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A Class of Functional Equations Characterizing Polynomials of Degree Two

Una Clase de Ecuaciones Funcionales que Caracteriza a los Polinomios de Grado 2

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Abstract

In this note, for any given real numbers a, b, c, we determine all the solutions $f : \mathbb{R} \longrightarrow \mathbb{R}$ of the functional equation

$$f(x - f(y)) = f(x) + f(f(y)) - axf(y) - bf(y) - c, \qquad (E(a, b, c))$$

for all $x, y \in \mathbb{R}$.
Key words and phrases: composite functional equations.

Resumen

En esta nota, para cualesquiera números reales a, b, c se determinan todas las soluciones $f : \mathbb{R} \longrightarrow \mathbb{R}$ de la ecuación funcional

$$f(x - f(y)) = f(x) + f(f(y)) - axf(y) - bf(y) - c, \qquad (E(a, b, c))$$

para todo $x, y \in \mathbb{R}$.
Palabras y frases clave: ecuación funcional compuesta.

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1 Introduction

In this note, we are concerned by the following problem:

1.1 Problem: *Let a, b, c be three real numbers. Determine all functions $f : \mathbb{R} \longrightarrow \mathbb{R}$ such that*

$$f(x - f(y)) = f(x) + f(f(y)) - axf(y) - bf(y) - c, \qquad (E(a,b,c))$$

for all x and y in $\mathbb{R}$.

We point out that the particular case $E(-1, 0, 1)$ was one of the problems proposed at the 40th International Mathematical Olympiad, held in Bucharest, Romania in 1999. The subject of this note is to propose a solution to this problem when $a \neq 0$. Precisely we shall prove the following

1.2 Proposition: *Suppose that $a \neq 0$. Then, the polynomial $f(t) = c + \frac{b}{2}t + \frac{a}{2}t^2$ is the unique nontrivial solution of Equation $(E(a,b,c))$.*

Therefore, we can say that the functional equations $(E(a,b,c))$ characterize the polynomials of degree two. We point out that no regularity condition is required for the functions f. It is an interesting problem to look for the functional equations characterizing the polynomials of degree $n \geq 3$. As it was noticed in the abstract, the problem studied here generalizes the problem of solving the equation $E(-1, 0, 1)$ which was proposed in the fortieth international mathematical olympiad that was held in Bucharest, Romania, from 10 to 22 July 1999. It is interesting to look at the case where $a = 0$. This case will be discussed in Subsections three and four, but under continuity conditions for the functions f.

2 Proof of 1.2

We can verify that the polynomial $f(t) = c + \frac{b}{2}t + \frac{a}{2}t^2$ is a solution of equation (E). Let us suppose that $a \neq 0$, and let f be a nontrivial solution of equation (E). Let $y \in \mathbb{R}$ and $x = f(y)$. Then we have

$$f(f(y)) = \frac{d + c}{2} + \frac{b}{2}f(y) + \frac{a}{2}(f(y))^2, \qquad (1)$$

where $d = f(0)$. Since f is not identically zero on $\mathbb{R}$, then we can find a real u such that $f(u) \neq 0$. Letting $y = u$ in the functional equation (E), we get

$$f(x - f(u)) - f(x) = f(f(u)) - bf(u) - c - af(u)x, \ \forall x \in \mathbb{R}. \qquad (2).$$

Since $a \neq 0$, then the function in the right hand side of (2) is a non-constant linear function. Thus, given $z \in \mathbb{R}$, there exists a unique $x \in \mathbb{R}$ with $z = f(x - f(u)) - f(x) = f(w) - f(x)$, say. Hence for this z, according to (E) and (1), we have

$$\begin{aligned}
f(z) &= f(f(w) - f(x)) \\
&= f(f(x)) - af(w)f(x) - bf(x) + f(f(w)) - c \\
&= \frac{d+c}{2} + \frac{b}{2}f(x) + \frac{a}{2}(f(x))^2 - af(w)f(x) - bf(x) - c \\
&+ \frac{d+c}{2} + \frac{b}{2}f(w) + \frac{a}{2}(f(w))^2 \\
&= d + \frac{b}{2}[f(w) - f(x)] + \frac{a}{2}\left[(f(w))^2 - 2f(w)f(x) + (f(x))^2\right] \\
&= d + \frac{b}{2}[f(w) - f(x)] + \frac{a}{2}[f(w) - f(x)]^2 = d + \frac{b}{2}z + \frac{a}{2}z^2.
\end{aligned} \tag{3}$$

Taking $z = f(y)$ for any y and using (1), we obtain $d = c$ and thus

$$f(z) = c + \frac{b}{2}z + \frac{a}{2}z^2, \tag{4}$$

for all $z \in \mathbb{R}$. □

3 The case where $a = 0$.

We shall solve Equation $(E(0, b, c))$ under some supplementary conditions. To this respect, we shall make use of the proposition 3.1 below which is a result owed to Dhombres (see [1], [2] and [3]). To state this proposition we need the following terminology: A function $g : \mathbb{R} \to \mathbb{R}$ is of type (λ, α, β), where $\lambda \in \mathbb{R}$ and $-\infty \leq \alpha < \beta \leq \infty$, if

$$g(x) = \begin{cases} \lambda x + (1-\lambda)\alpha & \text{if} x < \alpha, \\ x & \text{if} \alpha \leq x \leq \beta, \\ \lambda x + (1-\lambda)\beta & \text{if} \beta < x. \end{cases}$$

We adopt natural conventions as, for example, when $\alpha = -\infty$, then there is no $x < -\infty$ case. A function $g : \mathbb{R} \to \mathbb{R}$ is of type (λ, δ) if it is given by $g(x) = \lambda x + \delta$, for every $x \in \mathbb{R}$. With these notations, we recall the following proposition (see [1], p. 322).

3.1 Proposition: *Let $\lambda > 0$ and consider the following functional equation:*

$$f(f(y)) = (\lambda + 1)f(y) - \lambda y, \ \forall y \in \mathbb{R}. \qquad (F(\lambda))$$

Then a continuous function $g : \mathbb{R} \to \mathbb{R}$ satisfies $(F(\lambda))$, if and only if g is of one of the following types:
$\lambda \neq 1$: *type (λ, α, β) or (λ, δ),*
$\lambda = 1$: *type $(1, \delta)$*

Now, we are in position to solve Equation $(E(0, b, c))$. Let us suppose that $b < 2$. We set $\lambda = 1 - \frac{b}{2}$. Let f be a continuous function such that $f(0) = c$ and satisfying $(E(0, b, c))$. We set $g(x) = x - f(x)$, for all $x \in \mathbb{R}$. Then an easy computation will show that g is a solution of Equation $(F(\lambda))$, where $\lambda = 1 - \frac{b}{2}$. We have $\lambda > 0$ by assumption, therefore we can use proposition 3.1 to deduce that if $b = 0$, then any continuous function f such that $f(0) = c$ and satisfying $(E(0, 0, c))$ must be a constant $f = c$ on $\mathbb{R}$, and that if $b \neq 0$, then any continuous function f such that $f(0) = c$ and satisfying $(E(0, b, c))$ must be either of type $f(x) = \frac{b}{2}x + c$, for all $x \in \mathbb{R}$ or of type

$$f(x) = \begin{cases} \frac{b}{2}(x - \alpha) & \text{if } x < \alpha, \\ 0 & \text{if } \alpha \leq x \leq \beta, \\ \frac{b}{2}(x - \beta) & \text{if } \beta < x, \end{cases}$$

for some $-\infty \leq \alpha < \beta \leq \infty$ with the natural conventions quoted above. But it is easy to see that this case occurs only when $c = 0$ with $-\infty = \alpha$ and $\beta = \infty$, so that in this case, we have $f(x) = \frac{b}{2}x$ for all $x \in \mathbb{R}$. One can see that this conclusion is still true when $b \geq 2$. Indeed, by [1], the conclusions of Proposition 3.1 remain valid even if $\lambda \leq 0$. Thus we have proved the following proposition: **3.2 Proposition:** *Every continuous function f such that $f(0) = c$ and satisfying $(E(0, b, c))$ must be of the type $f(x) = \frac{b}{2}x + c$, for all $x \in \mathbb{R}$.*

4 The case where $a = b = c = 0$.

Here, we are concerned by the following functional equation:

$$f(x - f(y)) = f(x) + f(f(y)), \quad \forall x, y \in \mathbb{R}. \qquad (E(0, 0, 0))$$

Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function verifying equation $(E(0, 0, 0))$, and set $d = f(0)$. Then we have $f(f(y)) = \frac{d}{2}$ for all $y \in \mathbb{R}$, and we get

$$f(x - d) = f(x) + \frac{d}{2} = f(x + d) + d, \quad \forall x \in \mathbb{R}. \qquad (2)$$

Letting $x = d$ in (2) we obtain that $f(2d) = 0$. Using the functional equation $(E(0,0,0))$, we get $0 = f(2d - f(2d)) = f(2d) + f(f(2d)) = \frac{d}{2}$. Therefore $d = 0$. Now, by setting $g(x) = x - f(x)$, $(\forall x \in \mathbb{R})$, we see that g is a continuous solution of the functional equation $(F(1))$. By Proposition 3.1, g must be of type $(1, \delta)$. Necessarily $\delta = 0$. Therefore f must be zero. So we have proved the following proposition:

4.1 Proposition: *Every continuous function f satisfying $(E(0,0,0))$ must be identically zero on $\mathbb{R}$.*

References

[1] Aczèl, J., Dhombres, J. G. *Functional Equations in Several Variables*, Encyclopedia of Mathematics and its Applications, Cambridge University press, 1989.

[2] Dhombres, J, G. *Some Aspects of Functional Equations*, Chulalongkorn University, Department of Mathematics, Bangkok 1979. MR: 80j,♯ 39001.

[3] Dhombres, J. G. *Itération linéaire d'ordre deux*, Publ. Math. Debrecen 24, 277- 87. MR: 58,♯ 6798 (1979).