ϕ (F_{11}) = 88

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Abstract

Here, we show that the numbers appearing in the title give the largest solution to the Diophantine equation

\[ ϕ(F_n) = a \frac{10^n - 1}{10 - 1}, \quad a ∈ \{1, \ldots, 9\}, \]

where ϕ is the Euler function and F_n is the n-th Fibonacci number.

Key words and phrases: Fibonacci numbers, Euler’s ϕ function, Diophantine equation.

Resumen

Aquí se muestra que los números que aparecen en el título dan la mayor solución a la ecuación diofántica

\[ ϕ(F_n) = a \frac{10^n - 1}{10 - 1}, \quad a ∈ \{1, \ldots, 9\}, \]

donde ϕ es la función de Euler y F_n es el n-ésimo número de Fibonacci.

Palabras y frases clave: número de Fibonacci, función ϕ de Euler, ecuación diofántica.

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For a positive integer \( n \) let \( \phi(n) \) be its Euler function. Let \( (F_n)_{n \geq 0} \) be the Fibonacci sequence given by \( F_0 = 0, F_1 = 1 \) and \( F_{n+2} = F_{n+1} + F_n \) for all \( n \geq 0 \). Recall that a positive integer is a rep-digit (in the decimal system) if it is of the form \( a(10^m - 1)/9 \) for some digit \( a \in \{1, \ldots, 9\} \). Here, we prove the following result.

**Theorem 1.** The largest positive integer solution \((n, m, a)\) of the equation

\[
\phi(F_n) = a \frac{10^m - 1}{10 - 1}, \quad a \in \{1, \ldots, 9\}
\]  

(1)

is \((n, m, a) = (11, 2, 8)\).

**Proof.** For a positive integer \( k \) let \( \mu_2(k) \) be the order at which 2 divides the positive integer \( k \). Since \( (10^m - 1)/9 \) is always odd, we get that if \((n, m, a)\) satisfy equation (1), then

\[
\mu_2(\phi(F_n)) = \mu_2\left(a \frac{10^m - 1}{9}\right) = \mu_2(a) \leq 3.
\]

One checks by hand that \( n = 11 \) gives the largest solution of equation (1) when \( n \leq 24 \). Assume now that \( n > 24 \). We show that there exists a prime factor \( p \) of \( F_n \) such that \( p \equiv 1 \pmod{4} \). Indeed, let \( (L_k)_{k \geq 0} \) be the Lucas sequence given by \( L_0 = 2, L_1 = 1 \) and satisfying the same recurrence relation \( L_{k+2} = L_{k+1} + L_k \) for all \( k \geq 0 \) as \( (F_k)_{k \geq 0} \) does. It is well-known that

\[
L_k^2 - 5F_k^2 = 4(-1)^k.
\]

(2)

If there exists a prime \( r \geq 5 \) dividing \( n \), then \( F_r \) is odd and \( F_r \mid F_n \). Let \( p \) be any prime factor of \( F_r \). Reducing relation (2) with \( k = r \) modulo \( p \), we get \( L_r^2 \equiv -4 \pmod{p} \). Since \( p \) is odd, this leads to the conclusion that \(-1\) is a quadratic residue modulo \( p \); hence, \( p \equiv 1 \pmod{4} \). Assume now that the largest prime factor of \( n \) is \( \leq 3 \). Note that \( 3^2 \) does not divide \( n \) since otherwise \( F_9 \mid F_n \), therefore \( \mu_2(\phi(F_n)) \geq \mu_2(\phi(F_9)) = \mu_2(\phi(34)) = \mu_2(16) = 4 \), which is impossible. Finally, if \( n = 2^a \cdot 3 \) or \( n = 2^a \), then, since \( n > 24 \), we get that either \( 12 \mid n \) or \( 32 \mid n \); hence,

\[
\mu_2(\phi(F_n)) \geq \min\{\mu_2(\phi(F_{12})), \mu_2(\phi(F_{32}))\} = \min\{\mu_2(\phi(2^{12})), \mu_2(\phi(3^{32}))\},
\]

\[
\phi(3 \cdot 7 \cdot 47 \cdot 2207) = 4,
\]

which is again impossible. Thus, we have shown that there exists a prime \( p \equiv 1 \pmod{4} \) which divides \( F_n \). Clearly, \( p - 1 \mid \phi(F_n) \) and so \( \mu_2(p - 1) \geq 2 \).
\[ \phi(F_{11}) = 88 \]

This shows that either there exists one other odd prime factor of \( F_n \), let’s call it \( q \), or \( p \) is the only odd prime factor of \( F_n \).

**Case 1. There exists an odd prime factor \( q \neq p \) of \( F_n \).**

Assume that \( n \) is odd. Since \( n \) is odd, relation (2) with \( k = n \) gives \( L_n^2 - 5F_n^2 = -4 \). Reducing the above equation modulo both \( p \) and \( q \), we get that \( L_n^2 \equiv -4 \pmod{p} \) and also \( L_n^2 \equiv -4 \pmod{q} \). In particular, \( -1 \) is a quadratic residue modulo both \( p \) and \( q \), which implies that both \( p \) and \( q \) are congruent to \( 1 \) modulo \( 4 \). Since \( (\mod 4) \) and discriminant \( \Delta \) must be a perfect square. Computing \( \Delta \) modulo 5 we immediately see that \( \Delta = 810 \).

Equation (1) becomes

\[ m = \frac{810}{2} = 405. \]

Since \( \phi(pq) \) is a quadratic residue modulo both \( p \) and \( q \), we get that \( \phi(F_n) \) is even. Then \( F_n = F_{2h} = F_hL_hL_{2h} \). Relation (2) with \( k = h \) together with the fact that \( F_h \) is odd implies that \( F_h \) and \( L_h \) are coprime. Furthermore, it is easy to see that \( m \) is odd. Indeed, assume that \( m = 2h \) is even. Then \( F_n = F_{4h} = F_hL_hL_{2h} \). Relation (2) with \( k = h \) together with the fact that \( F_h \) is odd gives that \( F_{2h} \) and \( L_{2h} \) are also coprime. Since \( h = n/4 > 6 \), we get that \( L_{2h} > F_{2h} > L_h > F_6 = 8 \). This argument shows that \( F_n \) has at least three odd prime factors, and since at least one of them (namely \( p \)) is congruent to \( 1 \) modulo \( 4 \), we get that \( \mu_2(\phi(F_n)) \geq 4 \), which is a contradiction. Hence, \( m \) is odd, therefore each prime factor of \( F_m \) is \( 1 \) modulo \( 4 \). Since \( p \equiv 1 \pmod{4} \) and \( q \equiv 3 \pmod{4} \), we get that \( F_m = p^\beta \) and \( L_m = q^\gamma \). Since \( m = n/2 > 12 \), it follows, from the known perfect powers in the Fibonacci and Lucas sequences [1], that \( \beta = \gamma = 1 \). Thus, \( F_n = pq \). Since clearly \( a = 8 \), equation (1) becomes

\[ 8 \cdot \frac{10^m - 1}{9} = \phi(F_n) = \phi(pq) = (p-1)(q-1) = pq + 1 - (p+q) = F_n + 1 - (p+q), \]

therefore

\[ p + q = F_n + 1 - 8 \cdot \frac{10^m - 1}{9}. \]

Since also \( pq = F_n \), we get that \( p \) and \( q \) are the two roots of the quadratic equation

\[ x^2 - \left( F_n + 1 - 8 \cdot \frac{10^m - 1}{9} \right) x + F_n = 0. \]

In order for the last equation above to have integer solutions \( p \) and \( q \), its discriminant \( \Delta \) must be a perfect square. Computing \( \Delta \) modulo 5 we imme-
diately get
\[
\Delta \equiv \left( F_n + 1 - 8 \frac{10^m - 1}{9} \right)^2 - 4F_n \pmod{5}
\]
\[
\equiv (F_n + 1 + 8 \cdot 9^{-1})^2 + F_n \pmod{5}
\]
\[
\equiv (F_n + 3)^2 + F_n = F_n^2 + 2F_n + 4 \pmod{5}.
\]

Clearly, \(F_n\) is not a multiple of 5 because \(n = 2m, m = n/2 > 12\) and \(F_m\) and \(L_m\) are both primes (note that \(F_5 = 5\)). The only value of \(b \in \{1, 2, 3, 4\}\) such that \(b^2 + 2b + 4\) is a perfect square modulo 5 is \(b = 3\). Thus, \(F_n \equiv 3 \pmod{5}\). The sequence \((F_k)_{k \geq 0}\) is periodic modulo 5 with period 20 and if \(F_n \equiv 3 \pmod{5}\), then \(n \equiv 4, 6, 7, 13 \pmod{20}\). Since \(n = 2m\) is even but not a multiple of 4, we get that \(n \equiv 6 \pmod{20}\). Hence, \(m \equiv 3 \pmod{10}\). Both \((F_k)_{k \geq 0}\) and \((L_k)_{k \geq 0}\) are periodic modulo 11 with period 10. Since \(m \equiv 3 \pmod{10}\), we get that \(p = F_m \equiv F_3 \equiv 2 \pmod{11}\) and \(q = L_m \equiv L_3 \equiv 4 \pmod{11}\). Thus, \(\phi(F_n) = (p-1)(q-1) \equiv 3 \pmod{11}\). Reducing now equation (1) modulo 11 we get
\[
3 \equiv 8((-1)^m - 1)9^{-1} \pmod{11},
\]
which leads to \(27 \equiv 0, -16 \pmod{11}\), which is impossible. This takes care of Case 1.

**Case 2.** \(p\) is the only odd prime factor of \(F_n\).

Write \(F_n = 2^\alpha p^\beta\). If \(2 \mid F_n\), then \(3 \mid n\). Put \(n = 3m\). Then \(F_n = F_{3m} = F_{m}(5F_{m}^2 + 3)\). One checks easily that \(gcd(F_m, 5F_{m}^2 + 3) = 1\) or 3. If \(3 \mid F_m\), then \(3 \neq p\) (because \(p \equiv 1 \pmod{4}\)), so \(F_n\) is divisible by two distinct primes, which is a contradiction. Thus, \(3 \mid F_m\), therefore \(F_m\) and \(5F_{m}^2 + 3\) are coprime. Since \(5F_{m}^2 + 3\) is odd, we get that \(p \mid 5F_{m}^2 + 3\), which in turn leads to the conclusion that \(F_m\) is a power of 2, which is impossible because \(m = n/2 > 12\) (the largest power of 2 in the Fibonacci sequence is \(F_6 = 8\)). Thus, \(\alpha = 0\). By the known perfect powers in the Fibonacci sequence again, we get that \(\beta = 1\). Hence, \(F_n = p\), therefore \(\phi(F_n) = p - 1 = F_n - 1\). We thus get the equation
\[
F_n - 1 = a \frac{10^m - 1}{9}.
\]
Furthermore, since \(F_n - 1 = p - 1\) is a multiple of 4, we get that \(a\) is a multiple of 4. Thus, \(a \in \{4, 8\}\). When \(a = 4\), we get that
\[
F_n = 4 \frac{10^m - 1}{9} + 1 = \frac{4 \cdot 10^m + 5}{9}.
\]
is a multiple of 5; hence, not a prime. Thus, \( a = 8 \). We now show that \( m \) is even. Indeed, assume that \( m \) is odd. Then \( 10^m \equiv -1 \pmod{11} \) which leads to the conclusion that the right hand side of equation (1) is congruent to 8 modulo 11. Hence, \( F_n \equiv 9 \pmod{11} \). The period of the Fibonacci sequence \((F_k)_{k \geq 0} \pmod{11}\) is 10. Checking the first 10 values one concludes that there is no Fibonacci number \( F_n \) which is congruent to 9 modulo 11. Hence, \( m \) is even. Since \( n > 24 \), we get that \( F_n > 10^2 \), therefore \( m \geq 3 \). Rewriting equation (1) as

\[
9F_n - 1 = 8 \cdot 10^m,
\]

we get that \( 9F_n - 1 \equiv 0 \pmod{64} \). The Fibonacci sequence \((F_k)_{k \geq 0} \) is periodic modulo 64 with period 96. Further, checking the first 96 values one gets that \( n \equiv 14, 37, 59 \pmod{96} \). Since \( n \) is odd, we get that \( n \equiv \pm 37 \pmod{96} \). But 96 is also the period of the Fibonacci sequence modulo 47, and if \( n \equiv \pm 37 \pmod{96} \), then \( F_n \equiv 5 \pmod{47} \). Reducing now equation (3) modulo 47 we get \( 44 \equiv 8 \cdot 10^m \pmod{47} \), which is equivalent to \( 29 \equiv (10^{m/2})^2 \pmod{47} \). However, this last congruence is false because 29 is not a quadratic residue modulo 47 as it can be seen since

\[
\left( \frac{29}{47} \right) = \left( \frac{47}{29} \right) = \left( \frac{18}{29} \right) = \left( \frac{2}{29} \right) = -1,
\]

because \( 29 \equiv 5 \pmod{8} \). In the above calculations, we used \( \left( \frac{p}{q} \right) \) for the Legendre symbol of \( p \) with respect to \( q \) (where \( q > 2 \) is prime) and its elementary properties. This takes care of Case 2 and completes the proof of Theorem 1.

\[\square\]

**Remark.** The argument from the beginning of the proof of Theorem 1 could be somewhat simplified using a result of McDaniel [3] who showed that if \( n \notin \{0, 1, 2, 3, 4, 6, 8, 16, 24, 32, 48\} \), then \( F_n \) has a prime factor which is congruent to 1 modulo 4. Furthermore, some of the arguments from the proof could also be simplified if one appeals to the **Primitive Divisor Theorem** for the Fibonacci sequence [2], which says that if \( n > 12 \), then there exists a prime factor \( p \mid F_n \) such that \( p \nmid F_m \) for any positive integer \( m < n \). We have however preferred to give a self-contained proof of our Theorem 1 up to the knowledge of perfect powers in the Fibonacci and Lucas sequence [1]. It would be interesting to give a completely elementary proof of Theorem 1 (i.e., without appealing to the results from [1]). We could not succeed in finding such an argument and we leave this as a challenge to the reader.
References

