \sigma(t_1)^{p-2}x_n^{p-1} \\
   x_n &= -\sigma(t_n)x_n^p + \sigma(t_2)^{p-2}x_n^{p-1} + \cdots + \sigma(t_n)x_n^{p-1} = 0.
\end{align*}
\] (7.8.3)

We note that \( \sigma(t_i) \in \mathfrak{m}_{\overline{R}}^2 \) for \( 1 \leq i \leq n - 1 \) and \( \sigma(t_n) \in \mathfrak{m}_{\overline{R}} \), with image \( i(t_n) \in \overline{K}_0 \), where \( i: \overline{K}_0 \rightarrow \overline{K}_0 \) is the fixed immersion. By Hensel’s lemma, every solution in \( \overline{K}_0 \) of the equation \( i(t_n)x_n^p + x_n = 0 \) lifts uniquely to a solution of (7.8.3). As \( \operatorname{Lie}(\mathfrak{g}_{R,\sigma}^\psi) \) has rank 1, by Lemma 7.6, these are all the solutions of (7.8.3). Let \( (\lambda_1, \ldots, \lambda_n) \) be a non-zero solution of (7.8.3). We have

\[
\lambda_n \in \mathfrak{m}_{\overline{R}}^2 \quad \text{and} \quad \lambda_i = -\lambda_{i-1}^{p-1}\sigma(t_i) \pmod{\mathfrak{m}_{\overline{R}}^2}. \] (7.8.4)

We put \( v = \lambda_1e_1 + \cdots + \lambda_ne_n \); so \( v \) is a basis of \( \operatorname{Lie}(\mathfrak{g}_{R,\sigma}^\psi) \) by 7.6. For \( 1 \leq i \leq n \), let \( f_i \) be the image of \( e_i \) in \( \operatorname{Lie}(\mathfrak{g}_{R,\sigma}^\psi) \). Then \( f_1, \ldots, f_n \) clearly generate \( \operatorname{Lie}(\mathfrak{g}_{R,\sigma}^\psi) \). By the explicit description above of \( \operatorname{Lie}(\mathfrak{g}_{R,\sigma}^\psi) \), we have \( f_n = -\lambda_n^{p-1}(\lambda_1f_1 + \cdots + \lambda_{n-1}f_{n-1}) \). Hence \( f_1, \ldots, f_{n-1} \) form a basis of \( \operatorname{Lie}(\mathfrak{g}_{R,\sigma}^\psi) \).

By the functoriality of Hasse-Witt maps, we have \( \varphi_{\mathfrak{g}_{\overline{R}}}(f_i) = f_{i+1} \) for \( 1 \leq i \leq n - 1 \), or equivalently,

\[
\varphi_{\mathfrak{g}_{\overline{R}}}(f_1, \ldots, f_{n-1}) = (f_1, \ldots, f_{n-1}) \cdot \begin{pmatrix} 0 & 0 & \cdots & 0 & -\lambda_{n-1}^{p-1}\lambda_1 \\
1 & 0 & \cdots & 0 & -\lambda_{n-1}^{p-1}\lambda_2 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \cdots & 1 & -\lambda_{n-1}^{p-1}\lambda_{n-1} \end{pmatrix}.
\]

In view of (7.8.4), we see that the above matrix has the form of (7.8.1) by setting \( \alpha = \lambda_{n-1}^{p-1} \in \mathfrak{m}_{\overline{R}}^2 \). The second part of statement (ii) follows immediately from Proposition 4.11(ii) and the description above of \( \varphi_{\mathfrak{g}_{\overline{R}}}(\sigma) \). \( \square \)

Now we can turn to the proof of 7.5.

7.9. Proof of Lemma 7.5. First, suppose that we have found a \( \sigma_2 \in \Sigma \) such that \( C_{\sigma_2} \neq 0 \) and \( \mathfrak{g}_{R,\sigma_2}^\psi \) is the universal deformation of \( \mathfrak{g}_{R_0}^\psi \). Since \( \Phi : \Sigma \to \mathfrak{D}_{\mathfrak{g}_{R_0}^\psi}(\overline{R}) \) is bijective by 7.8(i), there exists a \( \sigma_1 \in \Sigma \) corresponding to the deformation \( [\mathfrak{g}_{R,\sigma_2}^\psi \oplus \mathbb{Q}_p/\mathbb{Z}_p] \in \mathfrak{D}_{\mathfrak{g}_{R_0}^\psi}(\overline{R}) \). It is clear that \( \mathfrak{g}_{R,\sigma_1} \simeq \mathfrak{g}_{R,\sigma_2} \).

Besides, the exact sequence (7.4.5) for \( \sigma_1 \) splits: so we have \( C_{\sigma_1} = 0 \). It remains to prove the existence of \( \sigma_2 \). We note first that \( \overline{K}_0 \) can be canonically imbedded into \( \overline{R} \), since it is perfect. Since \( \overline{R} \) is formally smooth over \( k \) and
(t₁, · · · , tₙ) is a p-basis of R' over k, by [EGA, 0₁IV 21.2.7], there is a σ ∈ Σ such that σ(tᵢ) (1 ≤ i ≤ n − 1) form a system of regular parameters of R' and
σ(tₙ) ∈ K₀ ∩ R'. We claim that σ₂ = σ answers the question. In fact, Lemma 7.8(ii) implies that G_{R',σ} is the universal deformation of G_{K₀}. It remains to verify that Cσ ̸= 0.
Let A = K₀[[π]] be a complete discrete valuation ring of characteristic p with
residue field K₀, T = Spec(A), ξ be the generic point of T, ξ be a geometric
over ξ, and I = Gal(ξ/ξ) the Galois group. We define a homomorphism of
K₀-algebras f* : R' → A by putting f*(σ(t₁)) = π and f*(σ(tᵢ)) = 0 for
2 ≤ i ≤ n − 1. This is possible, since (σ(t₁), · · · , σ(tₙ−₁)) is a system of
regular parameters of R'. Let f : T → S be the homomorphism of schemes
corresponding to f*, and G_T = G_{R',σ} × S T. By the functoriality of Hasse-Witt
maps,
\[ b_T = \begin{pmatrix}
0 & 0 & · · · & 0 & -\pi \\
1 & 0 & · · · & 0 & 0 \\
0 & 1 & · · · & 0 & 0 \\
· & · & · & · & · \\
0 & 0 & · · · & 1 & -f^*(σ(tₙ))
\end{pmatrix} \in M_{n×n}(R') \]
is a matrix of ϕ_{G_T}. By definition (5.4), the Hasse invariant of G_T is h(G_T) = 1.
In particular, G_T is generically ordinary. Let U'_σ ⊆ S be the ordinary locus
of G_{R',σ}. We have f(ξ) ∈ U'_σ. By the functoriality of fundamental groups, f
induces a homomorphism of groups
\[ \pi_1(f) : I = \text{Gal}(ξ/ξ) → \pi_1(U'_σ, f(ξ)) \simeq \pi_1(U'_σ, π). \]
Let G_T be the connected part of G_T, and G_T^{et} be the étale part of G_T. Then
G_T^{et} ⊆ Q_p/Z_p. We have an exact sequence of F_p[I]-modules
\[ 0 → G_T^{et}(1)(ξ) → G_T(1)(ξ) → G_T^{et}(1)(ξ) → 0, \]
which determines a cohomology class C_T ∈ H^1(I, G_{T}(1)(ξ)). We notice that
G_T(1)(ξ) is isomorphic to G_{R',σ}^{et}(1)(π) as an abelian group, and the action of I
on G_T(1)(ξ) is induced by the action of π₁(U'_σ, π) on G_{R',σ}^{et}(1)(π). Therefore,
C_T is the image of Cσ by the functorial map
\[ H^1(\pi_1(U'_σ, π), G_{R',σ}^{et}(1)(π)) → H^1(I, G_{T}^{et}(1)(ξ)). \]
To verify that C_T ̸= 0, it suffices to check that Cσ ̸= 0. We consider the polynomial
P(X) = X^n + f*(σ(tₙ))X^{n−1} + πX ∈ A[X]. According to 5.12, it suffices to find a α ∈ K₀ ⊆ A such that P(α) is a uniformizer of A. But by the choice of σ, we have σ(tₙ) ∈ K₀ and σ(tₙ) ̸= 0, so f*(σ(tₙ)) ̸= 0 lies in K₀.
Let α be a p^{n−1}(p−1)-th root of −f*(σ(tₙ)) in K₀. Then we have α ∈ K₀, and
P(α) = απ is a uniformizer of A. This completes the proof of 7.5.

Documenta Mathematica 14 (2009) 397–440
8. End of the Proof of Theorem 1.3

In this section, $k$ denotes an algebraically closed field of characteristic $p > 0$.

8.1. First, we recall some preliminaries on Newton stratification due to F. Oort. Let $G$ be an arbitrary BT-group over $k$, $S$ be the local moduli of $G$ in characteristic $p$, and $\mathbf{G}$ be the universal deformation of $G$ over $S$ (3.8). Put $d = \dim(G)$ and $c = \dim(G^\vee)$. We denote by $\mathcal{N}(G)$ the Newton polygon of $G$ which has endpoints $(0,0)$ and $(c+d,d)$. Here we use the normalization of Newton polygons such that slope 0 corresponds to étale BT-groups and slope 1 corresponds to groups of multiplicative type.

Let $\mathcal{N}\mathcal{P}(c+d,d)$ be the set of Newton polygons with endpoints $(0,0)$ and $(c+d,d)$ and slopes in $(0,1)$. For $\alpha, \beta \in \mathcal{N}\mathcal{P}(c+d,d)$, we say that $\alpha \preceq \beta$ if no point of $\alpha$ lies below $\beta$; then “$\preceq$” is a partial order on $\mathcal{N}\mathcal{P}(c+d,d)$. For each $\beta \in \mathcal{N}\mathcal{P}(c+d,d)$, we denote by $V_\beta$ the subset of $S$ consisting of points $x$ with $\mathcal{N}(G_x) \preceq \beta$, and by $V_\beta^c$ the subset of $S$ consisting of points $x$ with $\mathcal{N}(G_x) = \beta$. By Grothendieck-Katz’s specialization theorem of Newton polygons, $V_\beta$ is closed in $S$, and $V_\beta^c$ is open (maybe empty) in $V_\beta$. We put

$$\diamondsuit(\beta) = \{(x,y) \in \mathbb{Z} \times \mathbb{Z} \mid 0 \leq y < d, y < x < c+d, (x,y) \text{ lies on or above the polygon } \beta\},$$

and $\dim(\beta) = \#(\diamondsuit(\beta))$.

Theorem 8.2 ([Oo2] Theorem 2.11). Under the above assumptions, for each $\beta \in \mathcal{N}\mathcal{P}(c+d,d)$, the subset $V_\beta^c$ is non-empty if and only if $\mathcal{N}(G) \preceq \beta$. In that case, $V_\beta$ is the closure of $V_\beta^c$ and all irreducible components of $V_\beta$ have dimension $\dim(\beta)$.

8.3. Let $G$ be a connected and HW-cyclic BT-group over $k$ of dimension $d = \dim(G) \geq 2$. Let $\beta \in \mathcal{N}\mathcal{P}(c+d,d)$ be the Newton polygon given by the following slope sequence:

$$\beta = \frac{1}{c+1}, \frac{1}{c+1}, \frac{1}{d-1}. \frac{1}{d-1}.$$ 

We have $\mathcal{N}(G) \preceq \beta$ since $G$ is supposed to be connected. By Oort’s Theorem 8.2, $V_\beta$ is a reduced dimensional closed subset of the local moduli $S$ of dimension $c(d-1)$. We endow $V_\beta$ with the structure of a reduced closed subscheme of $S$.

Lemma 8.4. Under the above assumptions, let $R$ be the ring of $S$, and

$$\begin{pmatrix}
0 & 0 & \cdots & 0 & -a_1 \\
1 & 0 & \cdots & 0 & -a_2 \\
0 & 1 & \cdots & 0 & -a_3 \\
\vdots & \ddots & \ddots & \vdots \\
0 & 0 & \cdots & 1 & -a_c
\end{pmatrix} \in M_{c \times c}(R)$$

be a matrix of the Hasse-Witt map $\varphi_G$. Then the closed reduced subscheme $V_\beta$ of $S$ is defined by the prime ideal $(a_1, \ldots, a_c)$. In particular, $V_\beta$ is irreducible.
Proof. Note first that \( \{a_1, \ldots, a_c\} \) is a subset of a system of regular parameters of \( R \) by 4.11(i). Let \( I \) be the ideal of \( R \) defining \( V_\beta \). Let \( x \) be an arbitrary point of \( V_\beta \), we denote by \( p_x \) the prime ideal of \( R \) corresponding to \( x \). Since the Newton polygon of the fibre \( G_x \) lies above \( \beta \), \( G_x \) is connected. By Lemma 4.4, we have \( a_i \not\in p_x \) for \( 1 \leq i \leq c \). Since \( V_\beta \) is reduced, we have \( a_i \not\in I \). Let \( \mathcal{P} = (a_1, \ldots, a_c) \), and \( V(\mathcal{P}) \) the closed subscheme of \( S \) defined by \( \mathcal{P} \). Then \( V(\mathcal{P}) \) is an integral scheme of dimension \( c(d-1) \) and \( V_\beta \subset V(\mathcal{P}) \). Since Theorem 8.2 implies that \( \dim V_\beta = c(d-1) \), we have necessarily \( V_\beta = V(\mathcal{P}) \). □

We keep the assumptions above. Let \((t_{i,j})_{1 \leq i \leq c, 1 \leq j \leq d}\) be a regular system of parameters of \( R \) such that \( t_{i,j} = a_i \) for all \( 1 \leq i \leq c \). Let \( x \) be the generic point of the Newton strata \( V_\beta \), \( k' = \kappa(x) \), and \( R' = \mathcal{O}_{S,x} \). Since \( R \) is noetherian and integral, the canonical ring homomorphism \( R \to \mathcal{O}_{S,x} \to R' \) is injective. The image in \( R' \) of an element \( a \in R \) will be denoted also by \( a \). By choosing a \( k \)-section \( k' \to R' \) of the canonical projection \( R' \to k' \), we get a (non-canonical) isomorphism of \( k \)-algebras \( R' \simeq [t_{1,d}, \ldots, t_{c,d}] \). Let \( k'' \) be an algebraic closure of \( k' \), and \( R'' = k''[[t_{1,d}, \ldots, t_{c,d}]] \). Then we have a natural injective homomorphism of \( k \)-algebras \( R' \to R'' \) mapping \( t_{i,j} \) to \( t_{i,j} \) for \( 1 \leq i \leq c \).

Let \( S'' = \text{Spec}(R'') \). \( \mathfrak{P} \) be its closed point. By the construction of \( S'' \), we have a morphism of \( k \)-schemes

\[
(8.4.1) \quad f : S'' \to S
\]
sending \( \mathfrak{P} \) to \( x \). We put \( \mathcal{G} = G \times_S S'' \). By the choice of the Newton polygon \( \beta \), the closed fibre \( \mathfrak{G}_x \) has a BT-subgroup \( \mathcal{H}_x \) of multiplicative type of height \( d-1 \). Since \( S'' \) is henselian, \( \mathfrak{G}_x \) lifts uniquely to a BT-subgroup \( \mathcal{H}_x \) of \( \mathcal{G} \). We put \( \mathfrak{G}'' = \mathfrak{G} / \mathcal{H}_x \). It is a connected BT-group over \( S'' \) of dimension 1 and height \( c+1 \).

**Lemma 8.5.** Under the above assumptions, \( \mathfrak{G}'' \) is the universal deformation in equal characteristic of its special fiber.

This lemma is a particular case of [Lau, Lemma 3.1]. Here, we use 4.11(ii) to give a simpler proof.

**Proof.** We have an exact sequence of BT-groups over \( S'' \)

\[
0 \to \mathcal{H} \to \mathcal{G} \to \mathfrak{G}'' \to 0
\]

which induces an exact sequence of Lie algebras \( 0 \to \text{Lie}(\mathcal{G}'') \to \text{Lie}(\mathcal{G}) \to \text{Lie}(\mathcal{H}) \to 0 \) compatible with Hasse-Witt maps. Since \( \mathcal{H} \) is of multiplicative type, we get \( \text{Lie}(\mathcal{H}) = 0 \) and an isomorphism of Lie algebras \( \text{Lie}(\mathfrak{G}'') \simeq \text{Lie}(\mathcal{G}) \). By the choice of the regular system \((t_{i,j})_{1 \leq i \leq c, 1 \leq j \leq d}\), there is a basis \((v_1, \ldots, v_c)\) of \( \text{Lie}(\mathcal{G}) \) over \( \mathcal{O}_{S''} \) such that \( \mathfrak{p}_{\mathfrak{G}''} \) is given by the matrix

\[
\mathfrak{g} = \begin{pmatrix}
0 & 0 & \cdots & 0 & -t_{1,d} \\
1 & 0 & \cdots & 0 & -t_{2,d} \\
0 & 1 & \cdots & 0 & -t_{3,d} \\
\vdots & \ddots & \ddots & \ddots & \ddots \\
0 & 0 & \cdots & 1 & -t_{c,d}
\end{pmatrix}
\]
Now the lemma results from Proposition 4.11(ii).

8.6. Proof of Theorem 1.3. The one-dimensional case is treated in 7.3. If \( \dim(G) \geq 2 \), we apply the preceding discussion to obtain the morphism \( f : S'' \to S \) and the BT-groups \( \mathcal{G} = G \times_S S'' \) and \( \mathcal{G}'' \), which is the quotient of \( \mathcal{G} \) by the maximal subgroup of \( \mathcal{G} \) of multiplicative type. Let \( U'' \) be the common ordinary locus of \( \mathcal{G} \) and \( \mathcal{G}'' \) over \( S'' \), and \( \zeta \) be a geometric point of \( U'' \). Then \( f \) maps \( U'' \) into the ordinary locus \( U \) of \( G \). We denote by

\[
\rho_G : \pi_1(U'', \zeta) \to \text{Aut}_{\mathbb{Z}_p}(T_p(\mathcal{G}, \zeta))
\]

the monodromy representation associated to \( \mathcal{G} \), and the same notation for \( \rho_{G''} \). By the functoriality of monodromy, we have \( \text{Im}(\rho_G) \subset \text{Im}(\rho_G) \). On the other hand, the canonical map \( \mathcal{G} \to \mathcal{G}'' \) induces an isomorphism of Tate modules \( T_p(\mathcal{G}, \eta) \cong T_p(\mathcal{G}''', \eta) \) compatible with the action of \( \pi_1(U'', \eta) \). Therefore, the group \( \text{Im}(\rho_G) \) is identified with \( \text{Im}(\rho_{G''}) \). Since \( \mathcal{G}'' \) is one-dimensional, we conclude the proof by Lemma 8.5 and Theorem 7.3.

References


Yichao Tian
Department of Mathematics
Princeton University
Princeton
New Jersey
08544
USA
yichao.tian@princeton.edu


