Representing lattices by homotopy groups of graphs

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Abstract. In this paper we represent every lattice by subgroups of free groups using the concept of the homotopy group of a graph.

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In this paper we present a method how to represent a given lattice $L$ as a sublattice of the subgroup lattice of a free group. The method is based on the idea of the homotopy group of a graph. Our construction is such that if the lattice $L$ is finite then the free group and all its subgroups representing the elements of $L$ are finitely generated. The first part of the proof is formulated in a more general way to enable further possible modifications of the proof replacing the free group by a finite group $G$.

The first proof that every lattice can be embedded into the subgroup lattice of a group was given in [Wh]. The proof was later simplified by the author of the present note in [Tů2] using the solution of the word problem for HNN-extensions.

Let $\Omega$ be a set. By a twist on $\Omega$ we mean a bijection $t: A \to B$ between two subsets $A, B$ of $\Omega$. The set $A$ is called the domain of $t$ and denoted by $\text{Dom}(t)$, while $B$ is called the range of $t$ and denoted by $\text{Rng}(t)$. If $t: A \to B$ is a twist on $\Omega$, then the inverse mapping $t^{-1}: B \to A$ is also a twist on $\Omega$ and called the inverse of $t$. The value of a twist $t$ at a point $a \in \text{Dom}(t)$ will be written as $at$.

By a twisting structure on $\Omega$ we mean a set $T = \{ t_i : i \in I \}$ of twists on $\Omega$ such that with every $t \in T$ the inverse $t^{-1}$ of $t$ is also contained in $T$.

The Cayley graph $\mathbf{G}(T)$ of a twisting structure $T$ is defined as follows. The vertex set of $\mathbf{G}(T)$ is $\Omega$. The edge set of $\mathbf{G}(T)$ is the set $E = \{(a, t) : a \in \text{Dom}(t), t \in T \}$. If $e = (a, t) \in E$, then $a$ is the initial vertex $\alpha(e)$ of $e$ and $at$ is the terminal vertex $\omega(e)$ of $e$. Since $T$ contains with every twist $t$ also the inverse $t^{-1}$ of $t$, with every edge $e = (a, t) \in E$ there is also the edge $(at, t^{-1}) \in E$. The edge $(at, t^{-1})$ is called the inverse of $e$ and denoted by $e^{-1}$. It is obvious that $\alpha(e^{-1}) = \omega(e)$ and $\omega(e^{-1}) = \alpha(e)$, thus the graph $\mathbf{G}(T)$ is a symmetric graph possibly with loops and parallel edges. We further define the value $\nu(e)$ of an edge $e = (a, t)$ as the twist $t$. Thus the Cayley graph $\mathbf{G}(T) = (\Omega, E, \alpha, \omega, \nu)$ of a twisting structure $T$ is a symmetric graph with edges valued by elements of $T$. It

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is the union of the graphs of all partial bijections $t \in T$. Note also that for every $t \in T$ and $a \in \Omega$ there is at most one edge of $G(T)$ with initial vertex $a$ and value $\nu(e) = t$.

By a congruence of a twisting structure $T$ on $\Omega$ we mean an equivalence relation $\pi$ on $\Omega$ satisfying the following condition:

\[(*) \quad \text{whenever } (a, b) \in \pi, \ t \in T \text{ and } a, b \in \text{Dom}(t), \text{ then also } (at, bt) \in \pi.\]

Thus a congruence of $T$ is a congruence of the partial algebra $(\Omega, T)$, where each $t \in T$ is considered to be a partial unary operation on $\Omega$. Obviously the set $C(T)$ of all congruences of the twisting structure $T$ on $\Omega$ is closed under arbitrary intersections and contains the least and the greatest equivalence relations on $\Omega$. Thus $C(T)$ when ordered by inclusion is a complete lattice. It is called the congruence lattice of $T$. The meet $\pi \land \rho$ of two congruences $\pi, \rho$ of $T$ is their set-theoretical intersection $\pi \cap \rho$, while their join $\pi \lor \rho$ in $C(T)$ is the least equivalence relation on $\Omega$ containing the set-theoretical union $\pi \cup \rho$ and satisfying the condition $(*)$.

The following simple representation result was proved in [Tů1].

**Theorem 1.** Every lattice $L$ can be represented as a sublattice of $C(T)$ for some twisting structure $T$. \(\square\)

For the sake of completeness we present the construction. Given a lattice $L$, we may assume that it has a least element 0. For any two non-zero elements $a < b$ of $L$ we define a twist $t_{ab}$ on $L$ with $\text{Dom}(t_{ab}) = \text{Rng}(t_{ab}) = \{0, a, b\}$ and such that

\[bt_{ab} = b, \ at_{ab} = 0, \ 0t_{ab} = a.\]

Moreover, if $a, b \in L$ are two non-comparable and non-zero elements, then we define a twist $t_{ab}$ on $L$ such that $\text{Dom}(t_{ab}) = \text{Rng}(t_{ab}) = \{a, b, a \lor b\}$ and

\[at_{ab} = a, \ bt_{ab} = a \lor b, \ (a \lor b)t_{ab} = b.\]

Let the twisting structure $T$ on $L$ consist of all twists of the form $t_{ab}$, where $0 \neq a, b \in L$. Then the mapping assigning to every $x \in L$ the partition of $L$ with one block the interval $[0, x]$ and the other blocks singletons is an embedding of $L$ into $C(T)$.

The main purpose of this note is to investigate a canonical mapping $\Phi$ from $C(T)$ into the subgroup lattice of the free group $F(T)$. Here $F(T)$ denotes the free group generated by the set $T$ of free generators. Under sufficiently general conditions on $T$ we can prove that the canonical mapping $\Phi$ is an embedding of any member from a large class of sublattices of $C(T)$. To this end we recall some basic ideas from combinatorial group theory related to homotopy groups of graphs. By a graph we mean a quadruple $(V, E, \alpha, \omega)$, where $V, E$ are non-empty sets and $\alpha, \omega : E \to V$ are two incidence functions, $\alpha(e)$ is called the initial vertex of an edge $e \in E$ and $\omega(e)$ is the terminal vertex of $e$. By a path in the
graph \((V, E, \alpha, \omega)\) we mean a sequence \(p = e_1e_2\cdots e_k\) of edges of \(E\) such that 
\[
\omega(e_i) = \alpha(e_{i+1}) \text{ for every } i = 1, 2, \ldots, k - 1.
\]
A path \(p = e_1e_2\cdots e_k\) is called a loop if \(\omega(e_k) = \alpha(e_1)\), and a loop \(p = e_1e_2\cdots e_k\) is called a loop at a vertex \(v \in V\) if \(\alpha(e_1) = v\).

We will generalize these concepts to the Cayley graph of a twisting structure \(T\). Let \(\pi\) be a congruence of \(T\). A sequence \(p = e_1e_2\cdots e_k\) of edges of the Cayley graph \(C(T)\) of \(T\) is called a \(\pi\)-path if \((\omega(e_i), \alpha(e_{i+1})) \in \pi\) for every \(i = 1, 2, \ldots, k - 1\). A path \(p = e_1e_2\cdots e_k\) is called a \(\pi\)-loop if \((\alpha(e_1), \omega(e_k)) \in \pi\) and it is called a \(\pi\)-loop at a vertex \(v \in \Omega\) if moreover \((v, \alpha(e_1)) \in \pi\). If \((\omega(e_i), \alpha(e_{i+1})) \in \tau\) for some relation \(\tau\) on \(\Omega\), then we say that \((\omega(e_i), \alpha(e_{i+1}))\) is a \(\tau\)-jump.

Suppose moreover that \(T \subset G\), where \(G\) is a group. Then we can assign to every \(\pi\)-path \(p = e_1e_2\cdots e_k\) its \(G\) value

\[
\nu(p) = \nu(e_1)\nu(e_2)\cdots \nu(e_k) \in G.
\]

The set of \(G\)-values of all \(\pi\)-loops at a vertex \(v \in \Omega\) in the Cayley graph \(C(T)\) of a twisting structure \(T\) is obviously a subgroup of \(G\). Indeed, \(1 \in G\) is the value of the empty \(\pi\)-path, if \(g = \nu(p)\) for a \(\pi\)-loop \(p = e_1e_2\cdots e_k\) at \(v\), then \(g^{-1}\) is the value of the inverse path \(p^{-1} = e_k^{-1}\cdots e_1^{-1}\). And if \(g = \nu(p), h = \nu(q)\), then \(gh = \nu(pq)\). Thus we can define a mapping

\[
\Phi_G : C(T) \rightarrow \text{Sub}(G)
\]

from the congruence lattice of \(T\) into the subgroup lattice of \(G\) by

\[
\Phi_G(\pi) = \{\nu(p) : p \text{ is a } \pi\text{-loop at } v\}.
\]

If \(\pi \subseteq \rho\) are two congruences of \(T\), then obviously any \(\pi\)-loop at \(v\) is also a \(\rho\)-loop at \(v\), thus we get the following simple lemma.

**Lemma 2.** The mapping \(\Phi_G\) is order-preserving. \(\square\)

We say that a twisting structure \(T\) is connected if its Cayley graph \(G(T)\) is connected. For connected twisting structures we have the following result.

**Theorem 3.** If \(T\) is a connected twisting structure, then the mapping \(\Phi_G\) is join-preserving.

**Proof:** First of all we describe the join \(\pi \lor \rho\) of two congruences \(\pi, \rho \in C(T)\). Set \(\sigma_0 = \pi \lor \rho\). If \(\sigma_{2i}\) is already defined for a natural number \(i\), we define

\[
\sigma_{2i+1} = \sigma_{2i} \cup \{(at, bt) : t \in T, a, b \in \text{Dom}(t), (a, b) \in \sigma_{2i}\},
\]

and

\[
\sigma_{2i+2} \text{ is the transitive closure of } \sigma_{2i+1}.
\]
Obviously, any congruence \( \sigma \) of \( T \) containing both \( \pi \) and \( \rho \) must contain also \( \sigma_n \) for any natural number \( n \). On the other hand,

\[
\sigma = \bigcup_n \sigma_n
\]

is an equivalence relation satisfying the condition (*), hence a congruence of \( T \). Thus \( \sigma = \pi \lor \rho \) in \( C(T) \).

Since \( \Phi_G \) is order-preserving, we get that

\[
\Phi_G(\pi) \lor \Phi_G(\rho) \subseteq \Phi_G(\pi \lor \rho)
\]

for any two congruences \( \pi, \rho \) of \( T \). To prove the opposite inclusion we have to show that the value \( \nu(p) \) of any \( (\pi \lor \rho) \)-path \( p \) at \( v \) is contained in the subgroup of \( G \) generated by \( \Phi_G(\pi) \cup \Phi_G(\rho) \).

So let \( p = e_1e_2\cdots e_k \) be an arbitrary \( (\pi \lor \rho) \)-loop at \( v \). Thus \( (\omega(e_i), \alpha(e_{i+1})) \in \pi \lor \rho \) for any \( i = 1, 2, \ldots, k-1 \) as well as \( (v, \alpha(e_1)), (\omega(e_k), v) \in \pi \lor \rho \). Since \( \pi \lor \rho = \bigcup_n \sigma_n \), there exists a natural number \( m \) such that

\[
(v, \alpha(e_1)), (\omega(e_k), v) \in \sigma_m, (\omega(e_i), \alpha(e_{i+1})) \in \sigma_m
\]

for every \( i = 1, 2, \ldots, k-1 \). Let us call such a \( (\pi \lor \rho) \)-loop \( p = e_1e_2\cdots e_k \) at \( v \) a \( \sigma_m \)-loop. We are going to prove that the value \( \nu(p) \) of any \( \sigma_m \)-loop at \( v \), \( m \geq 1 \), belongs to the subgroup of \( G \) generated by the values of \( \sigma_{m-1} \)-loops at \( v \).

If \( m \) is odd, let \( i \leq k \) be such that \( (\omega(e_i), \alpha(e_{i+1})) \in \sigma_m \setminus \sigma_{m-1} \). Thus there exist a twist \( t \in T \) and \( a, b \in \sigma_{m-1} \) such that \( at = \omega(e_i) \) and \( bt = \alpha(e_{i+1}) \). Consider the loop \( p' = e_1e_2\cdots e_i(at, t^{-1})(b, t)e_{i+1}\cdots e_k \). Since \( \omega(e_i) = \alpha(at, t^{-1}) \), \( (\omega(at, t^{-1}), \alpha(b, t)) = (a, b) \in \sigma_{m-1} \) and \( \omega(b, t) = bt = \alpha(e_{i+1}) \), \( p' \) is also a \( \sigma_m \)-loop at \( v \) and the number of \( \sigma_m \setminus \sigma_{m-1} \)-jumps in \( p' \) is one less than the number of \( \sigma_m \setminus \sigma_{m-1} \)-jumps in \( p \). Similarly, if \( (v, \alpha(e_1)) \in \sigma_m \setminus \sigma_{m-1} \), then again there are a twist \( t \in T \) and \( (a, b) \in \sigma_m \setminus \sigma_{m-1} \) such that \( at = v \) and \( bt = \alpha(e_1) \). Again the loop \( p' = (at, t^{-1})(b, t)p \) is a \( \sigma_m \)-loop at \( v \) (since \( (v, t^{-1}) = (at, t^{-1}) \)) and the number of \( \sigma_m \setminus \sigma_{m-1} \)-jumps in \( p' \) is one less than the number of \( \sigma_m \setminus \sigma_{m-1} \)-jumps in \( p \). The case \( (\omega(e_k), v) \in \sigma_m \setminus \sigma_{m-1} \) is treated in exactly the same way. In all cases, \( \nu(p') = \nu(p) \) and the number of \( \sigma_m \setminus \sigma_{m-1} \)-jumps in the path \( p' \) is one less than the number of \( \sigma_m \setminus \sigma_{m-1} \)-jumps in \( p \). By a simple induction on the number of \( \sigma_m \setminus \sigma_{m-1} \)-jumps in \( p \) we prove that for every \( \sigma_m \)-loop \( p \) at \( v \) there exists a \( \sigma_{m-1} \)-loop \( p'' \) at \( v \) such that \( \nu(p'') = \nu(p) \). Thus if \( m \) is odd, then the value \( \nu(p) \) of any \( \sigma_m \)-loop \( p \) at \( v \) is equal to the value of a \( \sigma_{m-1} \)-loop at \( v \).

If \( m > 0 \) is even and \( p = e_1e_2\cdots e_k \) a \( \sigma_m \)-loop at \( v \) that is not a \( \sigma_{m-1} \)-loop, then either there exists some \( i = 1, 2, \ldots, k-1 \) such that \( (\omega(e_i), \alpha(e_{i+1})) \in \sigma_m \setminus \sigma_{m-1} \) or \( (v, \alpha(e_1)) \in \sigma_m \setminus \sigma_{m-1} \) or \( (\omega(e_k), v) \in \sigma_m \setminus \sigma_{m-1} \). Let the first of the three possibilities occur. Since \( \sigma_m \) is the transitive closure of \( \sigma_{m-1} \), there are elements \( \omega(e_i) = a_1, a_2, \ldots, a_l = \alpha(e_{i+1}) \) such that \( (a_j, a_{j+1}) \in \sigma_{m-1} \). Since the Cayley graph \( G(T) \) of \( T \) is connected, there are paths \( q_i \in G(T) \) of \( T \) from \( v \) to \( a_i \). Then
$p' = e_1 e_2 \cdots e_i q_1^{-1} q_2^{-1} \cdots q_{i-1}^{-1} q_i e_{i+1} \cdots e_k$ is again a $σ_m$-loop at $v$ in which the number of $σ_m \setminus σ_m-1$-jumps is one less than the number of $σ_m \setminus σ_m-1$-jumps in $p$. Moreover, $ν(p') = ν(p)$. The other two cases are treated in exactly the same way. Thus also in this case the value of any $σ_m$-loop at $v$ coincides with the value of a $σ_m-1$-loop at $v$.

Hence the value $ν(p)$ of any $(π \lor ρ)$-loop at $v$ equals the value $ν(p')$ of a $σ_0$-loop $p'$ at $v$. Recall that $σ_0 = π \lor ρ$. Let $p' = f_1 f_2 \cdots f_l$. For every $i = 1, 2, \ldots, l$ let $q_i$ be a path in $G(T)$ from $v$ to $ω(f_i)$. Then

$$p'' = f_1 q_1^{-1} q_1 f_2 q_2^{-1} q_2 f_3 \cdots q_{l-1}^{-1} q_{l-1} f_l$$

is also a $σ_0$-loop at $v$. Obviously, $ν(p'') = ν(p)$. Finally, observe that each $q_{i-1} f_i q_i^{-1}, i = 2, \ldots, l - 1$ is either a $π$-loop or a $ρ$-loop, since $(ω(f_{i-1}), α(f_i)) ∈ σ_0 = π \lor ρ$. Thus $ν(q_{i-1} f_i q_i^{-1}) ∈ Φ_G(π) \lor Φ_G(ρ)$.

Similarly, we prove that also $ν(f_1) q_1^{-1} ∈ Φ_G(π) \lor Φ_G(ρ)$ and $ν(q_{l-1} f_l) ∈ Φ_G(π) \lor Φ_G(ρ)$. Thus $ν(p'') ∈ Φ_G(π) \lor Φ_G(ρ)$.

In the rest of the paper we restrict ourselves to the case that $G$ is the free group $F$ freely generated by $T$. Let $L$ be any non-empty collection of partitions on the set $Ω$ closed under finite meets. We say that a twisting structure $T$ on $Ω$ is balanced with respect to $L$ if for every twist $t ∈ T$, an element $a ∈ Ω$ and any two $x, y ∈ Dom(t)$, whenever $(a, x) ∈ π ∈ L$ and $(a, y) ∈ ρ ∈ L$, then there exists some $z ∈ Dom(t)$ such that $(a, z) ∈ π \land ρ ∈ L$. The following lemma from [Tūl] gives us a way to construct balanced sets.

**Lemma 4.** Let $X$ be a set and $Ω = S_X$, the group of all permutations of $X$ of a finite type (i.e. generated by transpositions). For a set $Y ⊆ X$ let $S_Y$ be the subgroup of $S_X$ consisting of all permutations $p$ such that $p(x) = x$ for every $x ∈ X \setminus Y$. Let $L$ be the set of partitions of $Ω$ into left cosets of subgroups $S_Y$, $Y ⊆ X$. Then every left coset of every $S_Y$, $Y ⊆ X$, is balanced with respect to $L$.

By modifying Example 2.5. and Proposition 2.8. of [Tūl] we get the following lemma.

**Lemma 5.** For every lattice $L$ there exist a set $Ω$, a twisting structure $T$ on $Ω$ with finite domains and a lattice embedding $ϕ : L → C(T)$ such that the twisting structure $T$ is balanced with respect to the lattice $L = Im(ϕ)$.

No we are ready to prove the following counterpart to Theorem 3.

**Theorem 6.** Let $T$ be a twisting structure on a set $Ω$ and $L ⊆ C(T)$ a sublattice of $C(T)$. Suppose moreover that the domains of the elements of $T$ are finite and that $T$ is balanced with respect to $L$. Then the restriction of the canonical mapping $ϕ_F$ to the lattice $L$ is meet-preserving.

**Proof:** Let $π ∈ C(T)$. First of all we prove that for any $π$-loop $p = e_1 e_2 \cdots e_k$ in $G(T)$ at $v$ such that $ν(e_1) ν(e_2) \cdots ν(e_k)$ is not a reduced word in $F$ there
exists a subpath \( p' = e_{i_1} \cdots e_{i_l} \) of \( p \) that is also a \( \pi \)-loop at \( v \) and the word \( \nu(e_{i_1}) \cdots \nu(e_{i_l}) \) is reduced. Indeed, if \( \nu(e_1)\nu(e_2) \cdots \nu(e_k) \) is not reduced, then there is some \( i = 1, 2, \ldots, k-1 \) such that \( \nu(e_i) = t = \nu(e_{i+1})^{-1} \). Thus there are some \( a, b \in \text{Dom}(t) \) such that \( e_i = (a, t) \) and \( e_{i+1} = (bt, t^{-1}) \). Moreover, since \( p \) is a \( \pi \)-path, we have \( (\omega(e_i), \alpha(e_{i+1})) = (at, bt) \in \pi \). Since \( \pi \in C(T) \), we have also \( (att^{-1}, btt^{-1}) = (a, b) \in \pi \). But we have also \( (\omega(e_{i-1}), \alpha(e_i)) = (\omega(e_{i-1}), a) \in \pi \) and \( (\omega(e_{i+1}), \alpha(e_{i+2})) = (b, \alpha(e_{i+2})) \in \pi \), we get \( (\omega(e_{i-1}), \alpha(e_{i+2})) \in \pi \). Thus we can delete from \( p \) the edges \( e_i, e_{i+1} \) and the remaining path \( p' \) is still a \( \pi \)-loop at \( v \). In this way we can subsequently delete from \( p \) pairs of subsequent edges with mutually inverse values to get a \( \pi \)-loop \( p' \) with required properties. Let us call such a \( \pi \)-path a reduced \( \pi \)-path.

Take now arbitrary congruences \( \pi, \rho \in L \). Since \( \Phi_F \) is order-preserving, we have

\[
\Phi_F(\pi) \cap \Phi_F(\rho) \supseteq \Phi_F(\pi \cap \rho).
\]

To prove the opposite inclusion take any reduced word \( w = t_1 \cdots t_k \in \Phi_F(\pi) \cap \Phi_F(\rho) \). Then by the previous paragraph there is a reduced \( \pi \)-loop \( p = e_1 e_2 \cdots e_k \) at \( v \) such that \( \nu(p) = w \). Hence \( \nu(e_i) = t_i \) for every \( i = 1, 2, \ldots, k \). Similarly, there is a reduced \( \rho \)-loop \( q = f_1 f_2 \cdots f_l \) at \( v \) such that \( \nu(f_i) = t_i \) for every \( i = 1, 2, \ldots, l \). Thus in particular, \((v, \alpha(e_1)) \in \pi\) and \((v, \alpha(f_1)) \in \rho\). Let us denote \( \alpha(e_1) = a \) and \( \alpha(f_1) = b \). Thus \( a, b \in \text{Dom}(t_1) \). Since \( \text{Dom}(t_1) \) is balanced with respect to \( \pi \), there exists some \( c \in \text{Dom}(t_1) \) such that \((v, c) \in \pi \cap \rho\). Hence also \((a, c) \in \pi\) and \((b, c) \in \rho\). Since both \( \pi \) and \( \rho \) are congruences of \( T \), we get also \((at_1, ct_1) \in \pi\) and \((bt_1, ct_1) \in \rho\). Moreover, \((at_1, \alpha(e_2)) = (\omega(e_1), \alpha(e_2)) \in \pi\) and \((bt_1, \alpha(f_2)) = (\omega(f_1), \alpha(f_2)) \in \rho\). We get \((ct_1, \alpha(e_2)) \in \pi\) and \((ct_1, \alpha(f_2)) \in \rho\). Denote by \( g_1 \) the edge \((c, ct_1)\). Thus \( g_1 e_2 \cdots e_k \) is another \( \pi \)-loop at \( v \) and \( g_1 f_2 \cdots f_k \) is another \( \rho \)-loop at \( v \). Moreover, \((v, \alpha(g_1)) = (v, c) \in \pi \cap \rho\), and \( \nu(g_1) = t_1 \).

By repeating the same procedure with \( ct_1, e_2 \) and \( f_2 \) in place of \( v, e_1 \) and \( f_1 \), we get another edge \( g_2 \) that can replace \( e_2 \) in \( p \) and \( f_2 \) in \( q \) and satisfies \( \nu(g_2) = \nu(e_2) = \nu(f_2) \) and \( (\omega(g_1), \alpha(g_2)) \in \pi \cap \rho\). After \( k \) steps we construct a \((\pi \cap \rho)\)-loop \( r = g_1 g_2 \cdots g_k \) at \( v \) with \( \nu(r) = w \). Hence \( w \in \Phi_F(\pi \cap \rho) \). \( \square \)

Finally, connectedness of \( T \) also implies injectivity of \( \Phi_F \). Since the method of the proof will be also used in the proof of Theorem 8, we introduce some definitions here. If \( \pi \) is a congruence of \( T \), we define the quotient \( T/\pi \) of \( T \) as follows. The twisting structure \( T/\pi \) will be defined on the set \( \Omega/\pi \) of blocks of \( \pi \). For any twist \( t \in T \) we define another twist \( t_\pi \) on \( \Omega/\pi \). The domain \( \text{Dom}(t_\pi) \) consists of all blocks of \( \pi \) intersecting the domain \( \text{Dom}(t) \). If \( a \in \text{Dom}(t) \), then we define \([a]t_\pi = [at]\), where \([x]\) denotes the block of \( \pi \) containing \( x \). The definition of \( t_\pi \) is correct since \( \pi \) is a congruence of \( T \). Hence \( t_\pi \) is also a bijection between two subsets of \( \Omega/\pi \) and \( \{t_\pi : t \in T\} \) is a twisting structure on \( \Omega/\pi \). Thus \( T/\pi \) is simply the quotient of the partial unary algebra \((\Omega, T)\) by the congruence \( \pi \).

It is also useful to mention that the graph of \( T/\pi \) is naturally isomorphic to a quotient of the graph of \( T \). The vertices of \( G(T_\pi) \) are blocks of the partition \( \pi \) on \( \Omega \). Whenever \( a, b \in \text{Dom}(t) \) are such that \( (a, b) \in \pi \), then the two edges \( (a, t) \)
and \((b, t)\) of \(G(T)\) are identified into a single edge \(([a], [at])\) of \(G(T/\pi)\). If we assign to each edge \(([a], [at])\) of \(G(T/\pi)\) the value \(t \in F\), then we see that the values of \(\pi\)-loops at \(v\) in the graph \(G(T)\) are in one-to-one correspondence with the values of ordinary loops at \([v]\) in the graph of \(G(T/\pi)\).

**Lemma 7.** If \(T\) is a connected twisting structure on \(\Omega\) and \(\pi < \rho\) two congruences of \(T\), then \(\Phi_F(\pi) \neq \Phi_F(\rho)\).

**Proof:** Let \((a, b) \in \rho \setminus \pi\). Since \(T\) is connected, there exist a reduced path \(p = e_1e_2\cdots e_k\) in \(G(T)\) from \(v\) to \(a\) and a reduced path \(q = f_1f_2\cdots f_l\) in \(G(T)\) from \(v\) to \(b\). Then \(pq^{-1}\) is a \(\rho\)-loop in \(G(T)\) at \(v\) but it is not a \(\pi\)-loop at \(v\). If \(pq^{-1}\) is not reduced, then there exists a twist \(t \in T\) such that \(e_k = (at^{-1}, t)\) and \(f_l = (bt^{-1}, t)\). Then also \((at^{-1}, bt^{-1}) \in \rho \setminus \pi\). So we can replace \(p\) by \(p' = e_1e_2\cdots e_{k-1}\) and \(q\) by \(q' = f_1f_2\cdots f_{l-1}\) to get a shorter \(\rho\)-loop \(p'q'^{-1}\) at \(v\) that is not a \(\pi\)-loop at \(v\). Hence we may assume that the \(\rho\)-loop \(r = pq^{-1}\) at \(v\) is already reduced and it is not a \(\pi\)-loop. But then \(\nu(r) \in \Phi_F(\rho) \setminus \Phi_F(\pi)\).

Putting together previous results we get the following theorem.

**Theorem 8.** Every lattice \(L\) can be embedded into the subgroup lattice of a free group \(F\). If the lattice \(L\) is finite, then the group \(F\) and all the subgroups of \(F\) representing elements of \(L\) can be taken finitely generated.

**Proof:** It remains to prove the second assertion. However, if the lattice \(L\) is finite, then the twisting structure \(T\) of Lemma 4 can be taken finite by [Tù1]. But then the group \(\Phi_F(\pi)\) is isomorphic to the homotopy group of the graph \(G(T/\pi)\), by the remarks preceding Lemma 7. Since the graph \(G(T/\pi)\) is finite, its homotopy group is a finitely free group.

**References**


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