Differentially trivial left Noetherian rings

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Abstract. We characterize left Noetherian rings which have only trivial derivations.

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0. Let $R$ be an associative ring with an identity element. A mapping $D : R \rightarrow R$ is called a derivation of $R$ if

$$D(x + y) = D(x) + D(y)$$

and

$$D(xy) = D(x)y + xD(y)$$

for all elements $x$ and $y$ in $R$. A ring having no non-zero derivations will be called here differentially trivial ([1]). Every differentially trivial ring is commutative.

Note that the class of differentially trivial rings is contained in the class of ideally differential rings, i.e. rings $R$ in which every ideal is invariant with respect to all derivations of $R$. The ideally differential rings were studied in [1–3].

In this paper we characterize differentially trivial Noetherian rings.

For convenience of the reader we recall some notation and terminology.

$R^+$ will always denote the additive group of $R$, $\mathcal{J}(R)$ the Jacobson radical of $R$ and $\mathcal{F}(R)$ the periodic part of $R^+$, $Q(R)$ the field of quotients of a commutative domain $R$, $\text{char}(R)$ the characteristic of $R$, $\text{Nil}(R)$ the prime radical of $R$, $\text{Ann}(x)$ the annihilator of $x$ in $R$, and $D_R(N) = \{c \in R \mid c + N \text{ is a regular element of } R/N\}$.

Throughout the paper $p$ is a prime and $\mathbb{Z}_{p^t}$ is the ring of integers modulo a prime power $p^t$.

Let us recall that a ring $R$ is called local if the factor ring $R/\mathcal{J}(R)$ is a skew field.

We will also use some other terminology from [4].

1. Let $R$ be a commutative Noetherian ring and $\text{Ass}(R)$ be the set of all prime ideals $M$ of $R$ for which there is a non-zero element $x$ such that

$$M = \text{Ann}(x).$$
By Corollary 2 of [5, Chapter II, § 2, n°2]
\[ \text{Ass}(R) \leq \text{Supp}(R) \]
(see Definition 5 of [5, Chapter IV, § 1, n°3]) and therefore by Corollary 1 of [5, Chapter IV, § 1, n°3] every minimal prime ideal of a commutative Noetherian ring \( R \) with zero-divisors is an annihilator. The subring of \( R \) generated by the identity element of \( R \) is called the prime subring of \( R \). If the field of quotients \( Q(R) \) of \( R \) is algebraic over its prime subfield we say that \( R \) is algebraic over its prime subring.

**Proposition 1** (see [6]). A (commutative) domain \( R \) is differentially trivial if and only if at least one of the following two cases takes place:

1. \( \text{char}(R) = 0 \) and \( R \) is algebraic over its prime subring;
2. \( \text{char}(R) = q > 0 \) and \( R = \{ a^q \mid a \in R \} \).

**Lemma 2.** Let \( Z \) be a prime subring of a commutative domain \( R \). If \( R \) is algebraic over \( Z \) then every non-zero prime ideal of \( R \) is maximal.

**Proof:** Put \( q = \text{char}(R) \). If \( q > 0 \) then \( Z \) is a finite field and, for every non-zero \( u \in R \), the transformation

\[ \phi : x \longrightarrow xu, \ x \in S, \]

is an injective endomorphism of the vector space \( S_Z, S = Z[u] \). Since \( R \) is algebraic over \( Z \), the space \( S_Z \) is finite-dimensional, \( \phi \) is an automorphism and \( u \) is invertible in \( R \). We have proved that \( R \) is a field in this case, and so we may assume that \( q = 0 \) and \( Z = \mathbb{Z} \) (the ring of integers). Now, let \( P \) be a non-zero prime ideal of \( R \) and \( I \) an ideal of \( R \) such that \( P \subseteq I \) and \( P \neq I \neq R \). Since \( R \) is algebraic over \( Z \), we have

\[ P \cap Z = pZ = I \cap Z \]

for some prime number \( p \) of \( Z \). Further, if \( u \in I \setminus P \) then there are \( n \geq 1 \) and \( a_0, \ldots, a_n \in Z \) such that

\[ a_0 + a_1u + \ldots + a_nu^n = 0, \]

\( a_0 \neq 0 \neq a_n \) and the numbers \( a_i \) are relatively prime (\( i = 0, \ldots, n \)). Clearly,

\[ a_0 \in I \cap Z = pZ, \]

\( p \) divides \( a_0 \) and

\[ a_1u + \ldots + a_nu^n = u(a_1 + \ldots + a_nu^{n-1}) \in P. \]

Thus

\[ a_1 + \ldots + a_nu^{n-1} \in P \subseteq I \]

and, again,

\[ a_1 \in I \cap Z = pZ. \]

Proceeding similarly further, we show that \( p \) divides all numbers \( a_0, \ldots, a_n \), a contradiction. This means that \( P \) is a maximal ideal in \( R \), as desired. \( \square \)
Proposition 3. Let $R$ be a differentially trivial Noetherian domain of characteristic $q$.

(i) If $q = 0$ then every non-zero prime ideal of $R$ is maximal.
(ii) If $q > 0$ then $R$ is a field.

Proof: (i). Just combine Proposition 1(1) and Lemma 2.
(ii). Let $P$ be a prime ideal of $R$. From Proposition 1(2) it follows that $P^q = P$. On the other hand,
\[ \bigcap_{n=1}^{\infty} P^n = \{0\} \]
by the Krull Theorem (see [7, Chapter IV, §7, Theorem 12]). Thus $P = \{0\}$ and we conclude that $R$ is a field. The proposition is proved. \qed

Remark 4. Let $R$ be a differentially trivial Noetherian domain of characteristic 0. With respect to Proposition 1(1), we may assume that
\[ \mathbb{Z} \subseteq R \subseteq Q(R) \subseteq A, \]
where $A$ is the field of algebraic complex numbers. Now, it follows from Proposition 3(i) that the integral closure $S$ of $R$ in $Q(R)$ is a Dedekind domain.

Lemma 5. Let $R$ be a differentially trivial Noetherian ring such that $R$ is not a domain and let the additive group $R^+$ be torsion-free. Then $R$ is a subdirect product of finitely many differentially trivial domains of characteristic 0.

Proof: If $\text{char}(R/P) = q > 0$ for some $P \in \text{Ass}(R)$ then there is an $x \in R$ such that $x \neq 0$, $P = (0 : x)$ and
\[ qxR = \{0\}, \]
and, therefore, $x \in \mathcal{F}(R)$, a contradiction. Thus $\text{char}(R/P) = 0$ for every $P \in \text{Ass}(R)$.

If $R/P$ is a field for every $P \in \text{Ass}(R)$ then $R$ is an Artinian ring by the Akizuki Theorem (see [7, Chapter IV, §2, Theorem 2]). Applying Corollary 2.12 of [6] we obtain that $R$ is the ring direct sum of finitely many differentially trivial fields of characteristic 0.

Suppose that the quotient ring $R/P$ is not a field for some $P \in \text{Ass}(R)$. If $\text{Ass}(R) = \{P\}$ then $P^n = \{0\}$ for some integer $n \geq 1$. Now, let $M$ be a nil ideal of $R$ such that $P/M$ is a minimal ideal of $R/M$. By Proposition 4.1.3(iii) of [4]
\[ \mathcal{D}_{R/M}(0) = \mathcal{D}_{R/M}(P/M), \]
and, therefore,
\[ \overline{a} \cdot P/M = P/M \]
for every $\overline{a} \in \mathcal{D}_{R/M}(P/M)$. Then, by Robson's Theorem (see [4, Theorem 4.1.9])
\[ R/M = S \oplus A_1 \oplus \ldots \oplus A_m \]
is a ring direct sum of a semiprime ring $S$ and finitely many local Artinian rings $A_1, \ldots, A_m$ ($m \in \mathbb{N}$). Since the factor ring $R/M$ is differentially trivial, each $A_i$ is a field ($i = 1, \ldots, m$) by Lemma 2.2 of [6], a contradiction. Consequently, $\text{Ass}(R) = \{P_1, \ldots, P_n\}$ for an integer $n \geq 2$.

Assume that $N = \mathcal{N}il(R) \neq \{0\}$ and put $S = \bigcap_{i=1}^{n} (R \setminus P_i)$. If $P_i \leq P_j$, where $i$ and $j$ are distinct integers and $1 \leq i, j \leq n$, and $u \in P_j \setminus P_i$ then there exist $k \geq 1$ and $a_0, \ldots, a_k$ in the prime subring of $R$ such that $a_0 \notin P_i$ and

$$a_0 + a_1 u + \cdots + a_k u^k \in P_i$$

(use Proposition 1(1)). Then, however, $a_0 \notin P_j$, and this is a contradiction with $\text{char}(R/P_j) = 0$. Consequently, all prime ideals $P_1, \ldots, P_n$ are pair-wise incomparable. By Proposition 10(ii) of [5, Chapter IV, §2, n°5] the total ring of quotients $A = S^{-1}R$ is Artinian, and by Theorem 4.1.4 of [4] the factor ring $R/N$ is a Goldie ring.

Let $M$ be a nil ideal of $R$ such that $N/M$ is a minimal ideal of $R/M$. By Proposition 4.1.3(iii) of [4] we have

$$\mathcal{D}_{R/M}(\overline{0}) = \mathcal{D}_{R/M}(N/M),$$

and, hence,

$$\overline{a} \cdot N/M = N/M$$

for every element $\overline{a} \in \mathcal{D}_{R/M}(\overline{0})$. Using Robson’s Theorem again, we conclude that the factor ring $R/M$ is a ring direct sum of a semiprime ring $X$ and finitely many local Artinian rings $B_1, \ldots, B_l$ ($l \in \mathbb{N}$).

Thus to complete the proof we show that a differentially trivial local Artinian ring $A = B_i$ of characteristic 0 is a field. Let $\pi : A \rightarrow A/J(A)$ be a canonical epimorphism and $K = A/J(A)$. By $P$ we denote the prime subring of $A$. Since $\text{char}(A) = 0$, $P$ is a field. The family $\Gamma$ of all subfields of $A$ ordered by inclusion has a maximal element $M$ by Zorn’s Lemma. If $\beta \in K$ is transcendental over $\pi(M)$ then every non-zero element of the polynomial ring $M[\beta]$ is not contained in $J(A)$. Therefore $M[\beta]$ is a field, a contradiction. Hence $K$ is an algebraic extension of $\pi(M)$.

Let $\overline{f}(Y) = Y^n + \alpha_1 Y^{n-1} + \ldots + \alpha_n \in \pi(M)[Y]$ be a minimal polynomial of $\eta \in K$ over $\pi(M)$. By $a_i$ we denote the inverse image of $\alpha_i$ in $M$ ($i = 1, \ldots, n$). Since $\overline{f}(Y)$ have no multiple roots, by Hensel’s Lemma (see e.g. [8, Chapter 10, Exercises 9 and 10]) the polynomial

$$f(Y) = Y^n + a_1 Y^{n-1} + \ldots + a_n \in M[Y]$$

has a unique root $z$ such that

$$\pi(z) = \eta.$$
This means that the ring $M[z]$ is isomorphic to the ring $\pi(M)[\eta]$ which is a field. The maximality of $M$ yields that $\eta \in \pi(M)$. Hence $\pi(M) = K$ and

$$A = \mathcal{J}(A) + M.$$  

Thus for every element $a$ of $A$ there are unique elements $j \in \mathcal{J}(A)$ and $m \in M$ such that

$$a = j + m.$$  

(1)

It is well known that $\mathcal{J}(A)$ is a nilpotent ideal with index of nilpotency, say, $t \geq 2$. Furthermore, $\text{Ann}(\mathcal{J}(A)) \neq \mathcal{J}(A)^{t-2}$ if $t > 2$. The map $\sigma : A \rightarrow A$ given by

$$\sigma(a) = bj,$$

where $b$ is a fixed element of $\mathcal{J}(A)^{t-2} \setminus (\text{Ann}\mathcal{J}(A))$ if $t > 2$, and

$$\sigma(a) = j$$

if $t = 2$ with $j$ as in (1), determines a non-zero derivation $\sigma$ of $A$, a contradiction. Hence $\mathcal{J}(B_i) = \mathcal{J}(A) = \{0\}$, $i = 1, \ldots, l$, a contradiction. This means that

$$\bigcap_{s=1}^{n} P_s = \mathcal{N}il(R) = \{0\}.$$  

By Proposition 10 of [9, §2.1], $R$ is a subdirect product of differentially trivial rings $R/P_s$ $(s = 1, \ldots, n)$. The lemma is proved.

\textbf{Lemma 6.} Let $R$ be a differentially trivial Noetherian ring such that $R$ is not a domain and let the additive group $R^+$ be torsion. Then

$$R \cong \bigoplus_{i=1}^{n} \mathbb{Z}_{p_i^{k_i}} (k_i \in \mathbb{N}).$$

\textbf{Proof:} By Proposition 3(ii) every non-zero prime ideal of $R$ is maximal. Consequently, $R$ is an Artinian ring (see e.g. [7, Chapter IV, §2]) and the result follows from Corollary 2.12 of [6].

\textbf{Lemma 7.} If $R$ is a differentially trivial semiprime Noetherian ring with the mixed additive group $R^+$ then

$$R = A \oplus B$$

is the ring direct sum of a differentially trivial ring $A$ of characteristic 0 and a differentially trivial ring $B$ of finite characteristic.

\textbf{Proof:} Let $\text{Ass}(R) = \{P_1, \ldots, P_n\}$. From $\mathcal{N}il(R) = \{0\}$ it follows that $n \geq 2$. Moreover, there are ideals $P, Q \in \text{Ass}(R)$ such that $\text{char}(R/P) = p$ for some
prime $p$ and $\text{char}(R/Q) = 0$. Let $\pi$ be the set of all primes $p$ such that there is an ideal $P \in \text{Ass}(R)$ with $\text{char}(R/P) = p$.

We will show that $\mathcal{F}(R)^+$ is a $\pi$-group. For doing this, suppose by contrary that $\mathcal{F}(R)^+$ contains some non-zero element $a$ of order $q$ and $q \notin \pi$. Then

$$a \cdot qR = \{0\}$$

and, consequently, $qR \leq \bigcup_{i=1}^n P_i$, a contradiction. Since $R$ is a Noetherian ring, the set $\pi$ is finite and, further, $\mathcal{F}(R)^+$ is a group of exponent $p_0 = \prod_{p \in \pi} p$. If $F_p$ is the Sylow $p$-subgroup of $\mathcal{F}(R)^+$ ($p \in \pi$) then

$$\mathcal{F}(R)^+ = F_p \oplus (\mathcal{F}(R) \cap pR)$$

is a group direct sum, where $\mathcal{F}(R) \cap pR$ is a $p'$-subgroup. Since the factor ring $R/pR$ is differentially trivial and $(F_p + pR)/pR \cong F_p$, in view of Proposition 3(ii) and Lemma 6 the ideal $\mathcal{F}(R)$ is a differentially trivial ring with the identity element $e$. Thus $eR \leq \mathcal{F}(R)$ and

$$R = eR \oplus (1 - e)R$$

is a ring direct sum. If $eR \neq \mathcal{F}(R)$ and $f \in \mathcal{F}(R) \setminus eR$ then

$$f = eu + (1 - e)v$$

for some elements $u, v \in R$, and thus

$$f = e \cdot f = eu,$$

a contradiction. Hence $eR = \mathcal{F}(R)$ and $(1 - e)R$ is a differentially trivial ring of characteristic 0. The lemma is proved. \hfill \box 

**Theorem 8.** Let $R$ be a Noetherian ring. Then $R$ is differentially trivial if and only if it is of one of the following types:

1) $R$ is a differentially trivial Noetherian domain (i.e. $R$ is algebraic over its prime subring if $\text{char}(R) = 0$, and $R = \{a^p \mid a \in R\}$ if $\text{char}(R) = p$);

2) $R$ is a subdirect product of finitely many differentially trivial Noetherian domains of characteristic 0;

3) $R \cong \bigoplus_{i=1}^n \mathbb{Z}_{p_i^{k_i}}$;

4) $R = F \oplus S$ is a ring direct sum, where $S$ is a ring of type 2) and $F$ is a ring of type 3).

**Proof:** ($\Rightarrow$). Let $R$ be a differentially trivial Noetherian ring. If $R$ is a domain then by Proposition 1 it is a ring of type 1).

Suppose that $R$ is not a domain. If the additive group $R^+$ is torsion-free (periodic, respectively) then $R$ is a ring of type 2) by Lemma 5 (of type 3) by Lemma 6, respectively).
Assume that the additive group $R^+$ is mixed. As a consequence of Lemma 5
\[ \mathcal{N}il(R) \leq \mathcal{F}(R). \]
If $\text{Ass}(R) = \{P\}$ then $\text{char}(R/P) = 0$ and
\[ P = \mathcal{N}il(R) = \mathcal{F}(R). \]
Let $a$ be a non-zero element of $\mathcal{F}(R)^+$ of finite order $p$. Then
\[ a \cdot pR = \{0\} \]
and in view of Corollary 2 of [5, Chapter IV, §1, n°3] $pR \leq P$, a contradiction.
Hence $\text{Ass}(R) = \{P_1, \ldots, P_n\}$ for an integer $n \geq 2$. Since the group $R^+$ is nonperiodic, $\text{char}(R/Q) = 0$ for some ideal $Q \in \text{Ass}(R)$. If $\text{char}(R/P_i) = 0$ for all $i \ (i = 1, \ldots, n)$ then
\[ \mathcal{F}(R) = \mathcal{N}il(R) \]
and for every non-zero element $a$ of $\mathcal{F}(R)$ of order $p$ we have
\[ a \cdot pR = \{0\}. \]
By Corollary 2 of [5, Chapter IV, §1, n°3]
\[ pR \leq P_s \]
for some integer $s \ (1 \leq s \leq n)$, a contradiction. Thus $R$ has an ideal $M \in \text{Ass}(R)$ such that $\text{char}(R/M) = p$.
Now it is clear that
\[ c \cdot \mathcal{N}il(R) = \mathcal{N}il(R) \]
for every element $c \in \mathcal{D}_R(0)$. From Theorem 2.2.15 of [4] it follows that $R/\mathcal{N}il(R)$ is a Goldie ring. Then by Proposition 4.1.3(ii) of [4] we obtain
\[ \mathcal{D}_R(0) = \mathcal{D}_R(\mathcal{N}il(R)). \]
Using Robson’s Theorem (see [4, Theorem 4.1.9]), one sees that $R$ is the ring direct sum of a semiprime ring $S$ and finitely many local Artinian rings $A_1, \ldots, A_k$ $(k \in \mathbb{N})$. In view of Corollary 2.12 of [6], $A_i$ is either a differentially trivial field or isomorphic to some $\mathbb{Z}_{p^k}$. Finally, we can apply Lemma 7.

($\Rightarrow$). It is obvious that $R$ of type 1), 2) or 4) is differentially trivial. We will show that $R$ of type 3) is differentially trivial. Since $R$ is a subdirect product of finitely many differentially trivial domains $R_1, \ldots, R_n$ of characteristic 0, Proposition 10 of [9, §2.1] yields that there are the ideals $P_1, \ldots, P_n$ of $R$ such that
\[ \bigcap_{i=1}^{n} P_i = \{0\} \text{ and } R_i = R/P_i \ (i = 1, \ldots, n). \]
By Theorem 2.2.15 of [4] $R$ is a Goldie ring. Then by Theorem 2.3.6 of [4] $S = D_{R}(0)$ is an Ore set and $S^{-1}R$ is an Artinian ring. $\text{Nil}(R) = \{0\}$ yields that $\mathcal{J}(S^{-1}R) = \{0\}$ and, consequently,

$$S^{-1}R = B_1 \oplus \ldots \oplus B_n$$

is the ring direct sum of fields $B_1, \ldots, B_n$ such that $Q(R/P_i) \cong B_i \ (i = 1, \ldots, n)$. Clearly, every derivation of $R$ can be extended to a derivation of $S^{-1}R$. Since the ring $S^{-1}R$ is differentially trivial we conclude that $R$ is as desired. $\square$

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**References**


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