A remark on localized weak precompactness in Banach spaces

MINORU MATSUDA

Abstract. We give a characterization of $K$-weakly precompact sets in terms of uniform Gateaux differentiability of certain continuous convex functions.

Keywords: $K$-weakly precompact set, uniform Gateaux differentiability

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We begin with the requisite definition. Throughout this paper $X$ denotes a real Banach space with topological dual $X^*$. If $g : X \to \mathbb{R}$ is a continuous convex function, for $x, y \in X$, we define $Dg(x, y)$ by

$$\lim_{t \to 0} \frac{g(x + ty) - g(x)}{t}$$

provided that this limit exists, and we also define the subdifferential of $g$ at $x \in X$ to be the set $\partial g(x)$ of all elements $x^*$ of $X^*$ satisfying that $(u, x^*) \leq g(x + u) - g(x)$ for any $u \in X$. Then $\partial g(x)$ is a non-empty weak*-compact convex subset of $X^*$ for every $x \in X$. The triple $(I, \Lambda, \lambda)$ refers to the Lebesgue measure space on $I (= [0, 1])$, $\Lambda^+$ to the sets in $\Lambda$ with positive $\lambda$-measure. We always understand that $I$ is endowed with $\Lambda$ and $\lambda$. We denote the set $\{\chi_E/\lambda(E) : E \in \Lambda^+\}$ by $\Delta(I)$. A function $f : I \to X^*$ is said to be weak*-measurable if $(x, f(t))$ is $\lambda$-measurable for each $x \in X$. If $f : I \to X^*$ is a bounded weak*-measurable function, we obtain a bounded linear operator $T_f : X \to L_1(I, \Lambda, \lambda)$ given by $T_f(x) = x \circ f$ for every $x \in X$, where $(x \circ f)(t) = (x, f(t))$ for every $t \in I$, and the dual operator of $T_f$ is denoted by $T_f^* : L_\infty(I, \Lambda, \lambda) \to X^*$.

According to Bator and Lewis [1], let us define the notion of localized weak precompactness in Banach spaces as follows.

Definition 1. Let $A$ be a bounded subset of $X$ and $K$ a weak*-compact subset of $X$. Then we say that $A$ is $K$-weakly precompact if every sequence $\{x_n\}_{n \geq 1}$ in $A$ has a pointwise convergent subsequence $\{x_{n(k)}\}_{k \geq 1}$ on $K$.

Then, in [1], they have made a systematic study of $K$-weakly precompact sets $A$ in Banach spaces and obtained various characterizations of such sets.

Succeedingly, in our paper [4], we also have obtained measure theoretic characterizations of $K$-weakly precompact sets $A$ by the effective use of a $K$-valued
weak* -measurable function constructed in the case where \( A \) is non-\( K \)-weakly precompact. In this paper we wish to add a characterization of \( K \)-weakly precompact sets in terms of uniform Gateaux differentiability of certain continuous convex functions, which is our aim. This can be regarded as a slight generalization and refinement of Corollary 10 in [1]. And it should be noted that even here this \( K \)-valued function also becomes an effective means to an end. Before giving our characterization theorem, let us define some special continuous convex functions on \( X \) as follows.

**Definition 2.** Let \( H \) be a non-empty bounded subset of \( X^* \). Then the continuous convex function associated with \( H \), which is denoted by \( g_H \), is defined by 
\[
g_H(x) = \sup \{(x, x^*) : x^* \in H\}
\]
for every \( x \in X \).

In what follows, all notations and terminology used and not defined are as in [1].

Let \( A \) be a bounded subset of \( X \), \( K \) a weak* -compact subset of \( X^* \), \( \{x_n\}_{n \geq 1} \) a sequence in \( A \) and \( Y \) the closed linear span of \( \{x_n : n \geq 1\} \) in \( X \). In the following, we always understand that \( Y \) is a such space. Let \( j : Y \to X \) be the inclusion mapping and \( j^* \) its dual mapping. For any non-empty subset \( H \) of \( K \), the continuous convex function \( g_H : Y \to \mathbb{R} \) satisfies \( \partial g_H(y) \subset \overline{\sigma}^{*} (j^*(K)) \) for each \( y \in Y \). Further let us note two preliminary facts for the proof of Theorem. One concerns separably related sets in the case where \( A \) is \( K \)-weakly precompact. Let \( \{x_n\}_{n \geq 1} \) be a sequence in \( A \) and suppose that there exists a subsequence \( \{x_{n(k)}\}_{k \geq 1} \) of \( \{x_n\}_{n \geq 1} \) such that \( \lim_{k \to \infty} (x_{n(k)}, x^*) \) exists for every \( x^* \in K \). Then this implies that \( \lim_{k \to \infty} (x_{n(k)}), y^* \) exists for every \( y^* \in \overline{\sigma}^{*} (j^*(K)) \). Hence, by considering the mapping \( L : \overline{\sigma}^{*} (j^*(K)) \to c \) (the Banach space of all convergent sequences of real numbers equipped with the supremum norm \( || \cdot ||_{\infty} \) defined by \( L(y^*) = \{(x_{n(k)}, y^*)\}_{k \geq 1} \)), we easily know that \( \overline{\sigma}^{*} (j^*(K)) \) is separably related to \( \{x_{n(k)} : k \geq 1\} \), since \( c \) is separable. The other concerns the construction of a \( K \)-valued weak* -measurable function \( h \) and a sequence \( \{x_n\}_{n \geq 1} \) in \( A \) in the case where \( A \) is non-\( K \)-weakly precompact. Then, although the construction of this function \( h \) and the sequence \( \{x_n\}_{n \geq 1} \) in \( A \) is exactly the same as in \( \S \) 3 of [4], for the sake of completeness, we state its outline briefly in the following. Since \( A \) is not \( K \)-weakly precompact, by the celebrated argument of Rosenthal [5], we have a sequence \( \{x_n\}_{n \geq 1} \) in \( A \) and real numbers \( r \) and \( \delta \) with \( \delta > 0 \) such that putting \( A_n = \{x^* \in K : (x_n, x^*) \leq r\} \) and \( B_n = \{x^* \in K : (x_n, x^*) \geq r + \delta\} \), \( (A_n, B_n)_{n \geq 1} \) is an independent sequence of pairs of weak* -closed subsets of \( K \) (that is, for every \( \{\varepsilon_j\}_{1 \leq j \leq k} \) with \( \varepsilon_j = 1 \) or \(-1\), \( \bigcap \{\varepsilon_j A_j : 1 \leq j \leq k\} \) is a non-empty set, where \( \varepsilon_j A_j = A_j \) if \( \varepsilon_j = 1 \) and \( \varepsilon_j A_j = B_j \) if \( \varepsilon_j = -1 \)). Putting \( \Gamma = \bigcap_{n \geq 1} (A_n \cup B_n) \), \( \Gamma \) is a non-empty weak* -compact subset of \( K \), since \( (A_n, B_n)_{n \geq 1} \) is independent. Define \( \varphi : \Gamma \to \mathcal{P}(N) \) (Cantor space, with its usual compact metric topology) by \( \varphi(x^*) = \{p : (x_p, x^*) \leq r\} (= \{p : A_p \ni x^*\}) \in \mathcal{P}(N) \). Then \( \varphi \) is a continuous surjection from \( \Gamma \) to \( \mathcal{P}(N) \) (here, \( \Gamma \) is endowed with the weak* -topology \( \sigma(X^*, X) \)) and so we have a Radon probability measure \( \gamma \) on \( \Gamma \) such that \( \varphi(\gamma) = \nu \) (the normalized Haar measure if we identify \( \mathcal{P}(N) \) with \( \{0, 1\}^N \)).
and \{f \circ \varphi : f \in L_1(\mathcal{P}(N), \Sigma_\nu, \nu)\} = L_1(\Gamma, \Sigma_\gamma, \gamma) where \Sigma_\nu (resp. \Sigma_\gamma) is the family of all \nu (resp. \gamma)-measurable subsets of \mathcal{P}(N) (resp. \Gamma). Further, consider a function \tau : \mathcal{P}(N) \rightarrow I defined by \tau(D) = \Sigma\{1/2^m : m \in D\} for every D \in \mathcal{P}(N). Then \tau is a continuous surjection such that \tau(\nu) = \lambda and \{u \circ \tau : u \in L_1(I, \Lambda, \lambda)\} = L_1(\mathcal{P}(N), \Sigma_\nu, \nu). Then, making use of the lifting theory, we have a weak*-measurable function \(h: I \rightarrow \Gamma (\subset K)\) such that

\[
\rho(x \circ h)(t) = (x, h(t)) \quad \text{for every } x \in X \text{ and every } t \in I,
\]

\[
\int_E (x, h(t)) \, d\lambda(t) = \int_{\varphi^{-1}(\tau^{-1}(E))} (x, x^*) \, d\gamma(x^*)
\]

for every \(E \in \Lambda\) and every \(x \in X\). Here \(\rho\) denotes a lifting on \(L_\infty(I, \Lambda, \lambda)\).

Now we are ready to state our characterization theorem (a localized version of Theorem 8 in [1]). Its main part is that (3) implies (1), whose proof is significant in the point that the characters of the \(K\)-valued function \(h\) and the sequence \(\{x_n\}_{n \geq 1}\) in \(A\) obtained above are used concretely and effectively. And there, we can get a result that for every \(y \in Y\) and every subsequence \(\{x_{n(k)}\}_{k \geq 1}\) of \(\{x_n\}_{n \geq 1}\), \(Dg_H(y, x_{n(k)})\) does not exist uniformly in \(k\), where \(H = h(I) (\subset K)\).

**Theorem.** Let \(A\) be a bounded subset of \(X\) and \(K\) a weak*-compact (not necessarily convex) subset of \(X^*\). Then the following statements about \(A\) and \(K\) are equivalent.

1. The set \(A\) is \(K\)-weakly precompact.
2. If \(\{x_n\}_{n \geq 1}\) is a sequence in \(A\) and \(g: Y \rightarrow \mathbb{R}\) is a continuous convex function such that \(\partial g(y) \subset \overline{co}(j^*(K))\) for every \(y \in Y\), then there exists a dense \(G_\delta\)-subset \(G\) of \(Y\) and a subsequence \(\{x_{n(k)}\}_{k \geq 1}\) of \(\{x_n\}_{n \geq 1}\) such that \(Dg(y, x_{n(k)})\) exists uniformly in \(k\) for each \(y \in G\).
3. If \(\{x_n\}_{n \geq 1}\) is a sequence in \(A\) and \(H\) is a non-empty subset of \(K\), then there exists an element \(y\) of \(Y\) and a subsequence \(\{x_{n(k)}\}_{k \geq 1}\) of \(\{x_n\}_{n \geq 1}\) such that \(Dg_H(y, x_{n(k)})\) exists uniformly in \(k\).

**Proof:** (1) \(\Rightarrow\) (2). The proof is analogous to that of the corresponding part of Theorem 8 in [1]. Suppose that (1) holds. Take any sequence \(\{x_n\}_{n \geq 1}\) in \(A\) and any continuous convex function \(g: Y \rightarrow \mathbb{R}\) such that \(\partial g(y) \subset \overline{co}(j^*(K))\) for every \(y \in Y\). As \(A\) is \(K\)-weakly precompact, we have a subsequence \(\{x_{n(k)}\}_{k \geq 1}\) of \(\{x_n\}_{n \geq 1}\) such that \(\lim_{k \to \infty} (x_{n(k)}, x^*)\) exists for every \(x^* \in K\). Therefore, by the first preliminary fact preceding Theorem, \(\overline{co}(j^*(K))\) is separably related to \(B(= \{x_{n(k)} : k \geq 1\})\). So it is separably related to \(aco(B)\) (the absolutely convex hull of \(B\)). Since \(\partial g(y) \subset \overline{co}(j^*(K))\) for every \(y \in Y\), by the same argument as in Theorem 3.14 and Proposition 3.15 of [2], we have a dense \(G_\delta\)-subset \(G\) of \(Y\) such that \(g\) is \(aco(B)\)-differentiable (cf. [2]) at every \(y \in G\), whence (2) holds.

(2) \(\Rightarrow\) (3). This follows immediately from the fact that \(\partial g_H(y) \subset \overline{co}(j^*(K))\) for every non-empty subset \(H\) of \(K\) and every \(y \in Y\).
(3) ⇒ (1). The proof of this part is crucial. Suppose that (1) fails. By the second preliminary fact preceding Theorem, we have a function \( h : I \to K \) and a sequence \( \{ x_n \}_{n \geq 1} \) in \( A \) as stated above. Take \( H = h(I) \), and let \( \{ U(n, k) : n = 0, 1, \ldots ; k = 0, \ldots , 2^n - 1 \} \) be a system of open intervals in \( I \) given by \( U(n, k) = (k/2^n, (k + 1)/2^n) \) if \( n \geq 0, 0 \leq k \leq 2^n - 1 \). Then we get that 
\[ \varphi^{-1}(\tau^{-1}(U(n, 2k))) \subset B_n \text{ and } \varphi^{-1}(\tau^{-1}(U(n, 2k + 1))) \subset A_n \] for \( n = 1, 2, \ldots \) and \( k = 0, \ldots , 2^{n-1} - 1 \). Further we note a following elementary fact: Let \( E \in \Lambda^+ \) and \( \{ n(i) \}_{i \geq 1} \) be a strictly increasing sequence of natural numbers. Then there exists a natural number \( i \) and a non-negative number \( q \) with \( 0 \leq 2q < 2^{n(i)} - 1 \) such that both \( E \cap U(n(i), 2q) \) and \( E \cap U(n(i), 2q + 1) \) are in \( \Lambda^+ \), which can be easily shown by an argument used in Lemma 2 of [3].

Now, let us show that for every subsequence \( \{ x_{n(k)} \}_{k \geq 1} \) of \( \{ x_n \}_{n \geq 1} \) and every \( y \in Y, Dg_H(y, x_{n(k)}) \) does not exist uniformly in \( k \). To this end, take any point \( y \) in \( Y \) and any subsequence \( \{ x_{n(k)} \}_{k \geq 1} \) of \( \{ x_n \}_{n \geq 1} \), and set \( y_k = x_{n(k)} \) for every \( k \). Consider a family of weak*-open slices of \( \overline{co^*} (j^*(T^*_h(\Delta(I)))) \) (= \( M \)) : \( \{ S(y, \delta/3i, M) : i \geq 1 \} \). Then we have that for every \( i \)
\[
S(y, \delta/3i, M) = \left\{ y^* \in M : (y, y^*) > \sup_{z^* \in M} (y, z^*) - \delta/3i \right\} = \left\{ y^* \in M : (y, y^*) > \text{ess-sup}_{t \in I} (j(y), h(t)) - \delta/3i \right\} = \left\{ y^* \in M : (y, y^*) > g_H(y) - \delta/3i \right\},
\]
since \( g_H(y) = \sup_{t \in I} (j(y), h(t)) = \text{ess-sup}_{t \in I} (j(y), h(t)) \) by virtue of (\( \alpha \)) above. So, letting \( E_i = \{ t \in I : (j(y), h(t)) > g_H(y) - \delta/3i \} \), we easily get that \( E_i \in \Lambda^+ \) and \( j^*(h(E_i)) \subset S(y, \delta/3i, M) \) for every \( i \). Hence, by the elementary fact stated above, there exists a natural number \( k(i) \) and a non-negative number \( q(i) \) with \( 0 \leq 2q(i) < 2^{n(k(i))} - 1 \) such that both \( E_i \cap U(n(k(i)), 2q(i)) \) and \( E_i \cap U(n(k(i)), 2q(i) + 1) \) are in \( \Lambda^+ \). For every \( i \), let \( F_i = E_i \cap U(n(k(i)), 2q(i)) \) and \( G_i = E_i \cap U(n(k(i)), 2q(i) + 1) \), and let \( u^*_i = j^*(T^*_h(\chi_{F_i}/\lambda(F_i))) \) and \( v^*_i = j^*(T^*_h(\chi_{G_i}/\lambda(G_i))) \). Then we have that for every \( i \)

(a) \( (y, u^*_i) > g_H(y) - \delta/3i \) and \( (y, v^*_i) > g_H(y) - \delta/3i \),

(b) \( (y_{k(i)}, u^*_i - v^*_i) \geq \delta \),

(c) \( g_H(y + y_{k(i)}/i, u^*_i) \) and \( g_H(y - y_{k(i)}/i, u^*_i) \) are uniformly larger than \( g_H(y - y_{k(i)}/i, u^*_i) \).

Indeed, we have that
\[
(y, u^*_i) = \left( j(y), T^*_h(\chi_{F_i}/\lambda(F_i)) \right) = \left\{ \int_{F_i} (j(y), h(t)) d\lambda(t) \right\}/\lambda(F_i) > g_H(y) - \delta/3i,
\]
since \( j^*(h(F_i)) \subset S(y, \delta/3i, M) \). Similarly, \( (y, v^*_i) > g_H(y) - \delta/3i \). Thus we have (a). And we can prove (b) as follows. In virtue of (\( \beta \)), we have that for
every $i$

\begin{align*}
(y_{k(i)}, u^*_i - v^*_i) \\
&= (j(y_{k(i)}), T^*_h(x_F/\lambda(F_i))) - (j(y_{k(i)}), T^*_h(x_{G_i}/\lambda(G_i))) \\
&= (j(x_{n(k(i))}), T^*_h(x_F/\lambda(F_i))) - (j(x_{n(k(i))}), T^*_h(x_{G_i}/\lambda(G_i))) \\
&= \left\{ \int_{F_i} (j(x_{n(k(i))}), h(t)) \, d\lambda(t) \right\} / \lambda(F_i) \\
&\quad - \left\{ \int_{G_i} (j(x_{n(k(i))}), h(t)) \, d\lambda(t) \right\} / \lambda(G_i) \\
&= \left\{ \int_{\varphi^{-1}(\tau^{-1}(F_i))} (j(x_{n(k(i))}), x^*) \, d\gamma(x^*) \right\} / \lambda(F_i) \\
&\quad - \left\{ \int_{\varphi^{-1}(\tau^{-1}(G_i))} (j(x_{n(k(i))}), x^*) \, d\gamma(x^*) \right\} / \lambda(G_i) \\
&\ge (r + \delta) - r = \delta,
\end{align*}

since $\varphi^{-1}(\tau^{-1}(F_i)) (\subset \varphi^{-1}(\tau^{-1}(U(n(k(i)), 2q(i)))) \subset B_{n(k(i))}$, $\varphi^{-1}(\tau^{-1}(G_i)) (\subset \varphi^{-1}(\tau^{-1}(U(n(k(i)), 2q(i) + 1)))) \subset A_{n(k(i))}$ and $\tau(\varphi(\gamma)) = \lambda$. As to (c), we have that for every $i$

$$g_H(y + y_{k(i)}/i) = \sup_{t \in I} (j(y + y_{k(i)}/i), h(t))$$

$$\ge \left\{ \int_{F_i} (j(y + y_{k(i)}/i), h(t)) \, d\lambda(t) \right\} / \lambda(F_i) = (y + y_{k(i)}/i, u^*_i).$$

Similarly, $g_H(y - y_{k(i)}/i) \ge (y - y_{k(i)}/i, v^*_i)$. Then, making use of (a), (b) and (c), we have that for every $i$

$$g_H(y + y_{k(i)}/i) + g_H(y - y_{k(i)}/i) - 2 \cdot g_H(y)$$

$$\ge (y + y_{k(i)}/i, u^*_i) + (y - y_{k(i)}/i, v^*_i) - \{(y, u^*_i + v^*_i) + 2\delta/3i\}$$

$$= (y_{k(i)}, u^*_i - v^*_i)/i - 2\delta/3i \ge \delta/3i.$$

Consequently, we have that for every $i$

$$\left\{ g_H(y + y_{k(i)}/i) + g_H(y - y_{k(i)}/i) - 2 \cdot g_H(y) \right\} / (1/i) > \delta/3,$$

which implies that $D g_H(y, x_{n(k)})$ does not exist uniformly in $k$. Thus the proof is complete. \hfill \Box

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References


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