A Research on Characterizations of Semi-\(T_{1/2}\) Spaces

Un Levantamiento sobre Caracterizaciones de Espacios Semi-\(T_{1/2}\)

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Abstract

The goal of this article is to bring to your attention some of the salient features of recent research on characterizations of Semi – \(T_{1/2}\) spaces.

Keywords and phrases: Topological spaces, generalized closed sets, semi-open sets, Semi – \(T_{1/2}\) spaces, closure operator.

Resumen

El objetivo de este trabajo es presentar algunos aspectos resaltantes de la investigación reciente sobre caracterizaciones de los espacios Semi – \(T_{1/2}\).

Palabras y frases clave: Espacios topológicos, conjuntos cerrados generalizados, conjuntos semi-abiertos, espacios Semi – \(T_{1/2}\), operador clausura.

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1 Introduction

The concept of semi-open set in topological spaces was introduced in 1963 by N. Levine [11] i.e., if \((X, \tau)\) is a topological space and \(A \subset X\), then \(A\) is semi-open \((A \in SO(X, \tau))\) if there exists \(O \in \tau\) such that \(O \subseteq A \subseteq Cl(O)\), where \(Cl(O)\) denotes closure of \(O\) in \((X, \tau)\). The complement \(A^c\) of a semi-open set \(A\) is called semi-closed and the semi-closure of a set \(A\) denoted by \(sCl(A)\) is the intersection of all semi-closed sets containing \(A\).

After the work of N. Levine on semi-open sets, various mathematicians turned their attention to the generalizations of various concepts of topology by considering semi-open sets instead of open sets. While open sets are replaced by semi-open sets, new results are obtained in some occasions and in other occasions substantial generalizations are exhibited.

In this direction, in 1975, S. N. Maheshwari and R. Prasad [12], used semi-open sets to define and investigate three new separation axiom called Semi-\(T_0\), Semi-\(T_1\) (if for \(x, y \in X\) such that \(x \neq y\) there exists a semi-open set containing \(x\) but not \(y\) or (resp. and) a semi-open set containing \(y\) but not \(x\)) and Semi-\(T_2\) (if for \(x, y \in X\) such that \(x \neq y\), there exist semi-open sets \(O_1\) and \(O_2\) such that \(x \in O_1\), \(y \in O_2\) and \(O_1 \cap O_2 = \emptyset\)). Moreover, they have shown that the following implications hold.

\[
\begin{align*}
T_2 & \rightarrow \text{Semi-}T_2 \\
\downarrow & \downarrow \\
T_1 & \rightarrow \text{Semi-}T_1 \\
\downarrow & \downarrow \\
T_0 & \rightarrow \text{Semi-}T_0
\end{align*}
\]

Later, in 1987, P. Bhattacharyya and B. K. Lahiri [1] generalized the concept of closed sets to semi-generalized closed sets with the help of semi-openness. By definition a subset of \(A\) of \((X, \tau)\) is said to be semi-generalized closed (written in short as sg-closed sets) in \((X, \tau)\), if \(sCl(A) \subset O\) whenever \(A \subset O\) and \(O\) is semi-open in \((X, \tau)\). This generalization of closed sets, introduced in [1], has no connection with the generalized closed sets as considered by N. Levine given in [10], although both generalize the concept of closed sets, this notions are in general independent. Moreover in [1], they defined the concept of a new class of topological spaces called Semi-\(T_{1/2}\) (i.e., the spaces where the class of semi-closed sets and the sg-closed sets coincide). It is proved that every Semi-\(T_1\) space is Semi-\(T_{1/2}\) and every Semi-\(T_{1/2}\) space is Semi-\(T_0\), although none of these applications is reversible.

The purpose of the present paper is to give some characterizations of Semi-\(T_{1/2}\) spaces, incluing a characterization using a new topology that M. Caldas
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and J. Dontchev [6] define as $\tau^\omega$-topology. These characterizations are obtained mainly through the introduction of a concept of a generalized set or of a new class of maps.

2 Characterizations of Semi-$T_{1/2}$ Spaces

Recall ([15]) that for any subset $E$ of $(X, \tau)$, $sCl^*(E) = \bigcap\{A : E \subset A(\in sD(X, \tau))\}$, where $sD(X, \tau) = \{A : A \subset X$ and $A$ is sg-closed in $(X, \tau)\}$ and $\text{SO}(X, \tau)^* = \{B : sCl^*(B^c) = B^c\}$.

Similarly to W. Dunham [8], P. Sundaram, H. Maki and K. Balachandram in [15] characterized the Semi-$T_{1/2}$ spaces as follows:

**Theorem 2.1.** A topological space $(X, \tau)$ is a Semi-$T_{1/2}$ space if and only if $\text{SO}(X, \tau) = \text{SO}(X, \tau)^*$ holds.

**Proof.** Necessity: Since the semi-closed sets and the sg-closed sets coincide by the assumption, $sCl(E) = sCl^*(E)$ holds for every subset $E$ of $(X, \tau)$. Therefore, we have that $\text{SO}(X, \tau) = \text{SO}(X, \tau)^*$.

Sufficiency: Let $A$ be a sg-closed set of $(X, \tau)$. Then, we have $A = sCl^*(A)$ and hence $A^c \in \text{SO}(X, \tau)$. Thus $A$ is semi-closed. Therefore $(X, \tau)$ is Semi-$T_{1/2}$. \qed

**Theorem 2.2.** A topological space $(X, \tau)$ is a Semi-$T_{1/2}$ space if and only if, for each $x \in X$, \{x\} is semi-open or semi-closed.

**Proof.** Necessity: Suppose that for some $x \in X$, \{x\} is not semi-closed. Since $X$ is the only semi-open set containing \{x\}^c, the set \{x\}^c is sg-closed and so it is semi-closed in the Semi-$T_{1/2}$ space $(X, \tau)$. Therefore \{x\} is semi-open.

Sufficiency: Since $\text{SO}(X, \tau) \subseteq \text{SO}(X, \tau)^*$ holds, by Theorem 2.1, it is enough to prove that $\text{SO}(X, \tau)^* \subseteq \text{SO}(X, \tau)$. Let $E \in \text{SO}(X, \tau)^*$. Suppose that $E \notin \text{SO}(X, \tau)$. Then, $sCl^*(E^c) = E^c$ and $sCl(E^c) \neq E^c$ hold. There exists a point $x$ of $X$ such that $x \in sCl(E^c)$ and $x \notin E^c(= sCl^*(E^c))$. Since $x \notin sCl^*(E^c)$ there exists a sg-closed set $A$ such that $x \notin A$ and $A \supset E^c$. By the hypothesis, the singleton \{x\} is semi-open or semi-closed.

Case 1. \{x\} is semi-open: Since \{x\} is a semi-closed set with $E^c \subset \{x\}^c$, we have $sCl(E^c) \subset \{x\}^c$, i.e., $x \notin sCl(E^c)$. This contradicts the fact that $x \in sCl(E^c)$. Therefore $E \in \text{SO}(X, \tau)$.

Case 2. \{x\} is semi-closed: Since \{x\} is a semi-open set containing the sg-closed set $A(\supset E^c)$, we have \{x\} $\supset sCl(A) \supset sCl(E^c)$. Therefore $x \notin sCl(E^c)$. This is a contradiction. Therefore $E \in \text{SO}(X, \tau)$.

Hence in both cases, we have $E \in \text{SO}(X, \tau)$, i.e., $\text{SO}(X, \tau)^* \subseteq \text{SO}(X, \tau)$. \qed
As a consequence of Theorem 2.2, we have also the following characterization:

**Theorem 2.3.** 
\((X, \tau)\) is Semi-\(T_{1/2}\), if and only if, every subset of \(X\) is the intersection of all semi-open sets and all semi-closed sets containing it.

**Proof.** Necessity: Let \((X, \tau)\) be a Semi-\(T_{1/2}\) space with \(B \subset X\) arbitrary. Then \(B = \bigcap\{\{x\}^c; x \notin B\}\) is an intersection of semi-open sets and semi-closed sets by Theorem 2.2. The result follows.

Sufficiency: For each \(x \in X\), \(\{x\}^c\) is the intersection of all semi-open sets and all semi-closed sets containing it. Thus \(\{x\}^c\) is either semi-open or semi-closed and hence \(X\) is Semi-\(T_{1/2}\).

**Definition 1.** A topological space \((X, \tau)\) is called a semi-symmetric space [3] if for \(x\) and \(y\) in \(X\), \(x \in sCl(\{y\})\) implies that \(y \in sCl(\{x\})\).

**Theorem 2.4.** Let \((X, \tau)\) be a semi-symmetric space. Then the following are equivalent.

(i) \((X, \tau)\) is Semi-\(T_0\).

(ii) \((X, \tau)\) is Semi-\(T_{1/2}\).

(iii) \((X, \tau)\) is Semi-\(T_1\).

**Proof.** It is enough to prove only the necessity of (i) \(\implies\) (iii). Let \(x \neq y\). Since \((X, \tau)\) is Semi-\(T_0\), we may assume that \(x \in O \subset \{y\}^c\) for some \(O \in SO(X, \tau)\). Then \(x \notin sCl(\{y\})\) and hence \(y \notin sCl(\{x\})\). Therefore there exists \(O_1 \in SO(X, \tau)\) such that \(y \in O_1 \subset \{x\}^c\) and \((X, \tau)\) is a Semi-\(T_1\) space.

In 1995 J. Dontchev in [7] proved that a topological space is Semi-\(T_D\) if and only if it is Semi-\(T_{1/2}\). We recall the following definitions, which will be useful in the sequel.

**Definition 2.** (i) A topological space \((X, \tau)\) is called a Semi-\(T_D\) space [9], if every singleton is either open or nowhere dense, or equivalently if the derived set \(Cl(\{x\}) \setminus \{x\}\) is semi-closed for each point \(x \in X\).

(ii) A subset \(A\) of a topological space \((X, \tau)\) is called an \(\alpha\)-open set [14], if \(A \subset Int(Cl(Int(A)))\) and an \(\alpha\)-closed set if \(Cl(Int(Cl(A))) \subset A\).

Note that the family \(\tau^\alpha\) of all \(\alpha\)-open sets in \((X, \tau)\) forms always a topology on \(X\), finer than \(\tau\).

**Theorem 2.5.** For a topological space \((X, \tau)\) the following are equivalent:

(i) The space \((X, \tau)\) is a Semi-\(T_D\) space.

(ii) The space \((X, \tau)\) is a Semi-\(T_{1/2}\) space.
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Proof. (i)$\rightarrow$(ii). Let $x \in X$. Then $\{x\}$ is either open or nowhere dense by (i). Hence it is $\alpha$-open or $\alpha$-closed and thus semi-open or semi-closed. Then $X$ is a Semi-$T_{1/2}$ space by Theorem 2.2.

(ii)$\rightarrow$(i). Let $x \in X$. We assume first that $\{x\}$ is not semi-closed. Then $X\{x\}$ is sg-closed. Then by (ii) it is semi-closed or equivalently $\{x\}$ is semi-open. Since every semi-open singleton is open, then $\{x\}$ is open. Next, if $\{x\}$ is semi-closed, then $\text{Int}(\text{Cl}(\{x\})) = \text{Int}(\{x\}) = \emptyset$ if $\{x\}$ is not open and hence $\{x\}$ is either open or nowhere dense. Thus $(X,\tau)$ is a Semi-$T_D$ space. \hfill $\square$

Again using semi-symmetric spaces we have the following theorem (see [3]).

**Theorem 2.6.** For a semi--symmetric space $(X,\tau)$ the following are equivalent:

(i) The space $(X,\tau)$ is a Semi-$T_0$ space.

(ii) The space $(X,\tau)$ is a Semi-$D_1$ space.

(iii) The space $(X,\tau)$ is a Semi-$T_{1/2}$ space.

(iv) The space $(X,\tau)$ is a Semi-$T_1$ space.

where a topological space $(X,\tau)$ is said to be a Semi-$D_1$ if for $x,y \in X$ such that $x \neq y$ there exists an $sD$-set of $X$ (i.e., if there are two semi-open sets $O_1, O_2$ in $X$ such that $O_1 \neq X$ and $S = O_1 \setminus O_2$ containing $x$ but not $y$ and an $sD$-set containing $y$ but not $x$.

As an analogy of [13], M. Caldas and J. Dontchev in [5] introduced the $\land_s$-sets (resp. $\lor_s$-sets) which are intersection of semi-open (resp. union of semi-closed) sets. In this paper [5], they also define the concepts of $g.\land_s$-sets and $g.\lor_s$-sets.

**Definition 3.** In a topological space $(X,\tau)$, a subset $B$ is called:

(i) $\land_s$-set (resp. $\lor_s$-set) if $B = B^{\land_s}$ (resp. $B = B^{\lor_s}$), where, $B^{\land_s} = \bigcap\{O: O \supseteq B, O \in SO(X,\tau)\}$ and $B^{\lor_s} = \bigcup\{F: F \subseteq B, F^c \in SO(X,\tau)\}$.

(ii) Generalized $\land_s$-set (= $g.\land_s$-set) of $(X,\tau)$ if $B^{\land_s} \subseteq F$ whenever $B \subseteq F$ and $F^c \in SO(X,\tau)$.

(iii) Generalized $\lor_s$-set (= $g.\lor_s$-set) of $(X,\tau)$ if $B^{\lor_s}$ is a $g.\land_s$-set of $(X,\tau)$.

By $D^{\land_s}$ (resp. $D^{\lor_s}$) we will denote the family of all $g.\land_s$-sets (resp. $g.\lor_s$-sets) of $(X,\tau)$.

In the following theorem ([5]), we have another characterization of the class of Semi-$T_{1/2}$ spaces by using $g.\lor_s$-sets.

**Theorem 2.7.** For a topological space $(X,\tau)$ the following are equivalent:

(i) $(X,\tau)$ is a Semi-$T_{1/2}$ space.

(ii) Every $g.\lor_s$-set is a $\lor_s$-set.
Proof. (i)→(ii). Suppose that there exists a $g\lor_s$-set $B$ which is not a $\lor_s$-set. Since $B^{\lor_s} \subseteq B$ ($B^{\lor_s} \neq B$), then there exists a point $x \in B$ such that $x \notin B^{\lor_s}$. Then the singleton $\{x\}$ is not semi-closed. Since $\{x\}^c$ is not semi-open, the space $X$ itself is only semi-open set containing $\{x\}^c$. Therefore, $\text{sCl}(\{x\}^c) \subseteq X$ holds and so $\{x\}^c$ is a sg-closed set. On the other hand, we have that $\{x\}$ is not semi-open (since $B$ is a $g\lor_s$-set, and $x \notin B^{\lor_s}$). Therefore, we have that $\{x\}^c$ is not semi-closed but it is a sg-closed set. This contradicts the assumption that $(X, \tau)$ is a Semi-$T_{1/2}$ space.

(ii)→(i). Suppose that $(X, \tau)$ is not a Semi-$T_{1/2}$ space. Then, there exists a sg-closed set $B$ which is not semi-closed. Since $B$ is not semi-closed, there exists a point $x$ such that $x \notin B$ and $x \in s\text{Cl}(B)$. It is easily to see that the singleton $\{x\}$ is a semi-open set or it is a $g\lor_s$-set. When $\{x\}$ is semi-open, we have $\{x\} \cap B \neq \emptyset$ because $x \in s\text{Cl}(B)$. This is a contradiction. Let us consider the case: $\{x\}$ is a $g\lor_s$-set. If $\{x\}$ is not semi-closed, we have $\{x\}^c = \emptyset$ and hence $\{x\}$ is not a $\lor_s$-set. This contradicts (ii). Next, if $\{x\}$ is semi-closed, we have $\{x\}^c \supseteq s\text{Cl}(B)$ (i.e., $x \notin s\text{Cl}(B)$). In fact, the semi-open set $\{x\}^c$ contains the set $B$ which is a sg-closed set. Then, this also contradicts the fact that $x \in s\text{Cl}(B)$. Therefore $(X, \tau)$ is a Semi-$T_{1/2}$ space. 

M. Caldas in [4], introduces the concept of irresoluteness and the so called ap-irresolute maps and ap-semi-closed maps by using sg-closed sets. This definition enables us to obtain conditions under which maps and inverse maps preserve sg-closed sets. In [4] the author also characterizes the class of Semi-$T_{1/2}$ in terms of ap-irresolute and ap-semi-closed maps, where a map $f : (X, \tau) \rightarrow (Y, \sigma)$ is said to be: (i) Approximately irresolute (or ap-irresolute) if, $s\text{Cl}(F) \subseteq f^{-1}(O)$ whenever $O$ is a semi-open subset of $(Y, \sigma)$, $F$ is a sg-closed subset of $(X, \tau)$, and $F \subseteq f^{-1}(O)$; (ii) Approximately semi-closed (or ap-semi-closed) if $f(B) \subseteq s\text{Int}(A)$ whenever $A$ is a sg-open subset of $(Y, \sigma)$, $B$ is a semi-closed subset of $(X, \tau)$, and $f(B) \subseteq A$.

**Theorem 2.8.** For a topological space $(X, \tau)$ the following are equivalent:
(i) $(X, \tau)$ is a Semi-$T_{1/2}$ space.
(ii) For every space $(Y, \sigma)$ and every map $f : (X, \tau) \rightarrow (Y, \sigma)$, $f$ is ap-irresolute.

**Proof.** (i)→(ii). Let $F$ be a sg-closed subset of $(X, \tau)$ and suppose that $F \subseteq f^{-1}(O)$ where $O \in SO(Y, \sigma)$. Since $(X, \tau)$ is a Semi-$T_{1/2}$ space, $F$ is semi-closed (i.e., $F = s\text{Cl}(F)$). Therefore $s\text{Cl}(F) \subseteq f^{-1}(O)$. Then $f$ is ap-irresolute.

(ii)→(i). Let $B$ be a sg-closed subset of $(X, \tau)$ and let $Y$ be the set $X$ with the topology $\sigma = \{\emptyset, B, Y\}$. Finally let $f : (X, \tau) \rightarrow (Y, \sigma)$ be the identity map.
By assumption $f$ is $ap$-irresolute. Since $B$ is $sg$-closed in $(X, \tau)$ and semi-open in $(Y, \sigma)$ and $B \subseteq f^{-1}(B)$, it follows that $sCl(B) \subseteq f^{-1}(B) = B$. Hence $B$ is semi-closed in $(X, \tau)$ and therefore $(X, \tau)$ is a $Semi-T_{1/2}$ space.

**Theorem 2.9.** For a topological space $(X, \tau)$ the following are equivalent:

(i) $(Y, \sigma)$ is a $Semi-T_{1/2}$ space.

(ii) For every space $(X, \tau)$ and every map $f : (X, \tau) \rightarrow (Y, \sigma)$, $f$ is $ap$-semi-closed.

**Proof.** Analogous to Theorem 2.8 making the obvious changes.

Recently, in [6], M. Caldas and J. Dontchev used the $g.\Lambda_s$-sets to define a new closure operator $C^{\wedge_s}$ and a new topology $\tau^{\wedge_s}$ on a topological space $(X, \tau)$. By definition for any subset $B$ of $(X, \tau)$, $C^{\wedge_s}(B) = \bigcap \{U : B \subseteq U, U \in D^{\wedge_s}\}$. Then, since $C^{\wedge_s}$ is a Kuratowski closure operator on $(X, \tau)$, the topology $\tau^{\wedge_s}$ on $X$ is generated by $C^{\wedge_s}$ in the usual manner, i.e., $\tau^{\wedge_s} = \{B : B \subseteq X, C^{\wedge_s}(B^c) = B^c\}$.

We conclude the work mentioning a new characterization of $Semi-T_{1/2}$ spaces using the $\tau^{\wedge_s}$ topology.

**Theorem 2.10.** For a topological space $(X, \tau)$ the following are equivalent:

(i) $(X, \tau)$ is a $Semi-T_{1/2}$ space.

(ii) Every $\tau^{\wedge_s}$-open set is a $\vee_s$-set.

**Proof.** See [6].

**References**


[6] M. Caldas, J. Dontchev, $\Lambda_s$-closure operator and the associated topology $\tau^{\Lambda_s}$ (In preparation)


