Remarks on fixed points of rotative Lipschitzian mappings

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Abstract. Let $C$ be a nonempty closed convex subset of a Banach space $E$ and $T : C \to C$ a $k$-Lipschitzian rotative mapping, i.e. such that $\|Tx - Ty\| \leq k \cdot \|x - y\|$ and $\|T^n x - x\| \leq a \cdot \|x - Tx\|$ for some real $k$, $a$ and an integer $n > a$. The paper concerns the existence of a fixed point of $T$ in $p$-uniformly convex Banach spaces, depending on $k$, $a$ and $n = 2, 3$.

Keywords: rotative mappings, fixed points

Classification: 47H09, 47H10

1. Introduction

Many authors discussed the problem concerning the existence of fixed points for different class of mappings defined on nonempty closed convex subsets $C$ of infinite dimensional Banach space $E$ and satisfying some metric conditions. The main problem was connected with establishing some conditions of geometrical nature implying the fixed point property for nonexpansive mappings $T : C \to C$ (i.e. mappings satisfying $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y$ in $C$). The usual assumptions are those of uniform convexity and normal structure.

In 1981, Goebel and Koter [6] defined the conditions of rotativeness (see below) and proved the following

**Theorem 1.** If $C$ is a nonempty closed convex subset of a Banach space $E$, then any nonexpansive rotative mapping $T : C \to C$ has a fixed point. □

Note that this result does not require weak compactness or even boundedness of $C$, or any special geometric structure on $C$.

Further on, the authors studied the existence of fixed points for some class of $k$-Lipschitzian ($k > 1$) and rotative mappings in Banach spaces ([7], [13]).

In this note we extend Goebel and Koter’s results for a real $p$-uniformly convex Banach space and give an estimate for the function $\gamma_3$ in a Hilbert space.

2. Preliminaries

Let $C$ be a nonempty closed convex subset of a Banach space $E$. A mapping $T : C \to C$ is called $(n, a)$-rotative if there exists an integer $n \geq 2$ and a real number $0 \leq a < n$ such that for any $x \in C$, $\|x - T^n x\| \leq a \cdot \|x - Tx\|$.
The simplest examples of rotative mappings are contractions and rotation of the Euclidean space $\mathbb{R}^n$ or any periodic nonexpansive mappings (i.e. $T^n = I$ for some $n \in \mathbb{N}$, where $I$ means identity mapping) in any Banach space.

**Definition 1.** Denote by $\Phi(n, a, k, C)$ the class of all mappings $T : C \to C$ which are $(n, a)$-rotative and satisfy the following condition

$$\forall x, y \in C \quad \|Tx - Ty\| \leq k \cdot \|x - y\|.$$  

A mapping $T \in \Phi(n, a, k, C)$ is said to be $k$-Lipschitzian $(n, a)$-rotative on $C$.

We shall now consider mappings of the family $\Phi(n, a, k, C)$ with $k > 1$. For fixed $n \in \mathbb{N}$ put

$$\gamma_n(a) = \inf \left\{ k > 1 : \text{there exists a set } C \text{ (closed convex) and} \right. \\
\left. \text{a mapping } T \text{ such that } T \in \Phi(n, a, k, C) \right. \\
\left. \text{and } F(T) = \emptyset \right\}$$

($F(T)$ denotes the set of all fixed points of $T$).

The definition of $\gamma_n(a)$ implies that for an arbitrary set $C$, if $T \in \Phi(n, a, k, C)$ and $k < \gamma_n(a)$, then $T$ has at least one fixed point. It was proved in [7] that for an arbitrary Banach space $E$ and for any $n \in \mathbb{N}$, we have $\gamma_n(a) > 1$ for all $a < n$. It is a qualitative result which raises a number of technical yet attractive questions concerning the precise values of $\gamma_n(a)$. Even the exact value of $\gamma_n(0)$ is of interest since it characterizes the fixed point behavior of mappings of period $n$ (see [11], [16] and [4], [8], [9], [10] for involutions, i.e. mappings $T$ for which $T^2 = I$).

### 3. About the function $\gamma_2(a)$

Now, we restrict our attention to the case $n = 2$. It was proved in [5] that for an arbitrary Banach space $E$

$$\gamma_2(a) \geq \gamma_B(a), \quad a \in [0, 2),$$

where

$$\gamma_B(a) = \max \left\{ \frac{1}{2} \left[ 2 - a + \sqrt{(2 - a)^2 + a^2} \right], \\
\frac{1}{8} \left[ a^2 + 4 + \sqrt{(a^2 + 4)^2 - 64 \cdot (a - 1)} \right] \right\}. $$
Surprisingly, it is possible to show that the first term provides a better estimate if $a \leq 2(\sqrt{2} - 1) \approx 0.828$, while the second is better for $a \in [2(\sqrt{2} - 1), 2)$.

No upper bound for $\gamma_2(a)$ with $a \in [0, 1]$ is known until now, while if $a \in (1, 2)$ we have $\gamma_2(a) \leq \frac{k_R \cdot (a+1)}{a-1}$, where $k_R$ is the minimal Lipschitz constant of the retraction of the unit ball onto the unit sphere in $E$ (see Example 1 in [13]). In general, the value of $k_R$ is unknown, so that the bound given above shows only that $\gamma_2(a) < +\infty$ for $a \in (1, 2)$. It is however essential that this fact is true in an arbitrary Banach space. In $C[0, 1]$ or $L^1[0, 1]$, we have $\gamma_2(a) \leq \frac{1}{a-1}$, $a \in (1, 2)$ (see Examples 1, 2 in [7] and Example 17.2 in [5]).

These results are illustrated in Figure 1.

Denote

$D_1 = \{(a, k) \in [0, 2) \times [0, +\infty) : k < \gamma_2(a)\};$

$D_2 = \{(a, k) \in (1, 2) \times (1, +\infty) : k \geq \frac{k_R \cdot (a+1)}{a-1}\};$

$D_3 = \{(a, k) \in (1, 2) \times (1, +\infty) : k \geq \frac{1}{a-1}\};$

$D_4 = [0, 2) \times [0, +\infty) \setminus (D_1 \cup D_3).$

If $T$ is $k$-Lipschitzian and $(2, a)$-rotative, where $(a, k) \in D_1$, then $T$ has at least one fixed point. In other words: the graph of the function $\gamma_2$ for an arbitrary
space $E$ lies above the region $D_1$. On the other hand, it lies always below the curve which is the lower bound of the region $D_2$ (in some spaces even below the lower bound of $D_3$). The existence of fixed points for mappings $T \in \Phi(2, a, k, C)$, where $(a, k) \in D_4$, remains an open problem.

However, in some spaces one can slightly raise the lower bound of the region $D_4$. Koter [13] proved the following theorem (in spaces with known modulus of convexity, see [5]).

**Theorem 2.** Let $C$ be a nonempty closed convex subset of a Banach space $E$ with the modulus of convexity $\delta_E$. If $T \in \Phi(2, a, k, C)$ and

$$1 - \delta_E(2/k) \leq \frac{2 - a}{k},$$

then $T$ has at least one fixed point. \hfill \Box

Since in the space $L^p$ (or $\ell^p$), $p \in (2, +\infty)$, we have $\delta_p(\varepsilon) = 1 - (1 - (\varepsilon/2)^p)^{1/p}$, routine calculations and the previous estimates (1) yield

**Corollary 1.** Let $C$ be a nonempty closed convex subset of the space $L^p$ (or $\ell^p$), $2 < p < +\infty$. If $T \in \Phi(2, a, k, C)$ and

$$k < \max \left\{ \gamma_B(a), [(2 - a)^p + 1]^{1/p} \right\}, \quad a \in [0, 2),$$

then $T$ has at least one fixed point. \hfill \Box

Hence, in the space $L^p$ (or $\ell^p$), $2 < p < +\infty$, we have

$$\gamma_2(a) \geq \max \left\{ \gamma_B(a), [(2 - a)^p + 1]^{1/p} \right\}, \quad a \in [0, 2).$$

Komorowski [12] shows that for a real Hilbert space $\mathcal{H}$ we have a better bound for $\gamma_2$, namely

$$\gamma_2(a) \geq \sqrt{\frac{5}{a^2 + 1}} = \gamma_\mathcal{H}(a), \quad a \in [0, 2)$$

(see Figure 2).

4. The function $\gamma_2$ in $p$-uniformly convex spaces

In this section we give some estimates of the function $\gamma_2$ by means of inequalities in Banach spaces.

Let $p > 1$ and denote by $\lambda$ a number in $[0, 1]$ and by $W_p(\lambda)$ the function $\lambda \cdot (1 - \lambda)^p + \lambda^p \cdot (1 - \lambda)$.

The functional $\| \cdot \|_p^p$ is said to be *uniformly convex* ([22]) on the Banach space if

$$(*) \text{ there exists a positive constant } c_p \text{ such that for all } \lambda \in [0, 1] \text{ and } x, y \in E \text{ the following inequality holds: }$$

$$\| \lambda \cdot x + (1 - \lambda) \cdot y \|_p^p \leq \lambda \cdot \| x \|_p^p + (1 - \lambda) \cdot \| y \|_p^p - c_p \cdot W_p(\lambda) \cdot \| x - y \|_p^p.$$
Xu [12] proved that the functional $\| \cdot \|^p$ is uniformly convex on the whole Banach space $E$ if and only if $E$ is $p$-uniformly convex, i.e. there exists constant $c > 0$ such that the modulus of convexity (see [5]) $\delta_E(\varepsilon) \geq c \cdot \varepsilon^p$ for all $0 \leq \varepsilon \leq 2$. We note that a Hilbert space $H$ is 2-uniformly convex (indeed $\delta_H(\varepsilon) = 1 - \sqrt{1 - (\varepsilon/2)^2} \geq (1/8) \cdot \varepsilon^2$) and $L^p$ (or $\ell^p$) ($1 < p < +\infty$) is max(2,$p$)-uniformly convex.

**Theorem 3.** Let $E$ be a Banach space with the norm satisfying (*) for some $p > 1$, let $C$ be a nonempty closed convex subset of $E$. If $T \in \Phi(2,a,k,C)$ and

$$k < \max \left\{ 1, \left[ \frac{1 + 2^p}{2^{p-2} \cdot (1 + a^p)} \right]^{1/p} \right\} \text{ if } c_p = 1,$$

or

$$k < \max \left\{ 1, \left[ \frac{c_p + 2^p}{2^{p-2} \cdot (2 - c_p) \cdot (1 + a^p)} \right]^{1/p}, \right. \left[ \frac{\sqrt[2p-1]{1 + a^p})^2 + 8 \cdot (1 - c_p) \cdot (2^p + c_p) - 2^{p-1} \cdot (1 + a^p)}{2 \cdot (1 - c_p)} \right]^{1/p} \left\} \right.$$  

if $0 < c_p < 1$ and $a \in [0, 2)$,

then $T$ has at least one fixed point.
Proof: If \( k < 1 \), then the Banach Contraction Principle implies that \( T \) has a fixed point. Thus we assume that \( k \geq 1 \). Let \( x \) be an arbitrary point in the set \( C \) and \( \varepsilon \) an arbitrary real positive number. Suppose that
\[
\|T^2x - Tx\|^p > (1 - \varepsilon) \cdot \|x - Tx\|^p
\]
and put \( z = (1/2)(Tx + T^2x) \). Then we have
\[
\|z - Tz\|^p = \|((1/2) \cdot (Tx + T^2x) - Tz\|^p
\]
\[
= \|((1/2) \cdot (Tx - Tz) + (1/2) \cdot (T^2x - Tz))\|^p
\]
\[
\leq (1/2) \cdot \|Tx - Tz\|^p + (1/2) \cdot \|T^2x - Tz\|^p
\]
\[
- c_p \cdot (1/2)^p \cdot \|T^2x - Tx\|^p
\]
\[
\leq (1/2) \cdot k^p \|((1/2) \cdot (x - Tx) + (1/2) \cdot (x - T^2x))\|^p
\]
\[
+ (1/2) \cdot k^p \cdot \|((1/2) \cdot (Tx - T^2x))\|^p - c_p \cdot (1/2)^p \cdot \|T^2x - Tx\|^p
\]
\[
\leq \{(1/4) \cdot k^p + (1/4) \cdot k^p \cdot a^p\} \cdot \|x - Tx\|^p
\]
\[
+ (1/2)^{p+1} \cdot k^p \cdot (1 - c_p) \cdot \|T^2x - Tx\|^p - c_p \cdot (1/2)^p \cdot \|T^2x - Tx\|^p.
\]
If \( c_p = 1 \), then by last inequality we have
\[
\|z - Tz\|^p \leq \{(1/4) \cdot k^p + (1/4) \cdot k^p \cdot a^p\} \cdot \|x - Tx\|^p
\]
\[
- (1/2)^p \cdot \|T^2x - Tx\|^p
\]
\[
\leq \{(1/4) \cdot k^p + (1/4) \cdot k^p \cdot a^p - (1/2)^p \cdot (1 - \varepsilon)\} \cdot \|x - Tx\|^p
\]
\[
= f(\varepsilon) \cdot \|x - Tx\|^p.
\]
Now, assume \( 0 < c_p < 1 \).

Case I. By the estimate
\[
\|T^2x - Tx\|^p \leq \left(\|T^2x - x\| + \|x - Tx\|\right)^p
\]
\[
\leq 2^{p-1} \cdot \left(\|T^2x - x\|^p + \|x - Tx\|^p\right)
\]
\[
\leq 2^{p-1} \cdot (a^p + 1) \|x - Tx\|^p,
\]
we have
\[
\|z - Tz\|^p \leq \{(1/4) \cdot k^p + (1/4) \cdot k^p \cdot a^p
\]
\[
+ (1/2)^{p+1} \cdot k^p \cdot (1 - c_p) \cdot 2^{p-1} \cdot (a^p + 1)
\]
\[
- (1/2)^p \cdot c_p (1 - \varepsilon)\} \cdot \|x - Tx\|^p
\]
\[
= g(\varepsilon) \cdot \|x - Tx\|^p.
\]
**Case II.** By the estimate

\[ \|T^2 x - Tx\|^p \leq k^p \cdot \|Tx - x\|^p \]

we have

\[
\|z - Tz\|^p \leq \left\{ \left( \frac{1}{4} \right) \cdot k^p + \left( \frac{1}{4} \right) \cdot k^p \cdot a^p + \left( \frac{1}{2} \right)^{p+1} \cdot k^{2p} \cdot (1 - c_p) - \left( \frac{1}{2} \right)^p \cdot c_p \cdot (1 - \varepsilon) \right\} \cdot \|x - Tx\|^p \\
= h(\varepsilon) \cdot \|x - Tx\|^p.
\]

If the assumptions of the theorem are satisfied, then there exists \( \varepsilon > 0 \) such that \( \max\{f(\varepsilon), g(\varepsilon), h(\varepsilon)\} < 1 \), and we may consider the following sequence

\[
x_1 = x, \\
x_{n+1} = Tx_n \quad \text{if} \quad \|T^2 x_n - Tx_n\|^p \leq (1 - \varepsilon) \cdot \|Tx_n - x_n\|^p,
\]

or

\[
x_{n+1} = \left( \frac{1}{2} \right) (Tx_n + T^2 x_n) \quad \text{if} \quad \|T^2 x_n - Tx_n\|^p > (1 - \varepsilon) \cdot \|Tx_n - x_n\|^p
\]

for \( n = 1, 2, \ldots \).

Now, we show the convergence of the sequence \( \{x_n\} \). Indeed,

\[
\|Tx_{n+1} - x_{n+1}\|^p \leq A \cdot \|Tx_n - x_n\|^p, \quad \text{for} \quad n \in \mathbb{N},
\]

where \( A = \max\{f(\varepsilon), g(\varepsilon), h(\varepsilon), 1 - \varepsilon\} < 1 \). Thus

\[
\|Tx_{n+1} - x_{n+1}\|^p \leq A^n \cdot \|Tx_1 - x_1\|^p \to 0,
\]

as \( n \to +\infty \), which shows that \( \{x_n\} \) is a Cauchy sequence. Let \( y = \lim_{n \to \infty} x_n \).

Since \( \|Tx_{n+1} - x_{n+1}\|^p \to 0 \) as \( n \to +\infty \), we have \( Ty - y = 0 \), and \( Ty = y \). \( \square \)

5. Applications

Note that in a Hilbert space \( \mathcal{H} \) we have the identity

\[
\| \lambda \cdot x + (1 - \lambda) \cdot y \|^2 = \lambda \cdot \|x\|^2 + (1 - \lambda) \cdot \|y\|^2 - \lambda \cdot (1 - \lambda) \cdot \|x - y\|^2
\]

for all \( x, y \) in \( C \) and \( 0 \leq \lambda \leq 1 \). In this case \( p = 2 \) and \( c_2 = 1 \). Thus by Theorem 3, we have the following corollary.
Corollary 2 ([12]). Let $\mathcal{H}$ be a Hilbert space and let $C$ be a nonempty closed convex subset of $\mathcal{H}$. If $T \in \Phi(2, a, k, C)$ and

$$k < \sqrt{\frac{5}{a^2 + 1}}, \ a \in [0, 2),$$

then $T$ has at least one fixed point. \hfill \Box

If $1 < p < 2$, then we have for all $x, y$ in $L^p$ (or $\ell^p$) and $\lambda \in [0, 1]$,

$$\|\lambda \cdot x + (1 - \lambda) \cdot y\|^2 \leq \lambda \cdot \|x\|^2 + (1 - \lambda) \cdot \|y\|^2 - (p - 1) \cdot \lambda \cdot (1 - \lambda) \cdot \|x - y\|^2,$$

(see [20], [14]). Thus by Theorem 3 we have the following estimate for $k$ in $L^p$ (or $\ell^p$) spaces ($1 < p < 2$):

$$k < \max \left\{ 1, \sqrt{\frac{3 + 2(a^2)}{(1 + a^2)(3 - p)}}, \sqrt{\frac{4(1 + a^2)^2 + 8(2 - p)(3 + p) - 2(1 + a^2)^2}{2(2 - p)}} \right\} = f_p(a), \ a \in [0, 2).$$

If $p \to 2^+$, then $f_p(a) \to f_2(a) = \gamma_\mathcal{H}(a)$. Moreover, $f_p(0) > 2$ for $2 > p > 9/5$. The case $p = 3/2$ is illustrated by means of computer graphic in Figure 3.
Thus in $L^p$ (or $\ell^p$), $1 < p < 2$, we have the following

**Corollary 3.** Let $C$ be a nonempty closed convex subset of $L^p$ (or $\ell^p$), $1 < p < 2$. If $T \in \Phi(2, a, k, C)$ and

$$k < \max \left\{ \gamma_B(a), \sqrt{\frac{3+2}{(1+a^2)(3-p)}}, \frac{\sqrt[4]{4(1+a^2)^2 + 8(2-p)(3+p)-2(1+a^2)}}{2(2-p)} \right\}$$

for $a \in [0, 2)$, then $T$ has at least one fixed point. □

For all $x, y$ in $L^p$ (or $\ell^p$) spaces, $2 < p < +\infty$, and all $\lambda \in [0, 1]$, we have

$$\|\lambda \cdot x + (1 - \lambda) \cdot y\|^p \leq \lambda \cdot \|x\|^p + (1 - \lambda) \cdot \|y\|^p - c_p \cdot W_p(\lambda) \cdot \|x - y\|^p,$$

where $c_p = (p-1) \cdot (1 - t_p)^{2-p}$, and $t_p$ is the unique zero of the function $j(x) = -x^{p-1} + (p-1) \cdot x + (p-2)$ on the interval $(1, +\infty)$, see for example [18], [14].

By numerical approximation we obtain $c_{2.1} \approx 0.948917$ and the case $p = 2.1$ is illustrated in Figure 4.
Thus by Corollary 1 and Theorem 3 we have

**Corollary 4.** Let $C$ be a nonempty closed convex subset of $L^p$ (or $\ell^p$), $2 < p < +\infty$. If $T \in \Phi(2,a,k,C)$ and

$$k < \max \left\{ \gamma_B(a), \left[ (2-a)^p + 1 \right]^{1/p}, \left[ \frac{c_p + 2p}{2p-2 \cdot (2 - c_p)(1 + a_p)} \right]^{1/p}, \left[ \frac{\sqrt{(2p-1) \cdot (1 + a_p) + 8 \cdot (1 - c_p) \cdot (2p + c_p) - 2p-1 \cdot (1 + a_p)}}{2 \cdot (1 - c_p)} \right]^{1/p} \right\}$$

for $a \in [0, 2)$, then $T$ has at least one fixed point. \(\square\)

Using the result of Prus, Smarzewski ([17], [19]) we obtain from Theorem 3 a fixed point theorem, for example, for Hardy and Sobolev spaces.

Let $H^p$, $1 < p < +\infty$, denote the **Hardy space** ([3]) of all functions $x$ analytic in the unit disc $|z| < 1$ of the complex plane and such that

$$\|x\| = \lim_{r \to 1^-} \left( \frac{1}{2\pi} \int_0^{2\pi} |x(re^{i\theta})|^p \, d\theta \right)^{1/p} < +\infty.$$

Now, let $\Omega$ be an open subset of $\mathbb{R}^n$. Denote by $W^{r,p}(\Omega)$, $r \geq 0$, $1 < p < +\infty$, the **Sobolev space** ([1, p. 149]) of distributions $x$ such that $D^\alpha x \in L^p(\Omega)$ for all $|\alpha| = \alpha_1 + \alpha_2 + \cdots + \alpha_n \leq k$ equipped with the norm

$$\|x\| = \left( \sum_{|\alpha| \leq k} \int_\Omega |D^\alpha x(\omega)|^p \, d\omega \right)^{1/p}.$$

Let $(\Omega_\alpha, \Sigma_\alpha, \mu_\alpha)$, $\alpha \in \Lambda$, be a sequence of positive measure spaces, where $\Lambda$ is finite or countable. Given a sequence of linear subspaces $X_\alpha$ in $L^p(\Omega_\alpha, \Sigma_\alpha, \mu_\alpha)$, we denote by $L_{q,p}$, $1 < p < +\infty$, $q = \max(2, p)$ ([15]), the linear space of all sequences

$$x = \{ x_\alpha \in X_\alpha : \alpha \in \Lambda \}$$

equipped with the norm

$$\|x\| = \left[ \sum_{\alpha \in \Lambda} (\|x_\alpha\|_{p,\alpha})^q \right]^{1/q},$$

where $\| \cdot \|_{p,\alpha}$ denotes the norm in $L^p(\Omega_\alpha, \Sigma_\alpha, \mu_\alpha)$.

Finally, let $L^p = L^p(S_1, \Sigma_1, \mu_1)$ and $L^q = L^q(S_2, \Sigma_2, \mu_2)$, where $1 < p < +\infty$, $q = \max(2, p)$ and $(S_i, \Sigma_i, \mu_i)$ are positive measure spaces. Denote by $L_q(L_p)$ the Banach space ([2, III.2.10]) of all measurable $L^p$-valued functions $x$ on $S_2$ with the norm

$$\|x\| = \left( \int_{S_2} (\|x(s)\|)^q \, \mu_2(ds) \right)^{1/q}.$$
These spaces are $q$-uniform convex with $q = \max(2, p)$ ([17], [19]) and the norm in these spaces satisfies

$$\|\lambda \cdot x + (1 - \lambda) \cdot y\|^q \leq \lambda \cdot \|x\|^q + (1 - \lambda) \cdot \|y\|^q - d \cdot W_q(\lambda) \cdot \|x - y\|^q$$

with a constant

$$d = d_p = \frac{p - 1}{8} \quad \text{for } 1 < p \leq 2 \quad \text{and} \quad d = d_p = \frac{1}{p \cdot 2^p} \quad \text{for } 2 < p < +\infty.$$ 

Hence it follows from Theorem 3 the following

**Corollary 5.** Let $C$ be a nonempty closed convex subset of the space $X$, where $X = H^p$ or $X = W^{r,p}(\Omega)$ or $X = L_{q,p}$ or $X = L_q(L_p)$ and $1 < p < +\infty$, $q = \max(2, p)$, $r \geq 0$. If $T \in \Phi(2, a, k, C)$ and

$$k < \max \left\{ \gamma_B(a), \left[ \frac{d_p + 2^q}{2^{q-2} \cdot (2 - d_p)(1 + a^q)} \right]^{1/q}, \sqrt{\left[ \frac{[2^{q-1} \cdot (1 + a^q) + 8 \cdot (1 - d_p) \cdot (2^q + d_p) - 2^{q-1} \cdot (1 + a^q)]}{2 \cdot (1 - d_p)} \right]}^{1/q} \right\}$$

for $a \in [0, 2)$, then $T$ has at least one fixed point. \(\square\)

6. **$\gamma_3$ in a Hilbert space**

We mentioned that the function $\gamma_n$ may have different form in different spaces. Now we have to establish an evaluation of the function $\gamma_3$ in a Hilbert space.

**Theorem 4.** Let $\mathcal{H}$ be a Hilbert space and let $C$ be a nonempty closed convex subset of $\mathcal{H}$. If $T \in \Phi(3, a, k, C)$ and

$$k < \max \left\{ \sqrt{(1/2) \cdot [\sqrt{9a^4 + 2a^2 + 41} - 3 \cdot a^2 + 1]}, \sqrt{(1/2) \cdot [(1 + a^2)^2 + 40 - (1 + a^2)]} \right\}, \quad a \in [0, 3),$$

then $T$ has at least one fixed point.

(Note that it is possible to show that the second term provides a better estimate if $\sqrt{2} < a < \sqrt{(1/2)(\sqrt{29} + 7)} \approx 2.48849.$)

**Proof:** Let $x$ be an arbitrary point in the set $C$ and $\varepsilon$ an arbitrary real positive number. Suppose that

$$\|Tx - T^3x\|^2 + \|T^2x - T^3x\|^2 > (1 - \varepsilon) \cdot \|x - Tx\|^2$$
and put
\[ z = \left(\frac{1}{3}\right)(Tx + T^2x + T^3x) = \left(\frac{1}{3}\right) \cdot Tx + \left(\frac{2}{3}\right) \cdot [(1/2)(T^2x + T^3x)]. \]

Then we have
\[
\begin{align*}
\|z - Tz\|^2 &= \|\left(\frac{1}{3}\right) \cdot Tx + \left(\frac{2}{3}\right) \cdot [(1/2)(T^2x + T^3x)] - Tz\|^2 \\
&= \|\left(\frac{1}{3}\right) \cdot (Tx - Tz) + \left(\frac{2}{3}\right) \cdot [(1/2)(T^2x + T^3x) - Tz]\|^2 \\
&= \left(\frac{1}{3}\right) \cdot \|Tx - Tz\|^2 + \left(\frac{2}{3}\right) \cdot \|[(1/2)(T^2x + T^3x) - Tz]\|^2 \\
&\quad - \left(\frac{2}{9}\right) \cdot \|Tx - (1/2)(T^2x + T^3x)\|^2 \\
&\leq \left(\frac{1}{3}\right) \cdot k^2 \cdot \|x - z\|^2 + \left(\frac{2}{3}\right) \cdot \|[(1/2)(T^2x + T^3x) - Tz]\|^2 \\
&\quad - \left(\frac{2}{9}\right) \cdot \|[(1/2)(T^2x + T^3x) - Tz]\|^2 \\
&\leq \left(\frac{1}{3}\right) \cdot k^2 \cdot \|x - (1/3) \cdot Tx - (2/3) \cdot [(1/2)(T^2x + T^3x)]\|^2 \\
&\quad + \left(\frac{2}{3}\right) \cdot \left\{ \left(\frac{1}{2}\right) \cdot k^2 \cdot \|Tx - z\|^2 + \left(\frac{1}{2}\right) \cdot k^2 \cdot \|T^2x - z\|^2 \\
&\quad - \left(\frac{1}{4}\right) \cdot \|T^2x - T^3x\|^2 \right\} \\
&\quad - \left(\frac{2}{9}\right) \cdot \left\{ \left(\frac{1}{2}\right) \cdot \|Tx - T^2x\|^2 + \left(\frac{1}{2}\right) \cdot \|Tx - T^3x\|^2 \\
&\quad - \left(\frac{1}{4}\right) \cdot \|T^2x - T^3x\|^2 \right\} \\
&= \left(\frac{1}{3}\right) \cdot k^2 \cdot \left\{ \left(\frac{1}{3}\right) \cdot \|x - Tx\|^2 + \left(\frac{2}{3}\right) \cdot \|x - (1/2)(T^2x - T^3x)\|^2 \\
&\quad - \left(\frac{2}{9}\right) \cdot \|Tx - (1/2)(T^2x - T^3x)\|^2 \right\} \\
&\quad + \left(\frac{2}{3}\right) \cdot \left\{ \left(\frac{1}{2}\right) \cdot k^2 \cdot \|Tx - (1/2)(T^2x + T^3x)\|^2 \\
&\quad + (1/2) \cdot \|[(1/3)(T^2x - Tx) + (2/3)[T^2x - (1/2)(T^2x + T^3x)]]\|^2 \\
&\quad - \left(\frac{1}{4}\right) \cdot \|T^2x - T^3x\|^2 \right\} \\
&\quad - \left(\frac{2}{9}\right) \cdot \left\{ \left(\frac{1}{2}\right) \cdot \|Tx - T^2x\|^2 + \left(\frac{1}{2}\right) \cdot \|Tx - T^3x\|^2 \\
&\quad - \left(\frac{1}{4}\right) \cdot \|T^2x - T^3x\|^2 \right\} \\
&= \left(\frac{1}{9}\right) \cdot k^2 \cdot \|x - Tx\|^2 + \left(\frac{2}{9}\right) \cdot k^2 \cdot \left\{ \left(\frac{1}{2}\right) \cdot \|x - T^2x\|^2 \\
&\quad + (1/2) \cdot \|x - T^3x\|^2 - \left(\frac{1}{4}\right) \cdot \|T^2x - T^3x\|^2 \right\} \\
&\quad - \left(\frac{2}{27}\right) \cdot k^2 \cdot \|Tx - (1/2)(T^2x - T^3x)\|^2 \\
&\quad + \left(\frac{4}{27}\right) \cdot k^2 \cdot \|Tx - (1/2)(T^2x - T^3x)\|^2 \\
&\quad + (1/3) \cdot k^2 \cdot \left\{ \left(\frac{1}{3}\right) \cdot \|T^2x - Tx\|^2 + (2/3) \cdot \|T^2x - (1/2)(T^2x + T^3x)\|^2 \right\}
\end{align*}
\]
we have

\[- (2/9) \cdot \| Tx - (1/2)(T^2x - T^3x) \|^2 \leq (1/6) \cdot \| T^2x - T^3x \|^2 \]

\[- (2/9) \cdot \left\{ (1/2) \cdot \| Tx - T^2x \|^2 + (1/2) \cdot \| Tx - T^3x \|^2 \right\} \]

\[- (1/4) \cdot \| T^2x - T^3x \|^2 \]

\leq \text{ (reduction) } \]

\[- \{ (1/9) \cdot k^4 + (1/9) \cdot k^2 \} \cdot \| x - Tx \|^2 + (1/9) \cdot k^2 \cdot a^2 \cdot \| x - Tx \|^2 \]

\[- + [(1/9) \cdot k^2 - (1/9)] \cdot \| x - T^2x \|^2 \]

\[- (1/9) \cdot \{ \| Tx - T^3x \|^2 + \| T^2x - T^3x \|^2 \}. \]

**Case I.** By the estimate

\[ \| x - T^2x \|^2 \leq 2 \cdot \left( \| x - T^3x \|^2 + \| T^3x - T^2x \|^2 \right) \]

\[ \leq 2 \cdot (a^2 + k^2) \cdot \| x - T^2x \|^2, \]

we have

\[ \| z - Tz \|^2 \leq \{ (1/9) \cdot k^4 + (1/9) \cdot k^2 \} \cdot \| x - Tx \|^2 + (1/9) \cdot k^2 \cdot a^2 \cdot \| x - Tx \|^2 \]

\[- + [(1/9) \cdot k^2 - (1/9)] \cdot 2 \cdot (a^2 + k^2) \cdot \| x - Tx \|^2 \]

\[- (1/9) \cdot \{ \| Tx - T^3x \|^2 + \| T^2x - T^3x \|^2 \} \]

\[ \leq \{ (1/9) \cdot k^4 + [(3/9) \cdot a^2 - (1/9)] \cdot k^2 - (2/9) \cdot a^2 \]

\[- (1/9) \cdot (1 - \varepsilon) \} \cdot \| x - T^2x \|^2 \]

\[ = G(\varepsilon) \cdot \| x - T^2x \|^2. \]

**Case II.** By the estimate

\[ \| x - T^2x \|^2 \leq 2 \cdot \left( \| x - T^2x \|^2 + \| Tx - T^2x \|^2 \right) \]

\[ \leq 2 \cdot (1 + k^2) \cdot \| x - T^2x \|^2, \]

we have

\[ \| z - Tz \|^2 \leq \{ (1/9) \cdot k^4 + (1/9) \cdot k^2 \} \cdot \| x - Tx \|^2 + (1/9) \cdot k^2 \cdot a^2 \cdot \| x - Tx \|^2 \]

\[- + [(1/9) \cdot k^2 - (1/9)] \cdot 2 \cdot (1 + k^2) \cdot \| x - Tx \|^2 \]

\[- (1/9) \cdot \{ \| Tx - T^3x \|^2 + \| T^2x - T^3x \|^2 \} \]

\[ \leq \{ (1/9) \cdot k^4 + (1/9)(1 + a^2) \cdot k^2 - (1/9) \cdot (1 - \varepsilon) \} \cdot \| x - T^2x \|^2 \]

\[ = H(\varepsilon) \cdot \| x - T^2x \|^2. \]
If the assumptions of the theorem are satisfied, then there exists $\varepsilon > 0$ such that $\max\{G(\varepsilon), H(\varepsilon)\} < 1$, and we may consider the following sequence

\begin{align*}
x_1 &= x, \\
x_{n+1} &= T^2x_n & \text{if} \\
&\quad \|Tx_n - T^3x_n\|^2 + \|T^2x_n - T^3x_n\|^2 \leq (1 - \varepsilon) \cdot \|x_n - Tx_n\|^2,
\end{align*}

or

\begin{align*}
x_{n+1} &= (1/3)(Tx_n + T^2x_n + T^3x_n) & \text{if} \\
&\quad \|Tx_n - T^3x_n\|^2 + \|T^2x_n - T^3x_n\|^2 > (1 - \varepsilon) \cdot \|x_n - Tx_n\|^2,
\end{align*}

$n = 1, 2, \ldots$ .

It is easy to see that this sequence is convergent. Indeed,

\[ \|Tx_{n+1} - x_{n+1}\|^2 \leq A \cdot \|Tx_n - x_n\|^2, \text{ for } n \in \mathbb{N}, \]

where $A = \max\{G(\varepsilon), H(\varepsilon), 1 - \varepsilon\} < 1$. Thus

\[ \|Tx_{n+1} - x_{n+1}\|^2 \leq A^n \cdot \|T^1x - x_1\|^2 \to 0 \]

as $n \to +\infty$, which proves that $\{x_n\}$ is a Cauchy sequence. Let $y = \lim_{n \to \infty} x_n$. Since $\|Tx_{n+1} - x_{n+1}\|^2 \to 0$ as $n \to +\infty$, we have $\|Ty - y\| = 0$ and $Ty = y$. \(\square\)

Kirk [11] showed that a mapping $T : C \to C$ ($C$ is a nonempty closed convex bounded subset of a reflexive Banach space with the normal structure) for which $T^n = I$ ($n > 1$) has a fixed point if $\|T^ix - T^iy\| \leq k \cdot \|x - y\|$, $x, y \in C$, $i = 1, 2, \ldots, n - 1$, where $k$ satisfies

\[ (n - 1)(n - 2) \cdot k^2 + 2(n - 1) \cdot k < n^2. \]

Thus a $k$-Lipschitzian mapping satisfying $T^n = I$ ($n > 1$) has fixed point if

\[ (n - 1)(n - 2) \cdot k^{2(n-1)} + 2(n - 1) \cdot k^{n-1} < n^2. \]

For $n = 3$, we have the estimate $k < (1/2) \cdot \sqrt[3]{88} - 4 \approx 1.1598$. Linhart [16] showed (in an arbitrary Banach space) that this mapping has a fixed point if

\[ \frac{1}{n} \cdot \sum_{i=n-1}^{2n-3} k^i < 1. \]

Hence, for $n = 3$ we have the estimate for $k < k_0 \approx 1.174$.

By Theorem 4 a $k$-Lipschitzian involution $T$ of order $n = 3$ in a Hilbert space (i.e. $T \in \Phi(3, 0, k, C)$) has fixed points if $k < \sqrt{(1/2)(\sqrt{41} + 1)} \approx 1.92394$. 
**Theorem 5.** Let $C$ be a nonempty closed convex bounded subset of a Hilbert space $\mathcal{H}$. If $T : C \to C$ is $k$-Lipschitzian with $k < \sqrt{(1/2)(\sqrt{41} + 1)}$ and $\|T^3x - T^3y\| \leq \|x - y\|$ for $x, y$ in $C$, then there exists a fixed point of $T$.

**Proof:** According to Browder-Göhde-Kirk’s fixed point theorem [5] the set $C^* = \{x \in C : x = T^3x\}$ is nonempty. The strict convexity of $\mathcal{H}$ implies that $C^*$ is convex. Obviously, we have $T(C^*) = C^*$ and $T^3 = I$ on $C^*$. Hence, by Theorem 4, we obtain our result. $\square$

7. Open problems

The main problem of rather technical nature is whether $\gamma_n$ is continuous. Other questions concern the evaluation of $\gamma_n(a)$. The evaluation given in Theorem 3 seem, in my opinion, to be not exact (for example, $k$-Lipschitzian involutions defined on a nonempty closed convex subset of a Hilbert space have a fixed point if $k < (1/2)(\pi + \sqrt{\pi^2 - 4}) \approx 2.78215$, see [13]). We do not even know whether there exist $a \in [0, 1]$ such that $\gamma_2(a) < +\infty$ (in any Banach space), i.e. whether there exist $T \in \Phi(2, a, k, C)$, $0 \leq a \leq 1$, without fixed points. The same question can be stated for the whole interval $[0, 2]$ in the case of a Hilbert space. Analogous questions can be formulated for the function $\gamma_3$.

**References**


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(Received August 26, 1996)