A short proof on lifting of projection properties in Riesz spaces

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Abstract. Let $L$ be an Archimedean Riesz space with a weak order unit $u$. A sufficient condition under which Dedekind $\sigma$-completeness of the principal ideal $A_u$ can be lifted to $L$ is given (Lemma). This yields a concise proof of two theorems of Luxemburg and Zaanen concerning projection properties of $C(X)$-spaces. Similar results are obtained for the Riesz spaces $B_n(T), n = 1, 2, \ldots$, of all functions of the $n$th Baire class on a metric space $T$.

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The purpose of this note is to give a short and concise proof of the following result established by Luxemburg and Zaanen ([3, Theorems 43.2 and 43.3]).

**Theorem.** Let $C(X)$ and $C_b(X)$, respectively, denote the Riesz spaces of all real continuous and continuous and bounded, respectively, functions on a topological space $X$. Then the following conditions are equivalent.

(i) $C(X)$ has the [principal] projection property.

(ii) $C(X)$ is Dedekind $\sigma$-complete.

(iii) $C_b(X)$ has the [principal] projection property.

(iv) $C_b(X)$ is Dedekind $\sigma$-complete.

As remarked in ([3, p. 283]), the only nontrivial implication is (iv) $\Rightarrow$ (ii). Our proof replaces a large part of the direct argument in [3] by an appeal to a lemma (see below), inspired by the classical proof of the Tietze extension theorem ([1, p. 158], the unbounded case).

Let $S$ be a nonempty set. In the rest of the paper $L$ denotes a Riesz subspace of the Riesz space $\mathbb{R}^S$ (pointwise ordering) containing the constant-one on $S$ function $e$, and $B_e$ denotes the set $\{f \in L : |f(s)| < 1, s \in S\}$. It is obvious that $B_e$ is a (nonlinear) sublattice of $A_e$. The symbol $\circ$ denotes composition of functions.

**Lemma.** If there exists a strictly increasing and continuous function $\phi$ from $\mathbb{R}$ onto $(-1, 1)$ such that both

(a) $\phi \circ f \in B_e$ for every $f \in L$, and

(b) $\phi^{-1} \circ g \in L$ for every $g \in B_e$,
then \( L \) and \( B_e \) are order isomorphic as partially ordered sets. In particular, Dedekind \([\sigma\text{-}]\) completeness of \( A_e \) implies Dedekind \([\sigma\text{-}]\) completeness of \( L \).

**Examples.** 1. If \( L = C(X) \) then every strictly increasing, continuous and onto function \( \phi : \mathbb{R} \to (-1,1) \) fulfills both (a) and (b), and the same holds for the Riesz spaces \( B_n(T) \), \( n = 1, 2, \ldots \), of all functions \( T \to \mathbb{R} \) of the \( n \)th class on a metric space \( T \).

2. If \( L \) consists of all continuous and piecewise functions on \([0,1]\), then \( \phi \) must be piecewise linear to fulfill the condition (a).

**Proof of Lemma:** By (a) and (b), \( L \) and \( B_e \) are order isomorphic as partially ordered sets (in the sense of the definition given in [3, p.186]) via the mapping \( \hat{\phi}(f) = \phi \circ f, f \in L \). Since, by ([3, Definitions 1.1 and 23.1]), Dedekind \([\sigma\text{-}]\) completeness both is invariant under such isomorphisms and is hereditated from \( A_e \) by \( B_e \), the result follows.

**Proof of Theorem** (the nontrivial implication (iv) \( \Rightarrow \) (ii)): It follows by Lemma and Example 1.

**Remark.** Since bounded functions of the \( n \)th Baire class \( B^0_n(T) \), \( n = 1, 2, \ldots \), endowed with the sup-norm form AM-spaces with units ([2, Theorem 12.3.7]), the notions of the [principal] projection property and Dedekind \([\sigma\text{-}]\) completeness coincide (by Theorem). Moreover, Lemma and Example 1 prove that \( B^0_n(T) \) and \( B_n(T) \) are Dedekind \([\sigma\text{-}]\) complete simultaneously. These observations yield the result similar to that of Theorem when \( C(X) \) is replaced by \( B_n(T) \) and \( C_b(X) \) by \( B^0_n(T) \).

**References**


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