Condensations of Cartesian products

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Abstract. We consider when one-to-one continuous mappings can improve normality-type and compactness-type properties of topological spaces. In particular, for any Tychonoff non-pseudocompact space $X$ there is a $\mu$ such that $X^\mu$ can be condensed onto a normal ($\sigma$-compact) space if and only if there is no measurable cardinal. For any Tychonoff space $X$ and any cardinal $\nu$ there is a Tychonoff space $M$ which preserves many properties of $X$ and such that any one-to-one continuous image of $M^\mu$, $\mu \leq \nu$, contains a closed copy of $X^\mu$. For any infinite compact space $K$ there is a normal space $X$ such that $X \times K$ cannot be mapped one-to-one onto a normal space.

Keywords: condensation, one-to-one, compact, measurable

Classification: 54C10, 54A10

0. Introduction

We consider only Tychonoff topological spaces and continuous mappings. A condensation is a one-to-one mapping onto. Throughout the paper $\kappa$ denotes the first Ulam-measurable cardinal, if such a cardinal exists.

It is well-known that many key topological properties are not multiplicative. However, for many examples of a given property $P$ and a space $(X, \tau)$ which has $P$, but $X^2$ does not, there is a weaker topology $\tau'$ on $X$ such that the square of $(X, \tau')$ does have $P$. In fact, many examples are produced starting with the space $(X, \tau')$. This observation motivated A.V. Arhangel’skii to raise the following questions. Is it true that for any Lindelöf space $X$ there is a condensation $f : X \to Z$ such that $Z^2$ is Lindelöf (see [1])? Is it true that the second power of any normal (hereditarily normal, paracompact, Lindelöf, pseudocompact, countably compact, etc.) space can be condensed onto a space with the same property? Can any power of a Lindelöf space be condensed onto a Lindelöf space ([1])? Is it true that $\mathbb{Q}^\mu$ can be condensed onto a Lindelöf (compact) space for any infinite $\mu$? These questions are in line with the most general problem concerning condensations: when can a space from class $A$ be condensed onto a space from $B$, for some $A$ and $B$? $B$ is “better” than $A$ in some sense.

R. Buzyakova answered several of these questions negatively. She constructed a normal countably compact space in [3] and a Lindelöf space in [4], whose squares cannot be condensed onto a normal space (A.N. Yakivchik constructed earlier in [10] a Hausdorff non-regular finally compact space whose square cannot be condensed onto a Hausdorff finally compact space). We generalize these results
in Corollary 1: for any space \( X \) and a cardinal \( \nu \) there is a larger space \( M \) which preserves many properties of \( X \) and contains many clopen copies of \( X \) in such a way, that for any \( \mu \leq \nu \) and for each condensation \( f : M^\mu \to Z \), \( Z \) contains a closed copy of \( X^\mu \). Thus, condensations cannot improve most non-multiplicative properties of arbitrary large (but a priori fixed) powers. If also all powers of \( X \) are \( \tau \)-compact for some \( \tau \), then there is an \( M \) such that for any \( \mu \), \( f(M^\mu) \) contains a closed copy of \( X^\mu \).

E.G. Pytkeev proved in [9] that any separable metrizable non \( \sigma \)-compact Borel space can be condensed onto \( I^\nu \). Since \( Q^\omega \) is Borel (as a one-to-one continuous image of \( N^\omega \), see [8]) and not \( \sigma \)-compact (\( N^\omega \) is closed in \( Q^\omega \)), \( Q^\omega \) can be condensed onto \( I^\nu \). Therefore \( Q^\mu \) can be condensed onto \( I^\mu \) for any infinite \( \mu \). This solves one of the mentioned questions. It turns out that a somewhat similar result holds for most Lindelöf spaces. We show in Theorem 1 that for any non-pseudocompact \( X \) with \( |X| < \kappa \), \( X^\mu \) can be condensed onto a \( \sigma \)-compact space for many \( \mu < \kappa \). On the contrary, if \( \kappa \) does exist, then no power of some non-pseudocompact spaces (of cardinality \( \geq \kappa \)) can be condensed onto a normal space (Corollary 3).

1. Condensation onto a \( \sigma \)-compact space

**Theorem 1.** Let \( X \) be a non-pseudocompact Tychonoff space and let \( |X| \) be non Ulam-measurable. Let \( |X| \leq \mu_0 < \kappa \) and for every \( k \in \omega \), \( \mu_{k+1} = \exp(\mu_k) \) and \( \mu = \sup\{\mu_k : k \in \omega\} \). Then \( X^\mu \) can be condensed onto a regular \( \sigma \)-compact space.

**Proof:** Let \( \alpha_0 = |\beta X| \) and for any \( k \in \omega \), \( \alpha_{k+1} = \exp(\alpha_k) \). Then for \( \alpha = \sup\{\alpha_n : n \in \omega\} \), \( \alpha = \mu \). Let \( f \in C(X, [0, \infty)) \) be such that for each \( i \in \omega \) there is \( b_i \in f^{-1}(i + 0.5) \). Let \( K = \beta X \), \( \tilde{K} = \{x \in K : f \text{ can be extended on } X \cup \{x\}\} \) and let \( \tilde{f} \) be an extension of \( f \) on \( \tilde{K} \). We denote \( K = \tilde{K} \times \prod\{K_\gamma : 1 \leq \gamma < \alpha\} \) and \( X = \prod\{X_\gamma : \gamma < \alpha\} \), where \( K_\gamma \) and \( X_\gamma \) are copies of \( K \) and \( X \) respectively. Then \( K \) is a \( T_1 \) regular \( \sigma \)-compact space.

For any \( i \in \omega \), let \( A_i = \{a_{ij} \in \omega : a_{i_0} = i\} \) be an increasing sequence such that for \( i \neq j \), \( A_i^+ \cap A_j^+ = \emptyset \) where \( A_i^+ = A_i \setminus \{a_{i_0}\} \). By induction, a mapping \( \phi : \omega \to \omega \) can be defined such that

1. if \( i \notin \bigcup\{A_i^+ : i \in \omega\} \), then \( \phi(i) = 0 \), and
2. if \( j \geq 1 \), then \( \phi(a_{ij}) = \phi(i) + j + 1 \).

Let \( C_0 = \tilde{f}^{-1}([0;1]) \) and for \( i \in \omega \), \( C_{i+1} = \tilde{f}^{-1}([i + \frac{1}{2}i + 2]) \setminus C_i \); \( C_i = C_i \times \prod\{K_\gamma : 1 \leq \gamma < \alpha\} \).
For \( i, j \in \omega, j \geq 1 \), let \( F_{ij,0} = b_{aij} \times \prod \{ K_\gamma : 1 \leq \gamma \leq \alpha_{\phi(aij)} \} \), and for \( 1 \leq \Delta < \alpha \), \( F_{ij,\Delta} = \prod \{ K_\gamma : \alpha_{\phi(aij)} \cdot \Delta < \gamma \leq \alpha_{\phi(aij)} \cdot (\Delta + 1) \} \) (here we use a product of ordinals, see [7]), then \( b_{aij} \times \prod \{ K_\gamma : 1 \leq \gamma < \alpha \} = \prod \{ F_{ij,\Delta} : \Delta < \alpha \} \).

For any \( i, j \in \omega \), \( j \geq 1 \) and \( \Delta \geq 1 \) we denote \( M_{ij,0} = b_{aij} \times \prod \{ X_\gamma : 1 \leq \gamma \leq \alpha_{\phi(aij)} \} \) and \( M_{ij,\Delta} = \prod \{ X_\gamma : \alpha_{\phi(aij)} \cdot \Delta < \gamma \leq \alpha_{\phi(aij)} \cdot (\Delta + 1) \} \). Then \( M_{ij,0} \subseteq F_{ij,0} \) and \( M_{ij,\Delta} \subseteq F_{ij,\Delta} \). Each \( M_{ij,\Delta} \), \( \Delta \geq 0 \), contains a closed discrete subset \( H_{ij,\Delta} \) of cardinality \( \alpha_{\phi(aij)} - 1 \) which is also \( C^* \)-embedded in \( F_{ij,\Delta} \). Indeed, \( M_{ij,0} \approx M_{ij,0} \times M_{ij,0} \).

The first factor contains a closed discrete subset of cardinality \( \alpha_{\phi(aij)} - 1 \) by a theorem from [6] (since \( M_{ij,0} \) is a \( \alpha_{\phi(aij)} \)-power of a non countably compact space \( X \)). The second factor contains a \( C^* \)-embedded subset of the same cardinality. The diagonal product of these subsets is a required set \( H_{ij,\Delta} \). Let us denote \( \tilde{H}_{ij,\Delta} = \overline{\bigcap_{\tau = 1}^{\Delta} F_{ij,\tau}} \). For each \( \tau, C_{i,|\tau|} \) denotes projection of \( C \) onto ordinals not greater than \( \tau \).

If \( i \in \omega, k \geq 1 \) and \( \phi(i) = 0 \), let

\[
C_{i0} = C_{i,|\leq \alpha_0} \setminus \prod \{ X_\gamma : \gamma \leq \alpha_0 \},
\]

and

\[
C_{ik} = \{ x \in (C_{i,|\leq \alpha_k} \setminus \prod \{ X_\gamma : \gamma \leq \alpha_k \}) : x_{|\leq \alpha_{k-1}} \in \prod \{ X_\gamma : \gamma \leq \alpha_{k-1} \} \}.
\]

If \( n, k \geq 1 \) and \( i = a_{jn} \), let

\[
C_{i0} = C_{i,|\leq \alpha_{\phi(i)}} \setminus (\prod \{ X_\gamma : \gamma \leq \alpha_{\phi(i)} \} \cup H_{jn,0}),
\]

and

\[
C_{ik} = \{ x \in (C_{i,|\leq \alpha_{\phi(i)+k}} \setminus \prod \{ \tilde{H}_{jn,\Delta} : \Delta < \alpha \})_{|\leq \alpha_{\phi(i)+k}} : x_{|\leq \alpha_{\phi(i)+k}} \}, \text{ and } x_{\phi(i)+k-1} \in \prod \{ X_\gamma : \gamma \leq \alpha_{\phi(i)+k-1} \}.
\]

Then for every \( i, j \in \omega, |C_{ij}| = \exp(\alpha_{\phi(i)+j}) = \alpha_{\phi(i)+j+1} \). Let also \( C_{ik} = C_{ik} \times \prod \{ K_\gamma : \alpha_{\phi(i)+k} < \gamma < \alpha \} \). Therefore, if \( \phi(i) = 0 \), then \( \{ C_{ik} : k \in \omega \} \) is a partition of \( C_{i} \setminus \mathcal{X} \). If \( \phi(i) \neq 0 \) and \( i = a_{jn} \), then \( \{ C_{ik} : k \in \omega \} \) is a partition of \( C_{i} \setminus (\mathcal{X} \cup \bigcup \{ \tilde{H}_{jn,\Delta} : \Delta < \alpha \}) \).

For \( i, j \in \omega, j \geq 1 \), let \( \psi_{ij,0} \) be a one-to-one mapping of \( H_{ij,0} \) onto \( C_{i(j-1)} \).

Such a mapping exists since \( |H_{ij,0}| = \alpha_{\phi(aij)-1} = \alpha_{(\phi(i)+j+1)-1} = \alpha_{\phi(i)+j} = |C_{i(j-1)}| \). This mapping can be extended to a continuous mapping \( \tilde{\psi}_{ij,0} : H_{ij,0} \rightarrow C_{i(j-1)} \). In the same way for \( i, j \in \omega, j \geq 1 \) and \( 1 \leq \Delta < \alpha \) there is a one-to-one continuous mapping \( \tilde{\psi}_{ij,\Delta} \) of \( H_{ij,\Delta} \) onto \( F_{ij,\Delta} \). This mapping can be extended to a continuous mapping \( \tilde{\psi}_{ij,\Delta} : H_{ij,\Delta} \rightarrow F_{ij,\Delta} \). For any \( i, j \in \omega, j \geq 1 \), let \( \tilde{\psi}_{ij} = \prod \{ \tilde{\psi}_{ij,\Delta} : \Delta < \alpha \} : \prod \{ H_{ij,\Delta} : \Delta < \alpha \} \rightarrow C_{i} \) and \( \psi_{ij} = \tilde{\psi}_{ij} |\mathcal{X} \). It then follows that \( \tilde{\psi}_{ij} \) is a mapping “onto” and that \( \psi_{ij} \) is a condensation of \( \prod \{ H_{ij,\Delta} : \Delta < \alpha \} \) onto \( C_{i(j-1)} \).
For $i, j \in \omega$, $j \geq 1$, let $D_{ij} = \text{Dom}(\tilde{\psi}_{ij})$, then $\tilde{\psi}_{ij}$ induces an upper semicontinuous decomposition $E_{ij}$ of $D_{ij}$ since $D_{ij}$ is compact. We define a decomposition $E$ of $\mathcal{K}$ as follows:

1. if $x \notin \bigcup\{D_{ij} : i, j \in \omega, j \geq 1\}$, then $xEy \leftrightarrow x = y$;
2. if $j_0 \geq 1$ and $x \in D_{i_0j_0}$, then $xEy$ if and only if $y \in D_{i_0j_0}$ and $xE_{i_0j_0}y$.

This decomposition is well defined and it is upper semicontinuous since $\{D_{ij} \subset \mathcal{K} : i, j \in \omega, j \geq 1\}$ is a locally finite family of disjoint closed subsets of $\mathcal{K}$. Then the quotient mapping $q : \mathcal{K} \rightarrow \mathcal{K}' = \mathcal{K}/E$ is closed, therefore $\mathcal{K}'$ is a $T_1$ regular $\sigma$-compact space. For $i \in \omega$, let $D_{i0} = \overline{C_i}$, $D_i = \bigcup\{D_{ij} : j \in \omega\}$, $\mathcal{K}_i = \bigcup\{D_j : j \leq i\}$ and $G_i = \bigcup\{\overline{C_j} : j \leq i\}$. By a theorem from [2] the space $\mathcal{K}$ is an inductive limit of its closed subsets $\mathcal{K}_i$ and also of the compacta $G_i$. The same is true for the space $\mathcal{K}'$ and sets $\mathcal{K}'_i = q(\mathcal{K}_i)$ and $G'_i = q(G_i)$ since $q$ is a quotient mapping. Let $D'_i = q(D_i)$, $D'_{ij} = q(D_{ij})$ and $\mathcal{X}' = q(\mathcal{X})$.

We claim that $q_{|\mathcal{X}'}$ is a condensation. To see this, note that from the definition of the decomposition $E$ it is sufficient to prove that $q_{|D_{ij} \cap \mathcal{X}'}$ is a condensation. But this is obvious since $E_{ij}$ is generated by a mapping $\tilde{\psi}_{ij}$ whose restriction $\psi_{ij}$ is a condensation. In general, $\mathcal{X}'$ is not a $\sigma$-compact space. The desired condensation of $\mathcal{X}'$ onto a $\sigma$-compact space will be a restriction $g_{|\mathcal{X}'}$ of a quotient map $g : \mathcal{K}' \rightarrow g(\mathcal{K}')$ which we define at the end of the proof. $g$ will be the limit of maps $g_i$, $i \in \omega$, which are defined below, in the sense of Lemma 1. It will be constructed in such a way that $g(\mathcal{X}') = g(\mathcal{K}')$ which ensures that $g(\mathcal{X}')$ is $\sigma$-compact. In the next paragraph we introduce an auxiliary notation which will be used in the definition of maps $g_i$.

Let $H$ be a closed subset of some topological space $M$, and let $h$ be a quotient mapping of $H$. Then $h$ induces a decomposition $E_H$ of $H$ and an associate decomposition $E_M$ of $M$ by the rules: if $x \notin H$, then $xE_Hy \leftrightarrow x = y$; if $x \in H$, then $xE_Hy \leftrightarrow y \in H$ and $xE_Hy$. The decomposition $E_M$ defines a quotient mapping of $M$, which we will denote by $h_{H,M}$. It is clear that if $h$ is closed then so is $h_{H,M}$, that $h_{H,M|M \setminus H}$ is a homeomorphism, and that $h_{H,M}(M \setminus H) \cap h_{H,M}(H) = \emptyset$.

Let us define quotient mappings $g_{-1}$, $g_{-1,0}$ and $g_i$, $g_{i,i+1}$ as follows:

1. $g_{-1} \equiv id_{\mathcal{K}'}$;
2. if $g_{i-1}$ is already defined, then $g_{i-1,i} = g_{i-1,i}g_{i-1}(D'_{i})g_{i-1}(\mathcal{K}')$ and $g_i = g_{i-1,i} \circ g_{i-1}$;
3. let $g_{i-1,i}g_{i-1}(D')$ be a quotient mapping corresponding to decomposition $E'_{i}$ of the space $g_{i-1}(D'_{i})$, where for $y \in \overline{C_i}$, $E'_{i}(g_{i-1}(q(y))) = \{g_{i-1}(q(y))\} \cup \{g_{i-1}(q(X)) : \text{there is } j \geq 1, x \in D_{i,j} \text{ and } \tilde{\psi}_{i,j}(x) = y\}$.

The following are the properties of the mappings $g_{i-1}$, $g_{i-1,i}$ for $i \in \omega$:

a. $g_{i-1}(\mathcal{K})$ is a $T_1$ normal space;
b. every compact $g_{i-1}(D'_{in})$ ($n \in \omega$) has a neighborhood $U_{i,n}$ in $g_{i-1}(\mathcal{K}')$ such that $\{U_{i,n} : n \in \omega\}$ is a discrete family in $g_{i-1}(\mathcal{K})$;
Now let mappings each $i, j \in \omega$, $g_{i-1}|D'_i$ is a homeomorphism;
(e) $g_{i-1}|D'_i$ is a homeomorphism in a closed subset of $g_{i-1}(K')$;
(f) $B_{i-1} = g_{i-1}(K')$ is compact for $i > 0$;
(g) $g_{i-1|i}B_{i-1}$ is a homeomorphism for $i > 0$.

First, let us check properties (a)–(g) for $i = 0$. (a) holds trivially. The family
$\{U_n \subset K' : n \in \omega\}$, where $U_{00} = q(f^{-1}(0; 1/3))$ and $U_{0i} = q(f^{-1}(boaj -1/3; boaj +1/3))$
for $i \geq 1$ satisfies (b). (c) follows from (b) and the fact that $D_0' = \bigoplus\{D'_0, n : n \in \omega\}$ and each $D'_0, n$ is compact. (d) holds trivially, (e) follows directly from (b)–(d).

Now let mappings $g_k, g_{k-1,k}$ be constructed for all $k \leq i - 1$ and satisfy properties
(a)–(e).

**Lemma 1.** Let a $T_1$ normal space $M$ be an inductive limit of an increasing
sequence of its closed subsets $M_n$, where $n \in \omega$. Let $\{h_{n,n+1} : n \in \omega\}$ be a family of
quotient mappings such that $\text{Dom}(h_{0,1}) = M$, $\text{Dom}(h_{n,n+1,2}) = \text{Ran}(h_{n,n+1})$
and $h_{n+1} = h_{n,n+1} \circ \ldots \circ h_{0,1}$. Let $\mathcal{M}$ be an equivalence relation on $M$ such that
$xM y \Leftrightarrow h_k(x) = h_k(y)$ for some $n \in \omega$. Let also for $n \in \omega$ sets $B_n = h_n(M_n)$ be
normal and closed subsets of $h_n(M)$ and $h_{n,n+1}|B_n$ be a homeomorphism onto a
closed subset of $B_{n+1}$. Then the image $H/\mathcal{M}$ of a natural quotient mapping $h$ of $M$ is a $T_1$
normal space.

**Proof of Lemma 1:** For any $x \in M$, $h^{-1}(h(x)) = \bigcup\{h_{n-1}(h_n(x)) : n \in \omega\}$. For
each $i \in \omega$, $h_{n+i}(h_{n+i}(x)) \cap M_n = h_{n-1}(h_n(x)) \cap M_n$, therefore $h^{-1}(h(x)) \cap M_n$
$h_{n-1}(h_n(x)) \cap M_n$. The latter set is closed in $M_n$, hence $h^{-1}(h(x))$ is closed in $M$
and $M/\mathcal{M}$ is a $T_1$ space.

Let $F, G$ be disjoint closed subsets of $M$ such that $h^{-1}(h(F)) = F$, $h^{-1}(h(G)) =
G$. Let $O_0$ and $U_0$ be functionally disjoint in $B_0$ neighborhoods of $h_0(F_0)$ and
$h_0(G_0)$ respectively. The sets $V_0 = h_0^{-1}(O_0) \cap M_0$ and $W_0 = h^{-1}(U_0) \cap M_0$
satisfy the following conditions for $n = 0$:
(1) $h^{-1}(h_n(V_n)) \cap M_n = V_n$, $h^{-1}(h_n(W_n)) \cap M_n = W_n$;
(2) $F_n \subset V_n$ and $G_n \subset W_n$ where $F_n = F \cap M_n$ and $G_n = G \cap M_n$;
(3) $\overline{h_n(V_n)}^{B_n} \cap \overline{h_n(W_n)}^{B_n} = \emptyset$;
(4) $V_n \supset V_{n-1}$ and $W_n \supset W_{n-1}$ for all $n \geq 1$.

Let $V_n$, $W_n$ be constructed for all $n < k$, $k \geq 1$, and satisfy (1)–(4). By (3)
$h_{k-1,k}(\overline{h_{k-1}(V_{k-1})}^{B_{k-1}}) \cap h_{k-1,k}(\overline{h_{k-1}(W_{k-1})}^{B_{k-1}}) = \emptyset$. From the definition
of $F$ and $G$ and by (1), (2) $h_{k-1,k}(\overline{h_{k-1}(V_{k-1})}^{B_{k-1}}) \cap h_k(G) = \emptyset$ and $h_k(F) \cap
h_{k-1,k}(\overline{h_{k-1}(W_{k-1})}^{B_{k-1}}) = \emptyset$, then $h_k(V_{k-1} \cup F_k) \cup \overline{h_k(G_k)B_k} = \emptyset$,
and these sets have functionally disjoint in $B_k$ neighborhoods $O_k$ and $U_k$ respectivley. Let $V_k = h_k^{-1}(O_k) \cap M_k$, $W_k = h_k^{-1}(U_k) \cap M_k$. $V_k$ and $W_k$ satisfy (1)–(4)
for $n = k$, therefore the construction of $V_n$, $W_n$ can be carried out for all $n \in \omega$. 
Now let $V = \cup\{V_k : k \in \omega\}$ and $W = \cup\{W_k : k \in \omega\}$. $V$ and $W$ are open in $M$ since $M$ is an inductive limit of $M_n$. By (1) $h^{-1}(h(V)) = V$ and $h^{-1}(h(W)) = W$; by (2) $F \subset V$ and $G \subset W$. Lemma 1 is proved.

Let $M = g_{i-1}(\mathcal{K}')$ and $M_n = g_{i-1}(G_n)$. Let $h_n$ be a natural quotient mapping for the decomposition $\mathcal{M}_n$ of the space $g_{i-1}(\mathcal{K}')$, where for $x \in M_n$, $x\mathcal{M}_n y \Leftrightarrow x E'_i y$ and for $x \notin M_n$, $x\mathcal{M}_n y \Rightarrow x = y$. Since any element of $\mathcal{M}_n$ is a subset of some element of $\mathcal{M}_{n+1}$, the composition mapping $h_{n-1,n} = h_n \circ h_{n-1}^{-1}$ also is a quotient mapping. $M = g_{i-1}(\mathcal{K}')$ is an inductive limit of compacta $M_n$ since $\mathcal{K}'$ is an inductive limit of compacta $G'_n$ and $g_{i-1}$ is a quotient mapping.

Since $\mathcal{M}_n|_M \equiv \mathcal{M}_{n+1}|_M$, $h_{n,n+1}|_{h_n(M_n)}$ is a homeomorphism for any $n \in \omega$. All conditions of the lemma are satisfied, therefore $h$ maps $M$ onto a normal space $\mathcal{M}/M_n \equiv E'_i/M_n$, $n \in \omega$. $\cup\{M_n : n \in \omega\} = M = g_{i-1}(\mathcal{K}')$ and $M = \text{Dom}(\mathcal{M})$, $g_{i-1}(\mathcal{K}') = \text{Dom}(E'_i)$, thus $\mathcal{M} \equiv E'_i$ and the quotient mappings $H$ and $g_i$ (which are generated by $\mathcal{M}$ and $E'_i$) coincide. Therefore $g_i(\mathcal{K}')$ is a $T_1$ normal space. Let us prove properties (b)–(e). For $U_{i0} = g_i(q(\tilde{f}^{-1}[0; i + \frac{1}{2}]))$ and $U_{ij} = g_i(q(\tilde{f}^{-1}(b_{aij} - \frac{1}{3}; b_{aij} + \frac{1}{3})))$ for $j \geq 1$, the family $\{U_{in} : n \in \omega\}$ satisfies (b). Equality $D_{i+1} = \cup\{D_{i+1,n} : n \in \omega\}$ and (c) follow from (b) and the fact that each subset $D_{i+1,n}$ is compact, and therefore $g_i(D_{i+1,n})$ is closed in $g_i(\mathcal{K}')$. Each $D_{j,n}$ is compact and $E'_i|D_{j,n}$ is a trivial decomposition into singletons, therefore (d) is true. (e) follows from (b)–(d).

Therefore, $g_{i-1,i}$ and $g_i$ can be constructed for all $i \in \omega$ and satisfy (a)–(e). Let us prove (f) and (g) for $i \geq 1$. $B_i = g_i(\mathcal{K}) = g_i(G'_i)$, hence $B_i$ is compact. Map $g_{i,i+1}$ is defined by the decomposition $E'_{i+1}$, $E'_{i+1}|B_i$, which is a decomposition into singletons, therefore $g_{i,i+1}|B_i$ is a homeomorphism.

Now let $M = \mathcal{K}'$, $h_n = g_n$, $h_{n,n+1} = g_{n,n+1}$ and $M_n = D_n$ for $n \in \omega$. Conditions of the lemma follows from (f), (g). The resulting mapping $g$ is defined by the decomposition $E'$ of $\mathcal{K}$: $x E' y \Leftrightarrow g_i(x) = g_i(y)$ for some $i \in \omega$, and $g$ maps $\mathcal{K}'$ onto a $T_1$ regular $\sigma$-compact space.

The conclusion of Theorem 1 follows from the following properties:

(h) $B_i \subset g_i(\mathcal{X}')$;

(k) $g_i|_{\mathcal{X}'}$ is a condensation.

Assume the contrary to (h). Then there is the minimal $i_0 \in \omega$ such that for some $x \in C_{i_0} \setminus \mathcal{X}'$, $g_{i_0}(q(x)) \neq g_i(x')$. If $i_0 = a_{i_0,k_0}$ and $x \in \tilde{H}_{j_0,k_0}$, then $\tilde{\psi}_{i_0,k_0}(x) \in C_{i_0}$, $j_0 < i_0$ and by the assumption $g_{i_0}(q(x)) \in g_{i_0}(q(C_{j_0} \setminus \mathcal{X}')) \subset g_{i_0}(x')$. That contradicts the minimality of $i_0$. If $x \notin \cup\{\tilde{H}_{j,k} : j < j_0, k \in \omega\}$, then $x \in C_{i_0,j_0}$ for some $j_0 \in \omega$. Since $\psi_{i_0,j_0+1}$ maps $\tilde{H}_{i_0,j_0}$ onto $C_{i_0,j_0}$ and from the definition of $E'_{i_0}$, $g_{i_0}(q(x)) \subset g_{i_0}(q(H_{i_0,j_0+1})) \subset g_{i_0}(x')$ and (h) is proved.

Suppose it is proved that $g_i|_{\mathcal{X}'}$ is a condensation for all $i < k$, $k \in \omega$. Since $g_k = g_{k-1,k} \circ g_{k-1}$, it is sufficient to prove that $g_{k-1,k}|g_{k-1}(\mathcal{X}')$ is a condensation.
By (d) $g_k|D'_{kj}$ is a homeomorphism for any $i \in \omega$. It is sufficient to prove that for any $j_0, j_1 \in \omega$, $0 < j_0 < j_1$, and $x_0 \in D_{k,j_0} \cap \mathcal{X}$, $x_1 \in D_{k,j_1} \cap \mathcal{X}$ and $y \in D_{k_0} \cap \mathcal{X}$ the following inequalities hold: $g_k(q(x_0)) \neq g_k(q(x_1)) \neq g_k(q(y)) \neq g_k(q(x_0))$. 

Let $\psi_{k,j_0}(x_0) \in C_{k,j_0-1}$, $\psi_{k,j_1}(x_1) \in C_{k,j_1-1}$, therefore $g_k(q(x_0)) \neq g_k(q(x_1))$ since $C_{k,j_0-1} \cap C_{k,j_1-1} = \emptyset$. From the definition of $\psi_{ij}$, $\tilde{\psi}_{k,j_0}$ maps $D'_{k,j_0} \cap \mathcal{X}$ in $C_{k,j_0-1} \in D'_{k_0} \setminus \mathcal{X}$ and $\tilde{\psi}_{k,j_1}$ maps $D'_{k,j_1} \cap \mathcal{X}$ in $C_{k,j_1-1} \in D'_{k_1} \setminus \mathcal{X}$. Hence other inequalities also hold. 

A cardinal $\mu$ is called $\tau$-measurable, if there is a $\tau$-centered ultrafilter on $\mu$, so the Ulam-measurable cardinals are exactly those which are $\omega$-centered. The same method allows us to prove the following

**Theorem 2.** Let $\mu_0$ be a non $\tau$-measurable cardinal and for every $k \in \omega$, $\mu_{k+1} = \exp(\mu_k)$ and $\mu = \sup\{\mu_k : k \in \omega\}$. Let $X_0$ be a Tychonoff non-pseudocompact space and $\{X_\alpha : 1 \leq \alpha \leq \mu\}$ be a family of spaces such that $\text{ext}(X_\alpha) \geq \tau$ for $1 \leq \alpha < \tau$ and $|X_\alpha| < \mu$ for $0 \leq \alpha < \mu$. Then $\prod\{X_\alpha : \alpha < \mu\}$ can be condensed onto a regular $\sigma$-compact space.

2. A case of $\tau$-compact spaces

For any cardinal $\tau$, let $\tilde{\tau}$ be the set of all isolated ordinals less then $\tau$. A space $X$ is called $\tau$-compact if each of its subsets of cardinality $\tau$ has a complete accumulation point in $X$. For any space $X$, a compactification $cX$, and cardinals $\tau_1$, $\tau_2$ let $M(X, cX, \tau_1, \tau_2) = ((\tau_1 + 1) \times (\tau_2 + 1) \times cX) \setminus (\tilde{\tau}_1 \times \tilde{\tau}_2 \times (cX \setminus X))$. This construction is related to the space $((\tau + 1) \times \beta X) \setminus (\tau \times (\beta X \setminus X))$ for certain $X$ and $\tau$ which was described by R. Buzyakova in [4].

We have shown in Section 1 that for many spaces $X$ there are certain powers $\mu$, which depend on $X$, such that $X^\mu$ can be condensed onto a $\sigma$-compact space. The original space can be as bad as we wish and fail all the properties of $\sigma$-compact spaces. Thus, in that situation condensations can improve topological properties of powers. In this section we prove somewhat reverse result by producing examples of good spaces $M$ whose (small) powers are so bad that they cannot even be improved by condensations. Let $\mu$ be an ordinal, and let $\tau_i$, $i = 1, 2, 3, 4$, be cardinals which depend on $\tau$ and on the size of $X$ as it is stated in Theorem 3. We denote $M = M(X, cX, \tau_1, \tau_2) \bigoplus M(X, cX, \tau_3, \tau_4)$ and $M_\nu \approx M$ for $\nu < \mu$. $M$ consists of a compact “skeleton” $K = \left\{\left((\tau_1 + 1) \times (\tau_2 + 1)\right) \setminus (\tilde{\tau}_1 \times \tilde{\tau}_2)\right\} \bigoplus \left\{((\tau_3 + 1) \times (\tau_4 + 1)) \setminus (\tilde{\tau}_3 \times \tilde{\tau}_4)\right\} \times cX$ and of many clopen copies of $X$. If $f : M^\mu \to Z$ is a condensation, then $f|_{K^\mu}$ is a homeomorphism since $K^\mu$ is compact. $K^\mu$ is only a part of $M^\mu$, but the copies of $X$ are inserted in $M$ in such a way that this restriction influences the whole map $F$ and we can ultimately find clopen copies $X_\nu$ of $X$ in $M_\nu$ for all $\nu < \mu$ such that $f$ restricted to $\prod\{X_\nu : \nu < \mu\}$ is a homeomorphism onto a closed subset of $Z$. Now suppose that $X^\mu$ is not normal (paracompact, etc.). Then $Z$ is not normal (paracompact, etc.) either. This means that $M^\mu$ cannot be condensed onto a normal (paracompact, etc.) space.
The fact that $M$ is good itself when $X$ is so follows from Lemma 2. Hence $M$ is the desired example.

**Lemma 2.** Let $X$ be a Tychonoff space and let $cX$ be a compactification of $X$. Let $M = M(X, cX, \tau_1, \tau_2) \bigoplus M(X, cX, \tau_3, \tau_4)$ for some cardinals $\tau_i$, $i = 1, 2, 3, 4$. Then $M$ is normal ($\tau$-paracompact, realcompact) iff $X$ is so and $M^{\mu}$ is pseudocompact iff $X^{\mu}$ is so.

Let a property $P$ be invariant of continuous mappings, of inverse perfect mappings and suppose $P$ is inherited by clopen subsets. Then $M^{\mu}$ satisfies $P$ iff so does $X^{\mu}$. In particular, $l(M^{\mu}) = \tau$ ($M^{\mu}$ is $\tau$-initially compact, $\sigma$-compact, $\tau$ is regular and $M^{\mu}$ is $\tau$-compact, respectively) iff the same is true for $X^{\mu}$.

**Proof:** $K = \{[((\tau_1+1) \times (\tau_2+1)) \setminus (\tilde{\tau}_1 \times \tilde{\tau}_2)] \bigoplus\{((\tau_3+1) \times (\tau_4+1)) \setminus (\tilde{\tau}_3 \times \tilde{\tau}_4)\} \times cX$ is compact and any neighborhood of $K$ in $M$ contains a neighborhood $U$ such that $M \setminus U$ is a union of finitely many clopen copies of $X$. This proves the first part of the lemma.

$K_1 = ((\tau_1+1) \times (\tau_2+1)) \bigoplus((\tau_3+1) \times (\tau_4+1))$ is compact and $K_1 \times X$ is dense in $M$. Therefore $(K_1)^{\mu} \times X^{\mu}$ is dense in $M^{\mu}$. Some clopen subset of $M^{\mu}$ can be projected onto $X$. By these reasons $M^{\mu}$ is pseudocompact iff so is $X^{\mu}$.

The space $M/(K \times cX)$ is obtained from $M$ by identifying a closed subset $K \times cX$ to a single point (see [5]). $K \times cX$ is compact, so the corresponding quotient map $q : M \to M/(K \times cX)$ is perfect. Let $p$ be a restriction of $q$ to $K_1 \times X$, then $p(K_1 \times X) = q(M)$. Let $p_\alpha$, $q_\alpha$ be the $\alpha$-th “copies” of $p$, $q$, $\alpha < \mu$ and $p = \Delta(p_\alpha : \alpha < \mu)$, $q = \Delta(q_\alpha : \alpha < \mu)$, then $M^{\mu} = q^{-1}(p((K_1 \times X)^{\mu}))$. $\square$

**Theorem 3.** Let $X^{\mu}$ be $\tau$-compact and let $\tau, \tau_i$ be regular cardinals, $i = 1, 2, 3, 4$, such that $\tau_1 > \tau_2 > \tau_3 > \tau_4 > \max\{|cX|, \tau\}$. Then for $M = M(X, cX, \tau_1, \tau_2) \bigoplus M(X, cX, \tau_3, \tau_4)$, $Y = M^{\mu}$ and any condensation $f : Y \to Z$ there is a closed subset $F$ of $Y$ homeomorphic to $X^{\mu}$ such that $f|_F$ is a homeomorphism onto a closed subset of $Z$. Also, any continuous function on $f(F)$ that can be extended to a function on $(cX)^{\mu}$ (when $f(F)$ is naturallyembedded in $(cX)^{\mu}$) can be extended on $Z$. In particular, if $X^{\mu}$ is pseudocompact and $cX = \beta X$, then $f(F)$ is $C$-embedded in $Z$.

**Proof:** Assume that $cf(\mu) \neq \tau_1, \tau_2$. Let $Y = \prod\{Y_\alpha : \alpha < \mu\}$, where each $Y_\alpha$ is homeomorphic to $M$. We denote $\tilde{Y} = \beta Y$, $\tilde{Z} = \beta Z$; $\tilde{f}$ is a continuous extension of $f$ from $\tilde{Y}$ to $\tilde{Z}$. For any $\alpha < \mu$, let $\pi_\alpha : Y \to Y_\alpha$ be a projection and let $\tilde{\pi}_\alpha$ be its extension from $\tilde{Y}$ onto $\tilde{Y}_\alpha = \beta Y_\alpha$. For $y \in \tilde{Y}_\alpha$ and $i = 1, 2, 3$, $\phi_i(y)$ is a projection onto $(\tau_1+1)$, $(\tau_2+1)$ or $cX$ respectively if $y \in M(X, cX, \tau_1, \tau_2)\tilde{Y}_\alpha$ or onto $(\tau_3+1)$, $(\tau_4+1)$ or $cX$ respectively if $y \in M(X, cX, \tau_3, \tau_4)\tilde{Y}_\alpha$. For $\alpha < \mu$ and $i = 1, 2, 3$, we denote $\psi_{\alpha,i} = \phi_i \circ \tilde{\pi}_\alpha$ and $\psi_3 = \Delta\{\psi_{\alpha,3} : \alpha < \mu\}$. For any combination $i, j$ of indexes 1, 2, 3, let $\phi_{ij} = \phi_i \phi_j$ and $\psi_{\alpha,ij} = \phi_{ij} \circ \tilde{\pi}_\alpha$. For $(\alpha, \beta) \in \tau_1 \times \tau_2$, let $Y_{\alpha\beta} = \{y \in \tilde{Y} : \text{if } \psi_{\gamma,3}(y) \in cX \setminus X \text{ for some } \gamma < \mu, \text{ then } \psi_{\gamma,12}(y) = (\alpha, \beta)\}$. If $\gamma < \mu$ then let $Y_{\alpha\beta}^\gamma = \{y \in Y_{\alpha\beta} : \psi_{\gamma,3}(y) \in cX \setminus X\}$.
Now let $\gamma < \mu$ be fixed. For any $\beta' \in \tilde{\tau}_2$, let $A_{\beta'} = \{y \in Y_{\alpha\beta'}^\gamma : \alpha \in \tilde{\tau}_1$ and there is $y' \in Y_{\alpha\beta'}^\gamma \cup Y$ such that $\psi_{\gamma,3}(y) \neq \psi_{\gamma,3}(y')$ and $\bar{f}(y) = \bar{f}(y')\}$. Let $\tau' = \max\{\tau, |cX|\}^+$, we claim that $|\{\psi_{\gamma,1}(A_{\beta'})\}| < \tau'$. For, assume the contrary. Then there is a monotonically increasing mapping $\phi$ from $\tau'$ in $\tilde{\tau}_1$, a point $c \in cX \setminus X$, sets $A = \{y : \delta < \tau'\}$ and $A' = \{y' : \delta < \tau'\}$ and a neighborhood $U$ of $c$ in $\tau_2 \times cX$ such that for any $\delta < \tau'$, $y, y' \in Y_{\phi(\delta)\beta'}^\gamma \cup Y$, $\psi_{\gamma,23}(y) = c$, $\psi_{\gamma,23}(y') \notin U$, and $\bar{f}(y) = \bar{f}(y')$ (it’s all possible because $\psi_{\gamma,23}(A_{\beta'}) \subset \{\beta'\} \times cX$ and $\{\beta'\} \times cX$ is open in $\tau_2 \times cX$, so $\psi_{\gamma,23}(A_{\beta'})$ has a base of cardinality $\leq cX < \tau'$ in $\tau_2 \times cX$). For any $y \in A$, let $\bar{y}_\delta$ be such a point from $Y$ that for any $\nu < \mu$, $\pi_\nu(\bar{y}_\delta) = \bar{\pi}_\nu(y)$ if $\bar{\pi}_\nu(y) \in cX$, otherwise let $\psi_{\nu,23}(\bar{y}_\delta) = \psi_{\nu,23}(\bar{y}_\delta)$ and $\psi_{\nu,1}(\bar{y}_\delta) = \psi_{\nu,1}(y) + \omega$. Let $\tilde{A} = \{\bar{y}_\delta : \delta < \tau'\}$. In the same way the set $\tilde{A}' = \{\bar{y}_\delta' : \delta < \tau'\}$ is defined. The set $\{(\bar{y}_\delta, \bar{y}_\delta') \in Y \times Y : \delta < \tau'\}$ has a complete accumulation point $(a, a')$ in $Y \times Y$. Since 23 and $Y \times Y$ is a homeomorphism onto a point $\tilde{\tau}_2$.

In the same way, for any $\gamma < \mu$ and $\alpha' < \tau_1$ there is an ordinal $\beta_{\alpha'}^\gamma < \tau_2$ such that $\psi_{\gamma,2}(A_{\alpha'}) \subset \beta_{\alpha'}^\gamma$, where $A_{\alpha'} = \{y \in Y_{\alpha'\beta}^\gamma : \beta \in \tilde{\tau}_2$ and there is $y' \in Y_{\alpha'\beta}^\gamma \cup Y$ such that $\psi_{\gamma,3}(y) \neq \psi_{\gamma,3}(y')$ and $\bar{f}(y) = \bar{f}(y')\}$. Since $cf(\mu) \neq \tau_1$, there is $\tilde{\alpha} < \tau_1$ and $\Gamma_1 \subset \mu$ such that $|\Gamma_1| = \mu$ and for any $\gamma \in \Gamma_1$, $\nu_\gamma \leq \tilde{\alpha}$. Since also $cf(\mu) \neq \tau_2$, there is $\tilde{\beta} < \tau_2$ and $\Gamma_2 \subset \Gamma_1$ such that $|\Gamma_2| = \mu$ and for any $\gamma \in \Gamma_2$, $\beta_{\alpha+1}^\gamma \leq \tilde{\beta}$. Now let $y \in Y$; for any $\gamma \in \Gamma_2$ we define $F_{\gamma} = (\tilde{\alpha} + 1) \times (\tilde{\beta} + 1) \times X$ and for any $\gamma \in \mu \setminus \Gamma_2, F_{\gamma} = \pi_\gamma(y) \Gamma_$. The set $F = \prod\{F_{\gamma} : \gamma \in \mu\}$ is homeomorphic to $X^\mu$ and $f|F$ is a homeomorphism onto a closed subset $f(F)$ of $Z$. Let $g$ be a continuous function on $(cX)^\mu$ and let $h$ be a map from $\tilde{F}^Y$ onto $(cX)^\mu$ such that $h(y) = \{\psi_{\gamma,3}(y) : \gamma \in \Gamma_2\}, y \in \tilde{F}^Y$. Then $h \circ f^{-1}|f(F)$ is a natural embedding of $f(F)$ in $X^\mu \subset (cX)^\mu$ by the properties of $f|F$. Since $\bar{f}(h^{-1}(x_1)) \cap \bar{f}(h^{-1}(x_2)) = \emptyset$ for $x_1 \neq x_2, x_1, x_2 \in (cX)^\mu$ by the choice of $F$, $h \circ f^{-1}$ is a continuous function from $\bar{f}(F)^\tilde{Z}$ onto $(cX)^\mu$. Therefore $g$ can be lifted to a continuous function on $\bar{f}(F)^\tilde{Z}$ and extended to a function on $\tilde{Z}$.

If $cf(\mu) = \tau_1$ or $cf(\mu) = \tau_2$, all the preceding arguments remain valid if $\tau_1$ and $\tau_2$ are replaced everywhere with $\tau_3$ and $\tau_4$ respectively. \(\square\)

**Corollary 1. a.** For any Tychonoff space $X$ and any cardinal $\nu$ there is a larger space $M$ which preserves many properties of $X$ listed in Lemma 2 and
such that for any $\mu \leq \nu$ and a condensation $f : M^\mu \to Z$, $Z$ contains a closed subset homeomorphic to $X^\mu$; if $X^\mu$ is pseudocompact, then this subset is also $C$-embedded in $Z$. In particular, $M^\mu$ cannot be condensed onto a normal (Lindelöf, $\sigma$-compact, etc.) space if $X^\mu$ is not normal (Lindelöf, $\sigma$-compact, etc.).

**b. If $X$ is countably compact in all powers or if there is a $|X|$-measurable cardinal, then $M$ satisfies the above properties for all $\nu$.**

**Proof:**

**a.** Let $\tau = |\beta X^\nu|^+$ and $\tau_1 = \tau^+$, $\tau_{i+1} = \tau_i^+$, $i = 1, 2, 3$. Clearly, $X^\mu$ is $\tau$-compact for any $\mu \leq \nu$, so $M = M(X, \beta X, \tau_1, \tau_2) \oplus M(X, \beta X, \tau_3, \tau_4)$ is a required space.

**b.** If $X$ is countably compact in all powers, let $\tau = |\beta X|^+$, $\tau_1 = \tau^+$, and for $i = 1, 2, 3$, $\tau_{i+1} = \tau_i^+$. Then $M = M(X, \beta X, \tau_1, \tau_2) \oplus M(X, \beta X, \tau_3, \tau_4)$ is as desired. If $\tau$ is the first $|X|$-measurable cardinal, then all powers of $X$ are $\tau$-compact, hence for $\tau_1 = \tau^+$, $\tau_{i+1} = \tau_i^+$, $i = 1, 2, 3$, $M = M(X, \beta X, \tau_1, \tau_2) \oplus M(X, \beta X, \tau_3, \tau_4)$ is as required. □

**Corollary 2.** For any infinite compactum $K$ there is a normal space $X$ such that $X \times K$ cannot be condensed onto a normal space.

**Proof:** Let $Y$ be a Dowker space and $\tau = max\{|\beta Y|, |K|\}^+$, $\tau_1 = \tau^+$, $\tau_{i+1} = \tau_i^+$, $i = 1, 2, 3$. The space $X = M(Y, \beta Y, \tau_1, \tau_2) \oplus M(Y, \beta Y, \tau_3, \tau_4)$ is normal by Lemma 2. $X \times K$ cannot be condensed onto a normal space by Theorem 3 since $X \times K = M(Y \times K, \beta Y \times K, \tau_1, \tau_2) \oplus M(Y \times K, \beta Y \times K, \tau_3, \tau_4)$. □

From Theorem 1 and Corollary 1 we derive the following

**Corollary 3.** The following are equivalent:

1. for any Tychonoff non-pseudocompact space $X$ there is $\mu$ such that $X^\mu$ can be condensed onto a normal space;
2. for any Tychonoff non-pseudocompact space $X$ there is $\mu$ such that $X^\mu$ can be condensed onto a regular $\sigma$-compact space;
3. there is no measurable cardinal.

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**References**

Condensations of Cartesian products


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