(α, β, θ, ∂, ℐ)-Continuous Mappings and their Decomposition

Aplicaciones (α, β, θ, ∂, ℐ)-Continuas y su Descomposición

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Abstract

In this paper we introduce the concept of (α, β, θ, ∂, ℐ)-continuous mappings and prove that if α, β are operators on the topological space (X, τ) and θ, θ∗, ∂ are operators on the topological space (Y, ϕ) and ℐ a proper ideal on X, then a function f : X → Y is (α, β, θ ∧ θ∗, ∂, ℐ)-continuous if and only if it is both (α, β, θ, ∂, ℐ)-continuous and (α, β, θ∗, ∂, ℐ)-continuous, generalizing a result of J. Tong. Additional results on (α, Int, θ, ∂, {∅})-continuous maps are given.

Key words and phrases: P-continuous, mutually dual expansions, expansion continuous

Resumen

En este artículo se introduce el concepto de aplicación (α, β, θ, ∂, ℐ)-continua y se prueba que si α, β son operadores en el espacio topológico (X, τ) y θ, θ∗, ∂ son operadores en el espacio topológico (Y, ϕ) y ℐ es un ideal propio en X, entonces una función f : X → Y es (α, β, θ ∧ θ∗, ∂, ℐ)-continua si y sólo si es (α, β, θ, ∂, ℐ)-continua y (α, β, θ∗, ∂, ℐ)-continua, generalizando un resultado de J. Tong. Se dan resultados adicionales sobre aplicaciones (α, Int, θ, ∂, {∅})-continuas.

Palabras y frases clave: P-continuas, expansiones mutuamente duales, expansión continua.

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1 Introduction

In [17] Kasahara introduced the concept of an operation associated with a topology \( \tau \) on set \( X \) as a map \( \alpha : \tau \rightarrow P(X) \) such that \( U \subset \alpha(U) \) for every \( U \in \tau \). In [30] J. Tong called this kind of maps, expansions on \( X \). In [24] Vielma and Rosas modified the above definition by allowing the operator \( \alpha \) to be defined on \( P(X) \); they are called operators on \( (X, \tau) \).

Preliminaries

First of all let us introduce a concept of continuity in a very general setting: In fact, let \( (X, \tau) \) and \( (Y, \varphi) \) be two topological spaces, \( \alpha \) and \( \beta \) be operators on \( (X, \tau) \), \( \theta \) and \( \partial \) be operators in \( (Y, \varphi) \) respectively. Also let \( I \) be a proper ideal on \( X \).

Definition 1. A mapping \( f : X \rightarrow Y \) is said to be \((\alpha, \beta, \theta, \partial, I)\)-continuous if for every open set \( V \in \varphi \), \( \alpha \left( f^{-1}(\partial V) \right) \setminus \beta f^{-1}(\theta V) \in I \).

We can see that the above definition generalizes the concept of continuity, when we choose: \( \alpha = \) identity operator, \( \beta = \) interior operator, \( \partial = \) identity operator, \( \theta = \) identity operator and \( I = \{ \emptyset \} \).

Also, if we ask the operator \( \alpha \) to satisfy the additional condition that \( \alpha(\emptyset) = \emptyset \), \( \partial \leq \theta \), then the constant maps are always \((\alpha, \beta, \theta, \partial, I)\)-continuous for any ideal \( I \) on \( X \).

1. In fact, let \( f : X \rightarrow Y \) be a map such that \( f(x) = y_0 \ \forall x \in X \). Let \( V \) be an open set in \( (Y, \varphi) \)

   \[ \text{• If } y_0 \in V, \text{ then } f^{-1}(\partial V) = X, \alpha \left( f^{-1}(\partial V) \right) = X, f^{-1}(\theta V) = X, \beta \left( f^{-1}(\theta V) \right) = X \text{ Then } \alpha \left( f^{-1}(\partial V) \right) \setminus \beta f^{-1}(\theta V) = \emptyset \in I \]

   \[ \text{• If } y_0 \notin V \text{ but } y_0 \in \partial V \text{ and } y_0 \in \theta V \text{ then } \]

   \[ f^{-1}(\partial V) = X \quad f^{-1}(\theta V) = X \quad \alpha \left( f^{-1}(\partial V) \right) = X \quad \beta f^{-1}(\theta V) = X \]

   and \( \alpha \left( f^{-1}(\partial V) \right) \setminus \beta f^{-1}(\theta V) = \emptyset \in I \)

   If \( y_0 \notin \partial V \) then

   \[ f^{-1}(\partial V) = \emptyset \quad f^{-1}(\theta V) = \emptyset \quad \alpha \left( f^{-1}(\partial V) \right) = \emptyset, \quad \beta f^{-1}(\theta V) \subset X \]

   and \( \alpha \left( f^{-1}(\partial V) \right) \setminus \beta f^{-1}(\theta V) = \emptyset \in I \)
If $y_0 \notin \partial V$ and $y_0 \in \theta V$ then

$$f^{-1}(\partial V) = \emptyset \quad f^{-1}(\theta V) = X$$

$$\alpha (f^{-1}(\partial V)) = \emptyset \quad \beta f^{-1}(\theta V) = X$$

and $\alpha (f^{-1}(\partial V)) \setminus \beta f^{-1}(\theta V) = \emptyset \in I$

Let us give a historical justification of the above definition:

1. In 1922, H. Blumberg [5] defined the concept of densely approached maps: For every open set $V$ in $Y, f^{-1}(V) \subset Int f^{-1}(V)$. Here $\alpha =$ identity operator, $\beta =$ Interior closure operator, $\partial =$ identity operator, $\theta =$ identity operator and $I = \{\emptyset\}$.

2. In 1932, S. Kempisty [14] defined quasi-continuous mappings: For every open set $V$ in $Y, f^{-1}(V)$ is semi open. Here $\alpha =$ identity operator, $\beta =$ Interior operator, $\partial =$ identity operator, $\theta =$ identity operator and $I =$ nowhere dense sets of $X$.

3. In 1961, Levine [18] defined weakly continuous mappings: For every open set $V$ in $Y, f^{-1}(V) \subset Int f^{-1}(cl V)$. Here $\alpha =$ identity operator, $\beta =$ Interior operator, $\partial =$ identity operator, $\theta =$ closure operator and $I =$ nowhere dense sets of $X$.

4. In 1966, Singal and Singal [27] defined almost continuous mappings: For every open set $V$ in $Y, f^{-1}(V) \subset Int f^{-1}(Intcl V)$. Here $\alpha =$ identity operator, $\beta =$ Interior operator, $\partial =$ identity operator, $\theta =$ interior closure operator and $I =$ nowhere dense sets of $X$.

5. In 1972, S. G. Crossley and S. K. Hildebrand [8] defined irresolute maps: For every semi open set $V$ in $Y, f^{-1}(V)$ is semi open. Here $\alpha =$ identity operator, $\beta =$ Interior operator, $\partial =$ identity operator, $\theta =$ identity operator and $I =$ nowhere dense sets of $X$.


7. In 1982, J. Tong [29] defined weak almost continuous mappings: For every open set $V$ in $Y, f^{-1}(V) \subset Int f^{-1}(Int Kercl V)$. Here $\alpha =$ identity operator, $\beta =$ Interior operator, $\partial =$ identity operator, $\theta =$ Interior Kernel closure operator and $I =$ nowhere dense sets of $X$.

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8. In 1982, J. Tong [29] defined \textit{very weakly continuous} maps. For every open set \( V \) in \( Y \), \( f^{-1}(V) \subset \text{Int} f^{-1}(\text{Ker} V) \). Here \( \alpha = \text{identity operator} \), \( \beta = \text{Interior operator} \), \( \partial = \text{identity operator} \), \( \theta = \text{Kernel closure operator} \) and \( I = \{\emptyset\} \).

9. In 1984, T. Noiri [22] defined \textit{perfectly continuous} maps: For every open set \( V \) in \( Y \), \( f^{-1}(V) \) is clopen. Here \( \alpha = \text{Closure operator} \), \( \beta = \text{Interior operator} \), \( \partial = \text{identity operator} \), \( \theta = \text{identity operator} \) and \( I = \{\emptyset\} \).

10. In 1985, D. S. Jankovic [13], defined \textit{almost weakly continuous} maps: For every open set \( V \) in \( Y \), \( f^{-1}(V) \subset \text{Int} \text{clf}^{-1}(\text{cl} V) \). Here \( \alpha = \text{Identity operator} \), \( \beta = \text{Interior closure operator} \), \( \partial = \text{identity operator} \), \( \theta = \text{closure operator} \) and \( I = \{\emptyset\} \).

In order to continue the justification of the above definition, let us consider a certain property \( P \) that is satisfied by a collection of open sets in \( Y \).

\textbf{Definition 2.} A map \( f : X \to Y \) is said to be \( P \)-continuous if \( f^{-1}(U) \) is open for each open set \( U \) in \( Y \) satisfying property \( P \).

Let \( \theta_P : P(Y) \to P(Y) \) be an operator in \((Y, \varphi)\) defined as follows:

\[
\theta_P(A) = \begin{cases} A & \text{if } A \text{ is open and satisfies property } P \\ Y & \text{otherwise} \end{cases}
\]

\textbf{Theorem 1.} A map \( f : X \to Y \) is \( P \)-continuous if and only if it is \((\text{id}, \text{int}, \theta_P, \text{id}, \{\emptyset\})\)-continuous.

\textit{Proof.} In fact, suppose that \( f \) is \( P \)-continuous and let \( V \) an open set in \((Y, \varphi)\).

\textit{Case 1.} If \( V \) satisfies property \( P \), \( \theta_P(V) = V \), then by hypothesis \( f^{-1}(V) \) is open and then \( f^{-1}(V) \subset \text{Int} f^{-1}(\theta_P(V)) = \text{Int} f^{-1}(V) \).

\textit{Case 2.} If \( V \) does not satisfy property \( P \), then \( f^{-1}(V) \subset \text{Int} f^{-1}(\theta_P(V)) = Y \).

Conversely, suppose that \( f^{-1}(V) \subset \text{Int} f^{-1}(\theta_P(V)) \) for each open set \( V \) in \((Y, \varphi)\). Take \( V \) an open set satisfying property \( P \), then \( \theta_P(V) = V \) and since \( f^{-1}(V) \subset \text{Int} f^{-1}(\theta_P(V)) = \text{Int} f^{-1}(V) \). We conclude that \( f^{-1}(V) \) is open and then \( f \) is \( P \)-continuous. \( \square \)

11. In 1970, K. R. Gentry and H. B. Hoyle [12] defined \( C \)-continuous functions: For every open set \( V \) in \( Y \) with compact complement, \( f^{-1}(V) \) is open.

12. In 1971, Y. S. Park [23] defined \( C^* \)-continuous function: For every open set \( V \) in \( Y \) with countably compact complement, \( f^{-1}(V) \) is open.


Definition 3. If $\beta$ and $\beta^*$ are operators on $(X, \tau)$, the intersection operator $\beta \wedge \beta^*$ is defined as follows

$$(\beta \wedge \beta^*)(A) = \beta(A) \cap \beta^*(A)$$

The operators $\beta$ and $\beta^*$ are said to be mutually dual if $\beta \wedge \beta^*$ is the identity operator.

Theorem 2. Let $(X, \tau)$ and $(Y, \varphi)$ be two topological spaces and $\mathcal{I}$ a proper ideal on $X$. Let $\alpha, \beta$ be operators on $(X, \tau)$ and $\partial, \theta$ and $\theta^*$ be operators on $(Y, \varphi)$. Then a function $f : X \rightarrow Y$ is $(\alpha, \beta, \theta, \partial, \mathcal{I})$-continuous if and only if it is both $(\alpha, \beta, \theta, \partial, \mathcal{I})$ and $(\alpha, \beta, \theta^*, \partial, \mathcal{I})$-continuous, provided that $\beta(A \cap B) = \beta(A) \cap \beta(B)$.

Proof. If $f$ is both $(\alpha, \beta, \theta, \partial, \mathcal{I})$ and $(\alpha, \beta, \theta^*, \partial, \mathcal{I})$-continuous, then for every open set $V$ in $(Y, \varphi)$

$$\alpha (f^{-1}(\partial V)) \setminus \beta f^{-1}(\theta V) \in \mathcal{I}$$

and

$$\alpha (f^{-1}(\partial V)) \setminus \beta f^{-1}(\theta^* V) \in \mathcal{I},$$
then
\[ [\alpha (f^{-1}(\partial V)) \setminus \beta f^{-1}(\theta V)] \cup [\alpha (f^{-1}(\partial V)) \setminus \beta f^{-1}(\theta^* V)] \in I. \]

But
\[ [\alpha (f^{-1}(\partial V)) \setminus \beta f^{-1}(\theta V)] \cup [\alpha (f^{-1}(\partial V)) \setminus \beta f^{-1}(\theta^* V)] \]
\[ = [\alpha (f^{-1}(\partial V)) \setminus \beta f^{-1}(\theta V) \cap \beta f^{-1}(\theta^* V)] \]
\[ = [\alpha (f^{-1}(\partial V)) \setminus \beta f^{-1}(\theta V \cap \theta^* V)] \]
then \( f \) is \((\alpha, \beta, \theta, \partial, I)\)-continuous.

Conversely, if \( f \) is \((\alpha, \beta, \theta, \partial, I)\)-continuous, then
\[ \alpha (f^{-1}(\partial V)) \setminus \beta f^{-1}((\theta \land \theta^*) V) \in I. \]

Now, by the above equalities we get that
\[ [\alpha (f^{-1}(\partial V)) \setminus \beta f^{-1}(\theta V)] \cup [\alpha (f^{-1}(\partial V)) \setminus \beta f^{-1}(\theta^* V)] \in I \]
which implies
\[ \alpha (f^{-1}(\partial V)) \setminus \beta f^{-1}(\theta V) \in I \quad \text{and} \quad \alpha (f^{-1}(\partial V)) \setminus \beta f^{-1}(\theta^* V) \in I \]
which means that \( f \) is both \((\alpha, \beta, \theta, \partial, I)\) and \((\alpha, \beta, \theta^*, \partial, I)\)-continuous. \( \square \)

**Corollary 1 (Theorem 1 in [30]).** Let \((X, \tau)\) and \((Y, \varphi)\) be two topological spaces and \( A \) and \( B \) be two mutually dual expansions on \( Y \). Then a mapping \( f : X \to Y \) is continuous if and only if \( f \) is \( A \)-expansion continuous and \( B \)-expansion continuous.

**Proof.** Take \( \alpha = \text{identity operator}, \ \beta = \text{Int}, \ \theta = A, \ \theta^* = B, \ \partial = \text{identity operator} \) and \( I = \{\emptyset\} \), then the result follows from Theorem 2. \( \square \)

**Corollary 2 (Corollary 28 in [10]).** Let \((X, \tau)\) and \((Y, \varphi)\) be two topological spaces. A mapping \( f : X \to Y \) is continuous if and only if \( f \) is almost continuous and \( f^{-1}(V) \subset \text{Int} f^{-1}(\partial_s V) \subset \text{Int} f^{-1}(\partial_s V) \) for each open set \( V \in \varphi \).

**Proof.** Almost continuous equals \((\text{id}, \text{Int}, \text{Int closure}, \text{id}, \{\emptyset\})\)-continuous. Since the operator \( \Lambda : P(X) \to P(X) \) where
\[ \Lambda(A) = (\partial_s A)^c = A \cup (\text{Int closure } A)^c \]
is mutually dual with the \text{Int closure} \( A \) operator, the result follows from Theorem 2. \( \square \)

In the set \( \Phi \) of all operators on a topological space \((X, \tau)\) a partial order can be defined by the relation \( \alpha < \beta \) if and only if \( \alpha(A) \subset \beta(A) \) for any \( A \in P(X) \).
Theorem 3. Let $(X, \tau)$ and $(Y, \varphi)$ be two topological spaces, $\mathcal{I}$ an ideal on $X$, $\alpha$ and $\beta$ operators on $(X, \tau)$ and $\partial, \theta$ and $\theta^*$ operators on $(Y, \varphi)$ with $\theta < \theta^*$. If $f : X \to Y$ is $(\alpha, \beta, \theta, \partial, \mathcal{I})$-continuous then it is $(\alpha, \beta, \theta^*, \partial, \mathcal{I})$-continuous, provided that $\beta$ is a monotone operator.

Proof. Since $f$ is $(\alpha, \beta, \theta, \partial, \mathcal{I})$-continuous, then for every open set $V$ in $(Y, \varphi)$ it happens that
\[ \alpha (f^{-1}(\partial V)) \setminus \beta f^{-1}(\theta V) \in \mathcal{I}. \]
Now we know that $\theta < \theta^*$, then for every $V \in \varphi$, $\theta(V) \subset \theta^*(V)$ and then $f^{-1}(\theta V) \subset f^{-1}(\theta^* V)$ and
\[ \beta f^{-1}(\theta V) \subset \beta f^{-1}(\theta^* V). \]
Therefore
\[ \alpha (f^{-1}(\partial V)) \setminus \beta f^{-1}(\theta^* V) \subset \alpha (f^{-1}(\partial V)) \setminus \beta f^{-1}(\theta V) \in \mathcal{I}, \]
then
\[ \alpha (f^{-1}(\partial V)) \setminus \beta f^{-1}(\theta^* V) \in \mathcal{I}, \]
which means that $f$ is $(\alpha, \beta, \theta^*, \partial, \mathcal{I})$-continuous. \(\square\)

Definition 4. An operator $\beta$ on the space $(X, \tau)$ induces another operator $\text{Int}\beta$ defined as follows
\[ (\text{Int}\beta)(A) = \text{Int}(\beta(A)) \]
Observe that $\text{Int}\beta < \beta$.

Definition 5. A function $f : X \to Y$ satisfies the openness condition with respect to the operator $\beta$ on $X$ if for every $B$ in $Y$, $\beta f^{-1}(B) \subset \beta f^{-1}(\text{Int}B)$.

Remark. If $\beta$ is the interior operator it is routine verification to prove that the openness condition with respect to $\beta$ is equivalent to the condition of being open.

Theorem 4. Let $(X, \tau)$ and $(Y, \varphi)$ be two topological spaces. If $f : X \to Y$ is $(\alpha, \beta, \theta, \partial, \mathcal{I})$ continuous and satisfies the openness condition with respect to the operator $\beta$, then $f$ is $(\alpha, \beta, \text{Int}\theta, \partial, \mathcal{I})$ continuous.

Proof. Let $V$ be an open set in $(Y, \varphi)$ we have that
\[ \alpha (f^{-1}(\partial V)) \setminus \beta f^{-1}(\theta V) \in \mathcal{I} \]

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since $f$ satisfies the openness condition with respect to the operator $\beta$, then

$$\beta f^{-1}(\partial V) \subset \beta f^{-1}(\text{Int}_V).$$

since

$$\alpha\left(f^{-1}(\partial V)\right) \setminus \beta f^{-1}(\text{Int}_V) \subset \alpha\left(f^{-1}(\partial V)\right) \setminus \beta f^{-1}(\partial V) \in \mathcal{I}$$

it follows that $f$ is $\left(\alpha, \beta, \text{Int}_\theta, \partial, \{\emptyset\}\right)$ continuous. \qed

\noindent \textbf{Corollary 3 (Theorem 2.3 [27]).} Let $(X, \tau)$ and $(Y, \varphi)$ be two topological spaces. If $f : X \rightarrow Y$ is weakly continuous and open then it is almost continuous.

\noindent \textit{Proof.} Let $\mathcal{I} = \{\emptyset\}$. $\alpha$ = identity operator, $\beta$ = Int, $\partial$ = identity operator and $\theta$ = closure operator then the result follows from Theorem 4. \qed

\noindent \textbf{Corollary 4.} Let $(X, \tau)$ and $(Y, \varphi)$ be two topological spaces. If $f : X \rightarrow Y$ is very weakly continuous and open, then it is weak almost continuous.

\noindent \textit{Proof.} Let $\mathcal{I} = \{\emptyset\}$, $\alpha$ = identity operator, $\beta$ = Int, $\partial$ = identity operator and $\theta$ = ker closure operator, then the result follows from Theorem 3. \qed

\section{Some results on $\left(\alpha, \text{Int}, \theta, \partial, \{\emptyset\}\right)$-continuous maps}

\noindent \textbf{Definition 6.} Let $\beta$ be an operator in a topological space $(X, \tau)$. We say that $(X, \tau)$ is $\beta - T_1$ if for every pair of points $x, y \in X$, $x \neq y$ there exists open sets $V$ and $W$ such that $x \in V$ and $y \notin \beta V$ and $y \in W$ and $x \notin \beta W$.

Observe that if $\beta$ is the closure operator $\text{Cl}$ then a space $(X, \tau)$ is $T_2$ if and only if it is $\text{Cl} - T_1$.

\noindent \textbf{Theorem 5.} Let $(X, \tau)$ and $(Y, \varphi)$ be two topological spaces, $\alpha$ an operator on $(X, \tau)$, $\theta$ and $\partial$ operators on $(Y, \varphi)$ and $(Y, \varphi)$ a $\theta - T_1$ space. If $f : X \rightarrow Y$ is $\left(\alpha, \text{Int}, \theta, \partial, \{\emptyset\}\right)$ continuous and $A \subset \alpha(A)$ for all $A \subset X$, then $f$ has closed point inverses.

\noindent \textit{Proof.} Let $q \in Y$ and let $a \in A = \{x \in X : f(x) \neq q\}$. Then there exists open sets $V$ and $V'$ in $(Y, \varphi)$ such that $f(a) \in V$ and $q \notin \theta V$. By hypothesis

$$\alpha\left(f^{-1}(\partial V)\right) \subset \text{Int}_f^{-1}(\partial V)$$
so there exists an open set $U$ in $(X, \tau)$ such that
\[ \alpha(f^{-1}(\partial V)) \subset U \subset f^{-1}(\theta V) \]
so $f(U) \subset \theta V$. If $b \in U \cap A^c$ then $f(b) \in \theta V$ and $f(b) = q \notin \theta V$ therefore $a \in U$ and $U \subset A$, therefore $\{x \in X : f(x) \neq q\}$ is open. \[\Box\]

**Corollary 5 (Theorem 6 in [31]).** Let $(X, \tau)$ and $(Y, \varphi)$ be two topological spaces. Let $f : X \to Y$ be a weakly continuous function. If $Y$ is Hausdorff then $f$ has closed point inverses.

**Proof.** Let $\alpha = \text{identity operator}$, $\beta = \text{Int}$, $\partial = \text{identity operator}$, $\theta = \text{Closure operator}$ and $I = \{\emptyset\}$, then the result follows from Theorem 5. \[\Box\]

**Theorem 6.** Let $(X, \tau)$ and $(Y, \varphi)$ be two topological spaces. $\alpha$ an operator on $(X, \tau)$, $\theta$ and $\partial$ operators on $(Y, \varphi)$, $A \subset \alpha(A) \forall A \subset X$. If $f : X \to Y$ is $(\alpha, \text{Int}, \theta, \partial, \{\emptyset\})$ continuous and $K$ is a compact subset of $X$, then $f(K)$ is $\theta$ compact on $Y$.

**Proof.** Let $\nu$ be an open cover of $f(K)$ and suppose without lost of generality that each $V \in \nu$ satisfies $V \cap f(K) \neq \emptyset$. Then for each $k \in K$, $f(k) \in V_k$ for some $V_k \in \nu$. Since $f$ is $(\alpha, \text{Int}, \theta, \partial, \{\emptyset\})$-continuous, for each $k \in K$ there exists an open set $W_k$ in $X$ such that
\[ \alpha(f^{-1}(\partial V_k)) \subset W_k \subset f^{-1}(\theta V_k). \]
Also since $f^{-1}(\partial V_k) \subset \alpha(f^{-1}(\partial V_k))$ for every $k \in K$ we have that the collection $\{W_k : k \in K\}$ is an open cover of $K$, so there exists $k_1, ..., k_n$ such that
\[ K \subset \bigcup_{i=1}^{n}(W_{k_i}). \]
Then $f(K) \subset \bigcup_{i=1}^{n}f(W_{k_i})$. Therefore
\[ f(K) \subset \bigcup_{i=1}^{n}\theta V_{k_i} \]
which means that $f(K)$ is $\theta$-compact. \[\Box\]

**Corollary 6 (Theorem 7 in [31]).** Let $(X, \tau)$ and $(Y, \varphi)$ be two topological spaces. Let $f : X \to Y$ be a weakly continuous map and $K$ a compact subset of $X$ then $f(K)$ is an almost compact subset of $Y$.

**Proof.** Let $\alpha = \text{identity operator on } X$, $\beta = \text{Int}$, $\theta = \text{closure operator on } Y$, $\partial = \text{identity operator}$ and $I = \{\emptyset\}$. \[\Box\]
Corollary 7 (Theorem 3.2 in [25]). Let \((X, \tau)\) and \((Y, \varphi)\) be two topological spaces. Let \(f : X \to Y\) be an almost continuous map and \(K\) a compact subset of \(X\), then \(f(K)\) is nearly compact.

Proof. Let \(\alpha =\) identity operator on \(X\), \(\beta = \text{Int}\), \(\theta =\) closure operator on \(Y\), \(\partial =\) identity operator and \(\mathcal{I} = \{\emptyset\}\. \)

References


