

An Application of the Gelfand-Mazur Theorem: the Fundamental Theorem of Algebra Revisited

*Una Aplicación del Teorema de Gelfand-Mazur:
el Teorema Fundamental del Algebra Revisitado*

J. M. Almira (jmalmira@ujaen.es)

Departamento de Matemáticas, Universidad de Jaén
E. U. P. Linares, C/ Alfonso X el Sabio, 28
23700 Linares (Jaén) SPAIN.

Abstract

The main goal of this note is to give a new elementary proof of the Fundamental Theorem of Algebra. This proof is based on the use of the well known Gelfand-Mazur Theorem.

Key words and phrases: Fundamental theorem of algebra, Gelfand-Mazur's theorem.

Resumen

El principal objetivo de esta nota es dar una nueva prueba elemental del Teorema Fundamental del Álgebra. Esta prueba se basa en el uso del bien conocido teorema de Gelfand-Mazur.

Palabras y frases clave: Teorema fundamental del álgebra, teorema de Gelfand-Mazur.

Gelfand-Mazur theorem is at the core of the theory of commutative Banach algebras and has many interesting applications [1], [9]. It claims that the only normed fields that there exists, up to Banach algebra isometries, are \mathbb{R} (the set of real numbers) and \mathbb{C} (the set of complex numbers), both equipped with their standard absolute value. This result was announced by Mazur in 1938 [6] and proved by Gelfand in 1941 [2]. In this note we explore how to use Gelfand-Mazur's result in order to give a new proof of the Fundamental Theorem of Algebra (FTA in all what follows)

The first thing we should note is that there are proofs of Gelfand-Mazur theorem which do not use the FTA nor any other result, like the well known Liouville's principle, which is at the heart of other demonstration of the FTA.

In fact, the most extended proof of Gelfand-Mazur's theorem uses Liouville's theorem (see [8]). Fortunately there are other proofs. Concretely, those by Kametami [4] and Rickart [7] are based on the continuity properties of the product in a Banach algebra and the fact that for every $n \in \mathbb{N}$ the polynomial $x^n - 1$ is decomposable in linear factors over the set of complex numbers, which is a result weaker than the FTA (and easy to prove if you know Euler's formula $e^{i\theta} = \cos \theta + i \sin \theta$). The proofs by Kametami and Rickart have also the advantage that they belong to the so called "elementary proofs" of Gelfand-Mazur's theorem. Indeed they can be explained at the second year undergraduate level in a mathematics faculty.

Now it is time to give our proof of the FTA.

Theorem 1 (FTA). *Every polynomial $p(z) = a_0 + a_1z + \dots + a_nz^n$ is decomposable as a product of linear factors*

$$p(z) = a_n \prod_{k=1}^n (z - \alpha_k), \quad \{\alpha_k\}_{k=1}^n \subset \mathbb{C}.$$

Proof. We prove that the only irreducible elements of the ring $\mathbb{C}[z]$ are the linear polynomials. Hence if we assume that $p(z) = a_0 + \dots + a_nz^n \in \mathbb{C}[z]$ is an irreducible polynomial with $a_n \neq 0$ we must show that $n = 1$.

Now, if $(p) = p(z)\mathbb{C}[z]$ denotes the ideal generated by $p(z)$, the ring $\mathbf{A} = \mathbb{C}[z]/(p)$ is a field since the irreducibility of $p(z)$ is equivalent to the maximality of the ideal (p) (see [3], [5]). Moreover, \mathbf{A} is also an n -dimensional \mathbb{C} -vector space with basis $\beta = \{1 + (p), z + (p), \dots, z^{n-1} + (p)\}$, so that \mathbf{A} is isomorphic to \mathbb{C}^n via the natural map $\pi : \mathbb{C}^n \rightarrow \mathbf{A}$, $\pi(a_0, \dots, a_{n-1}) = a_0 + \dots + a_{n-1}z^{n-1} + (p)$.

Let us consider the norm $\|\cdot\|_* : \mathbf{A} \rightarrow \mathbb{R}^+$ given by $\|a\|_* = \|L_a\|$, where $L_a : \mathbf{A} \rightarrow \mathbf{A}$ is the linear operator given by $L_a(b) = a \cdot b$ and $\|L_a\|$ denotes the standard norm of L_a . Clearly, $(\mathbf{A}, \|\cdot\|_*)$ is a normed field, since

$$\|a \cdot b\|_* = \|L_{a \cdot b}\| = \|L_a L_b\| \leq \|L_a\| \|L_b\| = \|a\|_* \|b\|_*.$$

It follows from Gelfand-Mazur's theorem that there exists an isometry of Banach algebras $\tau : \mathbf{A} \rightarrow \mathbb{C}$. Of course, this implies that $n = 1$ since τ is also an isomorphism of \mathbb{C} -vector spaces and $\dim_{\mathbb{C}} \mathbf{A} = n$. This ends the proof. \square

References

- [1] F. F. Bonsall, J. Duncan, *Complete normed algebras*, Springer-Verlag, New-York, 1973.
- [2] I. Gelfand, *Normierte Ringe*, Mat. Sbornik N. S. **9** (51) (1941) 3–24.
- [3] I. N. Herstein, *Topics in Algebra*, John Wiley and Sons, 1975.
- [4] S. Kametami, *An elementary proof of the fundamental theorem of normed fields*, J. Math. Soc. Japan **4** (1) (1952) 96–99.
- [5] S. Lang, *Undergraduate Algebra*, Springer-Verlag, 1987.
- [6] S. Mazur, *Sur les anneaux linéaires*, C. R. Acad. Sci. Paris **207** (1938) 1025–1027.
- [7] C. E. Rickart, *An elementary proof of a fundamental theorem in the theory of Banach algebras*, Michigan Math. J. **5** (1958) 75–78.
- [8] W. Rudin, *Real and complex analysis*, McGraw-Hill Inc., 1976.
- [9] W. Zelazko, *Banach Algebras*, Elsevier, 1973.