Rectangular modulus, Birkhoff orthogonality and characterizations of inner product spaces

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Abstract. Some characterizations of inner product spaces in terms of Birkhoff orthogonality are given. In this connection we define the rectangular modulus $\mu_X$ of the normed space $X$. The values of the rectangular modulus at some noteworthy points are well-known constants of $X$. Characterizations (involving $\mu_X$) of inner product spaces of dimension $\geq 2$, respectively $\geq 3$, are given and the behaviour of $\mu_X$ is studied.

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1. Introduction

In the present paper we shall give, at the beginning, natural generalizations of some known characterizations of inner product spaces (i.p.s. for short). By introducing a parameter $\lambda > 0$ we obtain, in the particular case $\lambda = 1$, the known results collected in D. Amir’s book [3, p. 79]. The characterizations are expressed in terms of Birkhoff orthogonality and the new conditions will be given in an “anti-symmetric” manner with respect to $\lambda$. In this direction one obtains a more general form (depending on $\lambda$) of M. Baronti’s Lemma 1 in [4] and a generalization of M. del Rio and C. Benitez’s Lemma 3 in [15].

These generalizations (especially Lemma 1 in the sequel) suggest to introduce a function $\mu_X : (0, \infty) \to \mathbb{R}$, with the property that $\mu_X(1)$ is the well-known rectangular constant of the normed space $X$. We call this function the rectangular modulus of $X$. The rectangular modulus is an increasing convex function and Lipschitz continuous of best Lipschitz constant 2. Moreover, $\mu_X(0+)$ is another well-known constant of the normed space $X$.

For any fixed $\lambda > 0$ a characterization of i.p.s. in terms of the rectangular modulus is also given. In the limit case when $\lambda \searrow 0$, the analogous characterization of i.p.s. is valid only for normed spaces of dimension $\geq 3$.

2. Preliminary results and notation

We denote by $(X, \| \cdot \|)$ a real normed space of dimension $\geq 2$. For $x \in X$ and $r > 0$ let $S_X(x, r) = \{ y \in X : \| x - y \| = r \}$ and $B_X(x, r) = \{ y \in X : \| x - y \| \leq r \}$, be the sphere respectively closed ball with center $x$ and radius $r$. The unit sphere
$S_X(0,1)$ and the closed unit ball $B_X(0,1)$ of the space $X$ will be denoted by $S_X$ and $B_X$ respectively. The symbol $\perp$ is used for Birkhoff orthogonality in $X$; namely $x \perp y$ if $\|x\| \leq \|x + ty\|$, for all $t \in \mathbb{R}$. Geometrically, this means that the line through $x$ in the $y$–direction supports the ball $B_X(0,\|x\|)$ at $x$. For $x, y \in X$, $x \neq y$, the closed line segment with vertices $x$ and $y$ is denoted by $[x; y]$. Any two-dimensional subspace of $X$ will be identified with $\mathbb{R}^2$ equipped with an appropriate norm and an orientation $\omega$. The orientation $\omega$ of the ordered pair $(x, y)$ of vectors (with $x + y \neq 0$ and $\|x\| = \|y\|$) is recorded by $x \prec y \prec -x$. Denote by $\perp^A$ the area orthogonality ([1], [3, p.65]) defined for $(\mathbb{R}^2, \| \cdot \|)$ by $x \perp^A y$ if the radius vectors $\pm x, \pm y$ divide the unit ball of $\mathbb{R}^2$ into four parts of equal area. The following known lemmas will be used in Section 3.

**Lemma A** ([2]). Let $S_{\mathbb{R}^2}$ be the unit sphere of $(\mathbb{R}^2, \| \cdot \|)$ and $s(\alpha)$ be the point of $S_{\mathbb{R}^2}$ which is to a given point $s(0)$ at an angle $0 \leq \alpha < 2\pi$, measured with a given orientation of the plane. Then for every $\lambda > 0$ the real continuous functions

$$\alpha \in [0, \pi) \to \|s(0) + \lambda s(\alpha)\|,$$

and

$$\alpha \in [0, \pi) \to \|s(0) - \lambda s(\alpha)\|,$$

are decreasing and increasing respectively.

If $(\mathbb{R}^2, \| \cdot \|)$ is strictly convex then the aforementioned functions are strictly monotonic.

In the two-dimensional normed space $X$ let $u^*, v^* \in S_X$ be such that $u^* \perp v^*$ and let us consider the corresponding $(u^*, v^*)$-coordinate system in which $u^*, v^*$ are versors. For $u, v \in X$ let $A_{u,v}$ be the area of the parallelogram $\{\alpha u + \beta v : \alpha, \beta \in [0,1]\}$ in the $(u^*, v^*)$-coordinate system. It is clear that if $r, s > 0$ then $A_{ru,sv} = rsA_{u,v}$.

**Lemma B** ([3, p.78]). Let $X$ be a two-dimensional normed space in which orthogonality is symmetric. Then $A_{u,v} = A_{u^*,v^*} = 1$, $\forall u, v \in S_X$, $u \perp v$.

### 3. Characterizations of inner product spaces and Birkhoff orthogonality

For $u, v \in S_X$, $u \neq \pm v$ and $\lambda > 0$ we define the function $\varphi_{\lambda,u,v} : (0, \infty) \to (0, \infty)$ by

$$\varphi_{\lambda,u,v}(t) = \frac{\lambda^2 + t}{\|\lambda u + tv\|}, \forall t > 0.$$

With the above notation we have the following generalization of Lemma 1 in [4].

**Lemma 1.** Let $u, v \in S_X$, $u \neq \pm v$ and $\lambda, t_0 > 0$ be fixed. The following are equivalent:

(a) $(\lambda u + t_0 v) \perp (u - \lambda v)$.
(b) $\varphi_{\lambda,u,v}(t_0) \geq \varphi_{\lambda,u,v}(t)$, $\forall t > 0$. 
Proof: If we suppose that (a) holds then we have
\[ \left( u - \frac{t_0}{\lambda^2 + t_0} (u - \lambda v) \right) \perp (u - \lambda v), \]
which implies
\[ (1) \quad \left\| u - \frac{t_0}{\lambda^2 + t_0} (u - \lambda v) \right\| \leq \left\| u - \frac{t}{\lambda^2 + t} (u - \lambda v) \right\|, \quad \forall t > 0. \]
and hence
\[ \frac{\lambda^2 + t_0}{\| \lambda u + t_0 v \|} \geq \frac{\lambda^2 + t}{\| \lambda u + tv \|}, \quad \forall t > 0. \]
Now, if (b) is satisfied then (1) holds and this shows that in the two-dimensional space \( X_2 \) generated by \( u \) and \( v \) the straight line containing the open line segment
\[ l = \left\{ u - \frac{t}{\lambda^2 + t} (u - \lambda v) : t > 0 \right\} \]
supports the ball \( B_X(0, \|w_0\|) \) at \( w_0 = u - t_0 (u - \lambda v)/(\lambda^2 + t_0) \). Then \( w_0 \perp (u - \lambda v) \) or equivalently \( (\lambda u + t_0 v) \perp (u - \lambda v) \). \( \square \)

Remark. If we consider the function \( \psi_{\lambda,u,v} : (0, \infty) \to (0, \infty) \) defined by
\[ \psi_{\lambda,u,v}(t') = \lambda \varphi_1/\lambda,u,v \left( \frac{1}{t'} \right) = \frac{\lambda^2 + t'}{\| t' u + \lambda v \|}, \]
then we easily deduce:

Lemma 1'. With the previous notation, let \( t'_0 > 0 \) be fixed. The following are equivalent:

(a') \( t'_0 u + \lambda v \) \( \perp (\lambda u - v) \).
(b') \( \psi_{\lambda,u,v}(t'_0) \geq \psi_{\lambda,u,v}(t'), \quad \forall t' > 0. \)

The next theorem is known for \( \lambda = 1 \), see Propositions 10.1–10.3, 10.3' and 10.4 in [3] (see also [4] and [15]).

Theorem 2. Let \( \lambda > 0 \) be fixed. The following are equivalent:

1) \( \forall u, v \in S_X, \ u \perp v \Rightarrow (\lambda u + v) \perp (u - \lambda v); \)
2) \( \forall u, v \in S_X, \ u \perp v \Rightarrow \| \lambda u + v \| = \| u - \lambda v \|; \)
3) \( \forall u, v \in S_X, \ u \perp v \Rightarrow \| \lambda u + v \| \leq \sqrt{1 + \lambda^2}; \)
4) \( \forall u, v \in S_X, \ u \perp v \Rightarrow \| \lambda u + v \| \geq \sqrt{1 + \lambda^2}; \)
5) \( \forall u, v \in S_X, \ u \perp v \Rightarrow \| \lambda u + v \| = \sqrt{1 + \lambda^2}; \)
6) the normed space \( X \) is an i.p.s.
Remarks. As we can see a little later the equivalences 3) ⇔ 4) ⇔ 5) are simple consequences of a result in [12]. The implication 5) ⇒ 6) is a strong result recently obtained (among other results) by C. Benitez, K. Przeslawski and D. Yost in [6]. We note that the weaker result 5′) ⇒ 6) was also proved and used in [18, pp. 388–389], where 5′) is given by
\[\forall u, v \in S_X, u \perp v \Rightarrow \|\lambda u + v\| = \sqrt{1 + \lambda^2}, \quad \|u + \lambda v\| = \sqrt{1 + \lambda^2},\]
\[\lambda > 0 \text{ being fixed.}\]

Proof of Theorem 2: We show that 1) ⇒ 2). Suppose that 1) is verified and let \(u, v \in S_X, u \perp v,\) and \(\lambda > 0\) be fixed. It follows that
\[\left(\lambda \frac{\|\lambda u + v\|}{\|u + \lambda v\|} + \frac{u - \lambda v}{\|u - \lambda v\|}\right) \perp \left(\frac{\lambda u + v}{\|\lambda u + v\|} - \lambda \frac{u - \lambda v}{\|u - \lambda v\|}\right).\]
If we put \(t = \|u - \lambda v\|/\|\lambda u + v\|\) then, by Lemma 1, we have:
\[\frac{\lambda^2 + 1}{\|\lambda(\lambda u + v)/\|\lambda u + v\| + (u - \lambda v)/\|u - \lambda v\|\|} \geq \frac{\lambda^2 + t}{\|\lambda(\lambda u + v)/\|\lambda u + v\| + t(u - \lambda v)/\|u - \lambda v\|\|},\]
and consequently
\[\frac{\lambda^2 + 1}{\|\lambda^2 u + \lambda v + (1/t)(u - \lambda v)\|} \geq \frac{\lambda^2 + t}{\|\lambda^2 u + \lambda v + u - \lambda v\|}.\]
From \(u \perp v\) one obtains
\[(\lambda^2 + 1)^2 \geq (\lambda^2 + t) \cdot \left(\lambda^2 + \frac{1}{t}\right)u + \lambda(1 - \frac{1}{t})v \geq (\lambda^2 + t) \left(\lambda^2 + \frac{1}{t}\right),\]
yielding
\[\left(\sqrt{t} - \frac{1}{\sqrt{t}}\right)^2 \leq 0 \iff t = 1.\]
This implies that \(\|\lambda u + v\| = \|u - \lambda v\|\).

Now we show that 2) implies the strict convexity of \(X\). Suppose that 2) is satisfied and, on the contrary, there exists a support line \(l\) of \(S_X\) such that \(l \cap S_X = [u_1, u_2], u_1 \neq u_2\). Then any \(u \in [u_1, u_2]\) can be written as \(u = u_t = u_1 + t(u_2 - u_1), t \in [0,1]\) and \(\|u_t\| = 1\). The function \(t \to \|u_1 + t(u_2 - u_1)\|, t \in \mathbb{R}\) is 1 on \([0,1]\], strictly increasing for \(t > 1\) and strictly decreasing for \(t < 0\). Denoting by \(v = (u_2 - u_1)/\|u_2 - u_1\|\) we have that \(u_t \perp v, \forall t \in [0,1]\), and the application
\[t \to \|\lambda u_t + v\| = \lambda \left\|u_1 + t(u_2 - u_1) + \frac{u_2 - u_1}{\lambda\|u_2 - u_1\|}\right\|, t \in (1 - \varepsilon_1, 1]\]
with sufficiently small $\varepsilon_1 > 0$ is strictly increasing. On the other hand, the application

$$t \rightarrow \|u_t - \lambda v\| = \left\| u_1 + t(u_2 - u_1) - \lambda \frac{u_2 - u_1}{\|u_2 - u_1\|} \right\|, \forall t \in (1 - \varepsilon_2, 1],$$

with small enough $\varepsilon_2 > 0$ is constant or strictly decreasing. But from 2) we have that $\|\lambda u_t + v\| = \|u_t - \lambda v\|$, $\forall t \in (1 - \min\{\varepsilon_1, \varepsilon_2\}, 1]$, a contradiction.

We prove that if 2) is satisfied then

$$u, v \in S_X \quad \text{and} \quad \|\lambda u + v\| = \|u - \lambda v\| \Rightarrow u \perp v.$$  \hspace{1cm} (2)

Suppose that 2) holds and, on the contrary, there exist $u, v' \in S_X$ such that $\|\lambda u + v'\| = \|u - \lambda v'\|$ and $u$ is not orthogonal to $v'$. In the space $X'_2$ generated by $u$ and $v'$ (understood as $(\mathbb{R}_2, \|\cdot\|)$) we choose the orientation such that $u \prec v' \prec -u$, $(v' \neq \pm u)$. Let $v \in S_X'$ be such that $u \perp v$ and $u \prec v \prec -u$. Then $v \neq v'$.

Supposing that $u \prec v' \prec v \prec -u$, by Lemma A and the strict convexity of $X$ we have

$$\|u - \lambda v'\| < \|u - \lambda v\|$$

respectively

$$\|\lambda u + v'\| = \lambda \|u + \frac{1}{\lambda} v'\| > \lambda \|u + \frac{1}{\lambda} v\| = \|\lambda u + v\|,$$

implying $\|\lambda u + v\| < \|u - \lambda v\|$, a contradiction. The case $u \prec v \prec v' \prec -u$ can be treated in a similar way.

Suppose now that 2) holds. Then 1) holds as well. Indeed, if $u, v \in S_X$, $u \perp v$ and $\lambda > 0$ is fixed then

$$\left\| \frac{\lambda u + v}{\|\lambda u + v\|} + \frac{u - \lambda v}{\|u - \lambda v\|} \right\| = \frac{\lambda^2 + 1}{\|\lambda u + v\|} = \left\| \frac{\lambda u + v}{\|\lambda u + v\|} - \lambda \frac{u - \lambda v}{\|u - \lambda v\|} \right\|.$$  

From (2) we have

$$\frac{\lambda u + v}{\|\lambda u + v\|} \perp \frac{u - \lambda v}{\|u - \lambda v\|},$$

which yields $(\lambda u + v) \perp (u - \lambda v)$.

Observe now that 2) implies the symmetry of orthogonality. Indeed, if $u, v \in S_X$ and $\lambda > 0$ then from 2) and (2) one obtains:

$$u \perp v \iff u \perp -v \iff \|\lambda u - v\| = \|u + \lambda v\| \iff$$

$$\iff \|\lambda v + u\| = \|v - \lambda u\| \iff v \perp u.$$  

Moreover, since $X$ is strictly convex, it follows that $X$ is also smooth (see [3, p. 78]).
In order to prove $3) \Rightarrow 4)$, it is sufficient to consider the case of two-dimensional spaces, i.e. $X$ may be considered $\mathbb{R}^2$ with the norm $\| \cdot \|$. It follows that $S_X$ is a rectifiable simple closed Jordan curve. Denoting

$$S_\lambda = \{ \lambda u + v : u, v \in S_X, u \perp v \},$$

it follows that $S_\lambda$ is also a closed rectifiable Jordan curve. A parametrization of $S_\lambda$ may be given as in J. Joly [12, p.304]. More precisely, let $u = u(\theta) = (u_1(\theta), u_2(\theta))$, $\theta \in [0, 2\pi)$ be the parametrization of $S_X$ in a rectangular system of axes with $u(0) \prec u(\theta) < -u(0)$, for all $\theta \in [0, \pi)$. Now, consider the vectors $u, v \in S_X$, $u \perp v$ such that $u \prec v \prec -u$. We have

$$u = u(\theta(\sigma)) = (u_1(\theta(\sigma)), u_2(\theta(\sigma))),$$
$$v = v(\nu(\sigma)) = (v_1(\nu(\sigma)), v_2(\nu(\sigma))),$$

where $\theta, \nu : [0, 4\pi) \to [0, 2\pi)$, are continuous increasing and surjective functions and $u_1, u_2, v_1, v_2$ are continuous functions with bounded variation. Moreover, $\sigma = \theta(\sigma) + \nu(\sigma)$ and the decomposition is unique. Then $S_\lambda$ can be rewritten

$$S_\lambda = \{ \lambda u(\theta(\sigma)) + v(\nu(\sigma)) : \sigma \in [0, 4\pi) \}.$$ 

Let $A$ be the area of the unit ball of $X$ and let $A_\lambda$ be the area enclosed by $S_\lambda$. Then with a similar computation as in [12], we have:

$$A_\lambda = \lambda^2 \int_{S_X} u_1 \, du_2 + \int_{S_X} v_1 \, dv_2 = (\lambda^2 + 1) A.$$ 

Now, from $3)$ and the continuity of the functions $u_1, u_2, v_1, v_2, \theta$ and $\nu$ we have:

$$\| \lambda u + v \| \geq \sqrt{1 + \lambda^2},$$

for all $u, v \in S_X$, $u \perp v$ proving that $3) \Rightarrow 4)$. Analogously $4) \Rightarrow 3)$ and finally we have $3) \iff 4) \iff 5)$.

We shall show that $2) \Rightarrow 5)$. Since the Birkhoff orthogonality in $X$ is symmetric, as it is well known, $\dim (X) \geq 3$ implies that $X$ is an i.p.s. ([11], [3, p. 143]), and in this case the result follows. Suppose $X$ is two-dimensional and for fixed $u^*, v^* \in S_X$, $u^* \perp v^*$, consider the $(u^*, v^*)$-coordinate system of $X$. Let $u, v \in S_X$, $u \perp v$ be given. Then the area $A_{\lambda u + v, u - \lambda v}$ can be computed by $A_{\lambda u + v, u - \lambda v} = |\Delta| \cdot A_{u, v}$, where

$$\Delta = \begin{vmatrix} \lambda & 1 & 1 \\ 1 & -\lambda & 1 \\ 0 & 0 & 1 \end{vmatrix} = -\lambda^2 - 1.$$
Now, from Lemma B, \( A_{\lambda u+v,u-\lambda v} = \lambda^2 + 1 \) in the \((u^*, v^*)\)-coordinate system. Since by 2) \(\Leftrightarrow\) 1), \(\lambda u + v \perp u - \lambda v\), we have
\[
A(\lambda u+v)/\|\lambda u+v\|, (u-\lambda v)/\|u-\lambda v\| = 1
\]
and again by 2) \(\|\lambda u + v\| = \|u - \lambda v\| = \sqrt{\lambda^2 + 1}\), \(\forall u, v \in S_X\), \(u \perp v\). From \(u \perp v \Leftrightarrow u \perp -v\) we obtain the desired result.

Now, by the quoted result in [6], we have 5) \(\Rightarrow\) 6). In fact in [6] it was proved that 5) implies the symmetry of Birkhoff orthogonality and that the Birkhoff orthogonality \(\perp\) implies the area orthogonality \(\perp^A\). By [15] it follows that \(X\) is an i.p.s. Since the implications 6) \(\Rightarrow\) 5) and 5) \(\Rightarrow\) 2) are trivial the theorem is completely proved.

\[\Box\]

4. The rectangular modulus of a normed space

For the normed space \(X\) the rectangular constant \(\mu(X)\) was defined in [12] by
\[
\mu(X) = \sup \{\mu[x, y] : x, y \in X \setminus \{0\}, x \perp y\},
\]
where
\[
\mu[x, y] = \sup_{s \in \mathbb{R}} \frac{\|x\| + |s|\|y\|}{\|x + sy\|}, \quad \forall x, y \in X \setminus \{0\}, x \perp y.
\]
Since \(x \perp y \Leftrightarrow x \perp -y\) we easily deduce that
\[
\mu(X) = \sup \left\{ \frac{1 + |s|\|y\|/\|x\|}{\|x + sy\|} : s \neq 0, x, y \in X \setminus \{0\}, x \perp y \right\}
\]
\[
= \sup \left\{ \frac{1 + t}{\|u + tv\|} : t > 0, u, v \in S_X, u \perp v \right\}.
\]
We define the rectangular modulus of \(X\) as the function \(\mu_X : (0, \infty) \to \mathbb{R}\)
\[
\mu_X(\lambda) = \sup \{\max\{\varphi_{\lambda, u, v}(t), \lambda\varphi_{1/\lambda, u, v}(t)\} : t > 0, u, v \in S_X, u \perp v\}
\]
\[
= \sup \left\{ \max \left\{ \frac{\lambda^2 + t}{\|\lambda u + tv\|}, \frac{1 + \lambda^2 t}{\|u + \lambda tv\|} \right\} : t > 0, u, v \in S_X, u \perp v \right\},
\]
for all \(\lambda > 0\). From the definition it is clear that \(\mu_X(1) = \mu(X)\). As it is well known the modulus of convexity of \(X\) ([7]), denoted by \(\delta_X\) and the modulus of smoothness of \(X\) ([13]), denoted by \(\rho_X\) satisfy Nordlander's type inequalities, i.e.
\[
\delta_X(\varepsilon) \leq \delta_H(\varepsilon) = 1 - \sqrt{1 - \varepsilon^2/4}, \quad \forall \varepsilon \in [0, 2]
\]
and
\[ \rho_X(\tau) \geq \rho_H(\tau) = \sqrt{\tau^2 + 1} - 1, \quad \forall \tau \geq 0, \]
where \( H \) is an i.p.s.

G. Nordlander [14] has conjectured that if \( \delta_X(\varepsilon) = 1 - \sqrt{1 - \varepsilon^2}/4 \) for a fixed \( \varepsilon \in (0, 2) \) then \( X \) is an i.p.s. J. Alonso and C. Benitez [2] proved that this assertion is true exactly for \( \varepsilon \in (0, 2) \setminus D \) where \( D = \{2 \cos(k\pi/(2n)) : k = 1, \ldots, n-1; n = 2, 3, \ldots\} \). Analogous results were obtained for the modulus of smoothness and for other known moduli. Generally, if \( \gamma_X \) denotes such a modulus and \( t \) is fixed then from \( \gamma_X(t) = \gamma_H(t) \) it follows that \( X \) is an i.p.s. except for a countable set of points \( t \) in the domain of \( \gamma_X \) ([21]).

The modulus of squareness \( \xi_X \) studied in [6], [16], [17], [18] satisfies also the inequality
\[ \xi_X(\beta) \geq \xi_H(\beta) = 1/\sqrt{1 - \beta^2}, \quad \forall \beta \in [0, 1). \]
Moreover, if \( \xi_X(\beta) = 1/\sqrt{1 - \beta^2} \), for a fixed \( \beta \in (0, 1) \) then \( X \) is an i.p.s.

For the rectangular modulus we have:

**Theorem 3.**
(a) If \( H \) is an i.p.s. then \( \mu_H(\lambda) = \sqrt{1 + \lambda^2}, \forall \lambda > 0. \)
(b) If \( X \) is a normed space and \( H \) is an i.p.s. then
\[ \mu_X(\lambda) \geq \mu_H(\lambda), \forall \lambda > 0. \]
(c) If \( \mu_X(\lambda) = \sqrt{1 + \lambda^2} \) for a fixed \( \lambda > 0 \) then \( X \) is an i.p.s.

**Proof:** (a) \( \mu_H(\lambda) = \]
\[= \sup \left\{ \max \left\{ \frac{\lambda^2 + t}{\|\lambda u + tv\|}, \frac{1 + \lambda^2 t}{\|u + tv\|} \right\} : t > 0, u, v \in S_H, u \perp v \right\} \]
\[= \sup \left\{ \max \left\{ \frac{\lambda^2 + t}{\sqrt{\lambda^2 + t^2}}, \frac{1 + \lambda^2 t}{\sqrt{1 + \lambda^2 t^2}} \right\} : t > 0 \right\}. \]

It is easily seen that the function \( f_\lambda : (0, \infty) \rightarrow \mathbb{R} \)
\[f_\lambda(t) = \frac{\lambda^2 + t}{\sqrt{\lambda^2 + t^2}} - \frac{1 + \lambda^2 t}{\sqrt{1 + \lambda^2 t^2}}, \quad t > 0 \]
satisfies the condition \( \text{sign} f_\lambda(t) = \text{sign}(1 - \lambda) \) and from \( f_\lambda(1) = 0, \forall \lambda > 0 \) we deduce that \( \mu_H(\lambda) = \sqrt{1 + \lambda^2}, \forall \lambda > 0. \)

(b) Let \( \lambda \in (0, \infty) \) be a fixed number. We can suppose that \( X \) is a two-dimensional normed space. By using formula (3) we conclude that
\[\inf \{\|\lambda u + v\| : u, v \in S_X, u \perp v\} \leq \sqrt{\lambda^2 + 1}\]
and this implies

\[
\mu_X(\lambda) \geq \sup \left\{ \frac{\lambda^2 + t}{\|\lambda u + tv\|} : t > 0, u, v \in S_X, u \perp v \right\}
\]

\[
\geq \sup \left\{ \frac{\lambda^2 + 1}{\|\lambda u + v\|} : u, v \in S_X, u \perp v \right\}
\]

\[
= \frac{\lambda^2 + 1}{\inf \{\|\lambda u + v\| : u, v \in S_X, u \perp v\}} \geq \frac{\lambda^2 + 1}{\sqrt{\lambda^2 + 1}} = \sqrt{\lambda^2 + 1}.
\]

In particular \( \mu(X) = \mu_X(1) \geq \sqrt{2} \), as in [12].

(c) \[
\mu_X(\lambda) = \sqrt{1 + \lambda^2}
\]
\[
\geq \sup \left\{ \max \left\{ \frac{\lambda^2 + 1}{\|\lambda u + v\|}, \frac{1 + \lambda^2}{\|u + \lambda v\|} \right\} : u, v \in S_X, u \perp v \right\}
\]
\[
\geq \frac{\lambda^2 + 1}{\|\lambda u + v\|}, \ \forall \ u, v \in S_X, u \perp v,
\]
\( \lambda > 0 \) being fixed. Hence \( \|\lambda u + v\| \geq \sqrt{\lambda^2 + 1}, \forall \ u, v \in S_X, u \perp v. \) By Theorem 2, 4) \( \Leftrightarrow \) 6), we have that \( X \) is an i.p.s. \( \square \)

**Remark.** Let us define the *-rectangular modulus* by the simpler formula

\[
\mu_X^*(\lambda) = \sup \{ \varphi_{\lambda,u,v}(t) : t > 0, u, v \in S_X, u \perp v \}
\]

\[
= \sup \left\{ \frac{\lambda^2 + t}{\|\lambda u + tv\|} : t > 0, u, v \in S_X, u \perp v \right\}, \ \forall \lambda > 0.
\]

It is clear (with similar proofs) that:

(a') \( \mu_H^*(\lambda) = \sqrt{\lambda^2 + 1}, \ \forall \lambda > 0, \ H \) being an i.p.s.;

(b') for each normed space \( X, \mu_X^*(\lambda) \geq \mu_H^*(\lambda) = \sqrt{\lambda^2 + 1}, \ \forall \lambda > 0; \)

(c') if \( \mu_X^*(\lambda) = \sqrt{1 + \lambda^2}, \) for a fixed \( \lambda > 0 \) then \( X \) is an i.p.s.

Some properties of the rectangular modulus are collected in

**Theorem 4.** (a) For each \( \lambda > 0 \)

\[
\mu_X(\lambda) = \max \{ \mu_X^*(\lambda), \lambda \mu_X^*(1/\lambda) \} \text{ and } \mu_X(\lambda) = \lambda \mu_X(1/\lambda).
\]

(b) The rectangular modulus (*)-rectangular modulus is an increasing and convex function on \((0, \infty).\)

(c) We have

\[
\mu_X(\lambda) \leq \max \{ \lambda + 2, 1 + 2\lambda \}, \ \forall \lambda > 0.
\]
Proof: (a) The first part of (a) easily follows from the definitions of $\mu_X$ and $\mu_X^*$. The second part of (a) follows from the first part.

(b) The modulus $\mu_X^*$ can be rewritten as

$$
\mu_X^*(\lambda) = \sup \left\{ \frac{\lambda + t/\lambda}{\|u + (t/\lambda)v\|} : t > 0, u, v \in S_X, u \perp v \right\}
$$

$$
= \sup \left\{ \frac{\lambda + t'}{\|u + t'v\|} : t > 0, u, v \in S_X, u \perp v \right\}, \lambda > 0.
$$

Consequently, $\mu_X^*$ and, by analogy, $\mu_X$ are increasing and convex functions as suprema of families of increasing and convex functions of variable $\lambda$.

(c) For $t \leq 2$, by $u \perp v$ we have:

$$
\frac{\lambda + t}{\|u + tv\|} \leq \frac{\lambda + 2}{\|u\|} = \lambda + 2, \forall \lambda > 0.
$$

For $t > 2$, by the triangle inequality one obtains

$$
\frac{\lambda + t}{\|u + tv\|} \leq \frac{\lambda + t}{t - 1} < \lambda + 2, \forall \lambda > 0.
$$

It follows that $\mu_X^*(\lambda) \leq \lambda + 2, \forall \lambda > 0$,

$$
\lambda \mu_X^*(1/\lambda) \leq \lambda(1/\lambda + 2) = 1 + 2\lambda, \forall \lambda > 0,
$$

and

$$
\mu_X(\lambda) \leq \max\{\lambda + 2, 1 + 2\lambda\}.
$$

In particular, the rectangular constant $\mu(X)$ satisfies the inequality:

$\mu(X) = \mu_X(1) \leq 3$ ([12]).

Remark. The inequality (4) is sharp. Indeed, let $X$ be the two-dimensional $l^1$-space and let $u_1 = (1, 0)$ and $v_1 = (-1/2, 1/2)$ be in $S_X$. We have

$$
\|u_1 + tv_1\| = \|1 - \frac{t}{2} + \frac{t}{2}\| \geq 1 = \|u_1\|, \forall t \in \mathbb{R},
$$

implying $u_1 \perp v_1$. It follows that

$$
\mu_X^*(\lambda) = \sup \left\{ \frac{\lambda + t}{\|u + tv\|} : t > 0, u, v \in S_X, u \perp v \right\}
$$

$$
\geq \frac{\lambda + 2}{\|u_1 + 2v_1\|} = \frac{\lambda + 2}{|1| + 1} = \lambda + 2, \forall \lambda > 0.
$$

Then $\mu_X^*(\lambda) = \lambda + 2, \forall \lambda > 0$, and consequently $\mu_X(\lambda) = \max\{\lambda + 2, 1 + 2\lambda\}, \forall \lambda > 0.$
Now, by Theorem 4 (b), (c) and Theorem 3 (b) it follows that there exists
\[ \mu_X(0^+) := \lim_{\lambda \downarrow 0} \mu_X(\lambda) \in [1, 2]. \]

The extension (by continuity) of \( \mu_X \) in origin (denoted by \( \overline{\mu}_X \)) remains an increasing and convex function on \([0, \infty)\). The function
\[ \lambda \rightarrow \overline{\mu}_X(\lambda) - \mu_X(0^+), \quad \forall \lambda \geq 0 \]
is convex, zero in origin and, consequently, the function
\[ \lambda \rightarrow \frac{\mu_X(\lambda) - \mu_X(0^+)}{\lambda}, \quad \lambda > 0, \]
is increasing on \((0, \infty)\).

By Theorem 4 (b), \( \mu_X \) is locally Lipschitz on \((0, \infty)\). Moreover it is Lipschitz continuous as it will be shown by the following theorem:

**Theorem 5.** *The rectangular modulus verifies the inequality*
\[ \mu_X(\lambda_2) - \mu_X(\lambda_1) \leq \mu_X(0^+)(\lambda_2 - \lambda_1) \leq 2(\lambda_2 - \lambda_1), \]
*for all \( \lambda_1, \lambda_2 > 0, \lambda_1 \leq \lambda_2, \) and the absolute constant 2 is the best possible.*

**Proof:** We have
\[ \mu_X(\lambda) - \mu_X(0^+) = \lambda \mu_X\left(\frac{1}{\lambda}\right) - \mu_X(0^+) \]
\[ = \frac{\mu_X(1/\lambda) - \mu_X(0^+)}{1/\lambda} + \mu_X(0^+)(\lambda - 1), \]
and
\[ \mu_X(\lambda_2) - \mu_X(\lambda_1) = \mu_X(\lambda_2) - \mu_X(0^+) - (\mu_X(\lambda_1) - \mu_X(0^+)) \]
\[ = \frac{\mu_X(1/\lambda_2) - \mu_X(0^+)}{1/\lambda_2} - \frac{\mu_X(1/\lambda_1) - \mu_X(0^+)}{1/\lambda_1} + \mu_X(0^+)(\lambda_2 - \lambda_1) \]
\[ \leq \mu_X(0^+)(\lambda_2 - \lambda_1) \leq 2(\lambda_2 - \lambda_1). \]

The constant 2 is attained for instance when \( X \) is the two-dimensional \( l^1 \)-space. \( \square \)

In the following, we are interested to know the properties of the constant \( \mu_X(0^+) \in [1, 2] \). At the beginning let us recall some notions:

The *radial projection constant* (\( [20] \)) of the space \( X \) is the best Lipschitz constant \( k(X) \) for the radial projection \( r : X \rightarrow B_X \) defined by
\[ r(x) = \begin{cases} 
  x, & \text{for } \|x\| \leq 1 \\
  x/\|x\|, & \text{for } \|x\| > 1.
\end{cases} \]
One of the representations of $k(X)$ is given in [4, p. 1075] by:

$$k(X) = \sup \left\{ \frac{1}{\|tu + v\|} : t \in \mathbb{R}, v \in S_X, u \perp v \right\}.$$ 

The radial projection constant is equal to other four constants of $X$, denoted by $MPB(X)$, $MPB^\prime(X)$, $\beta(X)$ respectively. For more information on this subject see [4], [5] and [8]–[10].

Recall that by Theorem 3, for a fixed $\lambda > 0$ and for a normed space $X$, with $\dim(X) \geq 2$ we have

$$\mu_X(\lambda) = \sqrt{1 + \lambda^2} \iff X \text{ is an i.p.s.}$$

In the limit case when $\lambda \searrow 0$ we are interested to see the relevance of the equality $\mu_X(0+) = 1$ to the geometry of $X$.

**Theorem 6.** (a) For any normed space $X$ we have:

$$\mu_X(0+) = k(X).$$

(b) The equality $\mu_X(0+) = 1$ is equivalent to the symmetry of Birkhoff orthogonality.

**Proof:** (a) A continuity argument and the equivalence $x \perp y \iff -x \perp y$ show that

$$\mu_X^*(0+) = \sup \left\{ \frac{t}{\|u + tv\|} : t > 0, u, v \in S_X, u \perp v \right\} = \sup \left\{ \frac{1}{\|t'u + v\|} : t' \in \mathbb{R}, v \in S_X, u \perp v \right\} = k(X).$$

But from $\lambda \mu_X^*(1/\lambda) \leq 1 + 2\lambda$, $\forall \lambda > 0$ it follows that:

$$\mu_X^*(0+) \leq \mu_X(0+) = \max \left\{ \mu_X^*(0+), \lim_{\lambda \searrow 0} \lambda \mu_X^*(1/\lambda) \right\} \leq \max\{\mu_X^*(0+), 1\} = \mu_X^*(0+) = k(X).$$

(b) The equality $\mu_X(0+) = 1$ is equivalent to $BMP(X) = 1$, which in its turn is equivalent to the symmetry of Birkhoff orthogonality ([19]).

**Remarks.** If $\dim(X) \geq 3$ then $\mu_X(0+) = 1$ implies that $X$ is an i.p.s. On the other hand, by a result of M.A. Smith [19], $1 \leq MPB(X) < 2$, $\iff X$ is uniformly non-square. It follows that $X$ is uniformly non-square $\iff 1 \leq \mu_X(0+) < 2$, and we expect that the rectangular modulus characterizes new geometric properties of $X$. Such geometric considerations will be given elsewhere.

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References


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