Purity and Direct Summands

Pureza y Sumandos Directos

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Abstract

A criteria for a pure submodule to be a direct summand is given and some applications are derived.
Key words and phrases: Pure submodule, flat module, regular ring.

Resumen

Se da un criterio para que un submódulo puro sea sumando directo y se deducen algunas aplicaciones.
Palabras y frases clave: Submódulo puro, módulo plano, anillo regular.

1 Preliminaries

In what follows \( R \) will denote an associative ring with identity and \( R \)-module will mean unitary left \( R \)-module. Recall that a short exact sequence of \( R \)-modules:

\[
0 \rightarrow N \rightarrow M \rightarrow F \rightarrow 0
\]

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is pure if it remains exact after being tensored with any right \( R \)-module. If \( N \) is a submodule of a \( R \)-module \( M \) and the canonical short exact sequence

\[
0 \to N \to M \to M/N \to 0
\]

is pure, then we say that \( N \) is a pure submodule of \( M \). It follows at once that:

**Lemma 1** Every direct summand is a pure submodule.

For completeness we sketch a proof of the following well known result

**Lemma 2** Let

\[
0 \to N \to P \to F \to 0
\]

be a short exact sequence of \( R \)-modules with \( P \) flat. The sequence is pure exact \( \iff \) \( F \) is flat.

**Proof:** Both implications can be obtained by diagram chasing. For example, assume \( F \) flat. We have to prove that

\[
0 \to M \otimes N \to M \otimes P \to M \otimes F \to 0
\]

is exact for any right \( R \)-module \( M \). Choose a short exact sequence

\[
0 \to S \to L \to M \to 0
\]

with \( L \) free. The result follows by diagram chasing applied to the following diagram with exact rows and columns:

\[
\begin{array}{c}
0 \\
\downarrow \\
S \otimes N \to L \otimes N \to M \otimes N \to 0 \\
\downarrow \\
S \otimes P \to L \otimes P \to M \otimes P \to 0 \\
\downarrow \\
0 \to S \otimes F \to L \otimes F \to M \otimes F \to 0 \\
\downarrow \\
0 \\
\end{array}
\]

Recall the characterization of a (Von Neumann) regular ring as a ring \( R \) such that every \( R \)-module is flat. From this and the above definition of purity it follows (noted by Gentile [2]) that:
Lemma 3  $R$ is a regular ring if and only if any submodule (of any $R$-module) is pure.

Recall also the following characterization of purity due to P. M. Cohn [1]: a submodule $N$ of an $R$-module $M$ is pure if and only if for any finite family $(x_i)_{i=1}^m$ of elements of $N$, any finite family $(y_j)_{j=1}^n$ of elements of $M$, and relations

$$x_i = \sum_j a_{ij}y_j \quad (a_{ij} \in R, \ i = 1, \ldots, m, \ j = 1, \ldots, n)$$

there exist $z_1, \ldots, z_n \in N$ such that

$$x_i = \sum_j a_{ij}z_j$$

2 Some purity results

The next theorem is a partial converse of Lemma 1:

Theorem 4  If $P$ is a projective $R$-module and $N$ a finitely generated pure submodule of $P$, then $N$ is a direct summand of $P$.

Proof: Suppose first that $P$ is free with basis $(e_j)_{j \in J}$. Choose a finite set $(x_i)_{i=1}^m$ of generators of $N$. We have

$$x_i = \sum_{j \in J_0} a_{ij}e_j \quad (i = 1, \ldots, m)$$

for some $a_{ij} \in R$ and finite $J_0 \subset J$. By purity there exist $z_j \in N (j \in J_0)$ such that

$$x_i = \sum_{j \in J_0} a_{ij}z_j$$

Define $\alpha: P \to N$ by $\alpha(e_j) = z_j$ if $j \in J_0$ and $\alpha(e_j) = 0$ if $j \notin J_0$. If $\beta: N \to P$ is the inclusion map, we have $\alpha\beta = 1_N$ and so $N$ is a direct summand of $P$. For the general case, there exists a free $R$-module $L$ such that $P$ is a direct summand of $L$. By the particular case $N$ is a direct summand of $L$

$$L = N \oplus N'$$
then $P = N \bigoplus (N' \cap P)$ and $N$ is a direct summand of $P$.

Now we give a criteria of purity:

**Proposition 5** Let $N$ be a submodule of a $R$-module $M$. If $N$ is projective and every map $f : N \rightarrow R$ can be extended to a map $f' : M \rightarrow R$, then $N$ is a pure submodule of $M$.

**Proof:** Consider the situation

$$x_i = \sum_j a_{ij}y_j$$

where $x_i \in N$, $y_j \in M$, $a_{ij} \in R$ for $i = 1, \ldots, m$, $j = 1, \ldots, n$. Being $N$ projective there exist a set of generators $(e_h)_{h \in H}$ of $N$ and a set $(f_h)_{h \in H}$ of linear functionals $f_h : N \rightarrow R$ such that for each $x \in N$, $f_h(x) = 0$ for almost all $h$, and

$$x = \sum_h f_h(x)e_h$$

By hypothesis $f_h$ extends to $f'_h : M \rightarrow R$ and then

$$f_h(x_i) = \sum_j a_{ij}f'_h(y_j) \quad \forall h \in H.$$ 

So

$$x_i = \sum_h f_h(x_i)e_h = \sum_j a_{ij}(\sum_h f'_h(y_j)e_h)$$

and $N$ is a pure submodule of $M$.

## 3 Applications

In this section we show the ubiquity of Theorem 4, obtaining results that arises in several different contexts.

The first application is a classical theorem due to Villamayor:

**Corollary 6** A finitely presented flat module is projective.
Proof: Let $F$ be a finitely presented and flat $R$-module. We have an exact sequence

\[ 0 \to N \to L \to F \to 0 \]

where $L$ is free and $N$ a finitely generated submodule of $L$. By Lemma 2, $N$ is a pure submodule of $L$ and, by Theorem 4, a direct summand of $L$ (i.e. the sequence splits). Hence $F$ is projective.

The next result is due to Kaplansky ([3], Th.1.11).

Corollary 7 If $P$ is a projective module over a regular ring then every finitely generated submodule of $P$ is a direct summand.

Proof: Since over a regular ring every submodule (of any module) is pure (Lemma 3), the result follows from Theorem 4.

As a final application we give a variation of a result due to Gentile ([2], Prop. 3.1). Recall that a left semihereditary ring is a ring such that every finitely generated submodule of a finitely generated projective module is also projective.

Corollary 8 A ring $R$ is regular if and only if it is left semihereditary and for any finitely generated projective $R$-module $P$, and any finitely generated submodule $N$ of $P$, every map $f: N \to R$ extends to $f: P \to R$.

Proof: Assume $R$ regular and let $N$ be a finitely generated submodule of a finitely generated projective $R$-module $P$. By Lemma 3 and Theorem 4, $N$ is a direct summand of $P$. It follows that $R$ is left semihereditary and the property of extension of maps holds. Conversely (recall the characterization of a regular ring as a ring such that any finitely generated left ideal is a direct summand) let $I$ be a finitely generated left ideal of $R$. Being $R$ semihereditary $I$ is projective and then by Proposition 5 it is a pure submodule of $R$. Finally, by Theorem 4, $I$ is a direct summand of $R$.

4 Final comment

Theorem 4, as one of the referees pointed out to the author, may also be obtained as a consequence of a result of O. Villamayor (see Lemma 2.2 in [4]).
References


