On locally $r$-incomparable families of infinite-dimensional Cantor manifolds

VITALIJ A. CHATYRKO

Abstract. The notion of locally $r$-incomparable families of compacta was introduced by K. Borsuk [KB]. In this paper we shall construct uncountable locally $r$-incomparable families of different types of finite-dimensional Cantor manifolds.

Keywords: Cantor manifolds, countable-dimensional, weakly infinite-dimensional, strongly infinite-dimensional

Classification: 54F45

0. Introduction

Throughout this note we shall consider only separable metrizable spaces. By dimension we mean the covering dimension $\text{dim}$.

A subset $L$ of a space $X$ is a partition in $X$ if there exist two non-empty open in $X$ subsets $U$ and $V$ such that $L = X \setminus (U \cup V)$. We say in this case that $X$ is separated by $L$.

An infinite-dimensional Cantor manifold is an infinite-dimensional compact space which cannot be separated by any finite-dimensional subspace.

There exist different types of infinite-dimensional Cantor manifolds. In particular, there exist countable-dimensional Cantor manifolds [Ch1], [O], weakly infinite-dimensional Cantor manifolds which cannot be separated by any countable-dimensional subspace (as recently showed by E. Pol [EP]) and even strongly infinite-dimensional Cantor manifolds which cannot be separated by any weakly infinite-dimensional subspace.

The last type of infinite-dimensional Cantor manifolds can be obtained as follows. It is well known that every strongly infinite-dimensional compact space contains an hereditarily strongly infinite-dimensional closed subset (see for example [R-S-W]). Every hereditarily infinite-dimensional compact space contains an infinite-dimensional Cantor manifold ([T]). Thus every strongly infinite-dimensional compact space contains hereditarily strongly infinite-dimensional Cantor manifold. Note that every hereditarily strongly infinite-dimensional Cantor manifold cannot be separated by any weakly infinite-dimensional subspace.

We shall call two compact spaces $A, B$ injectively different if $A$ does not embed into $B$ and vice versa. A family $\mathcal{A}$ of compacta is injectively different if every two different elements $A, B \in \mathcal{A}$ are injectively different.

E. Pol proved the following
Theorem 0.1 ([EP]). There exists an injectively different family \( A (|A| = 2^{\aleph_0}) \) of hereditarily infinite-dimensional Cantor manifolds.

Remark 0.1. The proof of Theorem 0.1 is based on the existence of hereditarily infinite-dimensional compact spaces. The existence of weakly infinite-dimensional hereditarily infinite-dimensional compact spaces is an open question ([RP1]). If we use in the proof of Theorem 0.1 an hereditarily strongly infinite-dimensional compactum (which exists) we shall obtain that the family \( A \) consists of hereditarily strongly infinite-dimensional Cantor manifolds.

Two compact spaces \( A, B \) are locally \( r \)-incomparable if any non-empty open subset of \( A \) does not embed into \( B \) and vice versa. A family \( A \) of compacta is locally \( r \)-incomparable if every two different elements \( A, B \in A \) are locally \( r \)-incomparable.

This notion was introduced by K. Borsuk. It is well known that for every \( n = 1, 2, \ldots \) there exists an uncountable locally \( r \)-incomparable family of \( n \)-dimensional AR-compacta (see for example [KB]). Recently this fact was used in order to define a fractional dimension function satisfying Menger’s axioms in the class of finite-dimensional locally compact spaces ([T-H]).

It is clear that every locally \( r \)-incomparable family of compacta is injectively different.

In this paper we shall construct uncountable locally \( r \)-incomparable families of named above types of infinite-dimensional Cantor manifolds.

1. Terminology and notation

The necessary information about notions and notations we use can be found in [A-P] and [E].

A space \( X \) is countable-dimensional (shortly c.d.) if \( X \) can be represented as a countable union of 0-dimensional subspaces.

A Cantor \( trInd \)-manifold of class \( \alpha, \alpha < \omega_1 \), is a compact space which cannot be separated by any partition \( L \) with \( trIndL < \alpha \).

It is known that for every \( \alpha < \omega_1 \) there exists a c.d. Cantor \( trInd \)-manifold of class \( \alpha \) ([Ch1], see also part 2).

A space \( X \) is A-weakly infinite-dimensional (shortly A-w.i.d.) if for each infinite sequence \( (A_1, B_1), (A_2, B_2), \ldots \) of pairs of disjoint closed subsets of \( X \) there exist partitions \( L_i \) between \( A_i \) and \( B_i \) in \( X \) such that \( \bigcap_{i=1}^{\infty} L_i = \emptyset \).

A space \( X \) is hereditarily A-w.i.d. if every subspace of \( X \) is A-w.i.d.

A space \( X \) is A-strongly infinite-dimensional (shortly A-s.i.d.) if it is not A-w.i.d.

Remind that each c.d. space is A-w.i.d. Moreover, a space which is the union of countably many c.d. (A-w.i.d.) subspaces is c.d. (A-w.i.d.).

If a space \( X \) is compact then one say that \( X \) is weakly infinite-dimensional (shortly w.i.d.) or strongly infinite-dimensional (shortly s.i.d.) respectively.
It is known that there exists a w.i.d. compact space $P$ which cannot be separated by any hereditarily A-w.i.d. subspace ([EP]). Note that $P$ cannot be separated by any countable-dimensional subspace. In particular $P$ is not c.d. Remind that the first example of a w.i.d. compactum which is not c.d. was given by R. Pol [RP2].

A compact space $X$ is hereditarily infinite-dimensional, shortly h.i.d. (hereditarily strongly infinite-dimensional, shortly h.s.i.d.), if each nonempty closed subset of $X$ is either 0-dimensional or infinite-dimensional (strongly infinite-dimensional).

The first example of h.i.d. compactum was given by D. Henderson [H1].

In [H2] D. Henderson has constructed a c.d. AR-compactum $H^\alpha$ with $\text{trInd}H^\alpha = \alpha$ for every $\alpha < \omega_1$. Remind this construction.

Let $H^I = I = [0, 1], p_1 = \{0\}$. Assume that for every $\beta < \alpha$ the compacta $H^\beta$ and the points $p_\beta \in H^\beta$ have already been defined. If $\alpha = \beta + 1$, then we set $H^{\beta+1} = H^\beta \times I$ and $p_{\beta+1} = (p_\beta, 0)$. If $\alpha$ is a limit ordinal, then $K_\beta$ is the union of the $H^\beta$ and a half-open arc $A_\beta$ such that $A_\beta \cap H^\beta = \{p_\beta\} = \{\text{endpoint of the arc } A_\beta\}, \beta < \alpha$. Let us define $H^\alpha$ as the one-point compactification of the free sum $\bigoplus_{\beta<\alpha} K_\beta$ and let $p_\alpha$ be the compactification point.

It is well known that every ordinal $\alpha$ may be represented in the form $\alpha = p(\alpha) + n(\alpha)$, where $p(\alpha)$ is a limit ordinal and $n(\alpha) < \omega$.

Note that the compactum $H^\alpha$, where $n(\alpha) \geq 1, \alpha < \omega_1$, cannot be separated by a point.

A dimension function $d$ is monotone if for any space $X$ and any subset $A \subset X$ closed in $X$, $dA \leq dX$.

2. Variation of Fedorchuk’s construction

Let $R$ be the real line, $Q \subset R$ be the rational numbers, $Irr \subset R$ be the irrational numbers and $I = [0, 1]$. The notation $Z \simeq Y$ will mean that spaces $Z$ and $Y$ are homeomorphic.

We shall follow [Ch2] as a variation of [F1], [F2]. Remind some definitions.

A continuous mapping $f : X \to Y$ is called fully closed if for any point $y \in Y$ and any finite covering $\{U_i : i = 1, 2, \ldots, s\}$ of $f^{-1}y$ by sets open in $X$, the set $\{y\} \cup (\bigcup_{i=1}^s f\#U_i)$ is open in $Y$. Here $f\#U = Y \setminus f(X \setminus U)$.

A continuous mapping $f : X \to Y$ is called ring-like if for any point $x \in X$ and arbitrary neighbourhoods $Ox$ and $Of_x$, the set $f\#Ox$ contains a partition between the point $fx$ and the set $Y \setminus Of_x$ in the space $Y$.

A continuous mapping $f : X \to Y$ is called monotone if for any point $y \in Y$ the set $f^{-1}y$ is connected.

A continuous mapping $f : X \to Y$ is called irreducible if for any non-empty open subset $O \subset X$ we have $f\#O \neq \emptyset$.

Consider a continuum $Y$ with a countable everywhere dense subset $P = \{a_1, a_2, a_3, \ldots\} \subset Y$ and fix an embedding $Y \subset I^\omega$. Define a mapping $f : (0, 1] \to I^\omega$ as follows. Namely
\[ f \left[ 1/(i+1), 1/i \right] : [1/(i+1), 1/i] \to I^\infty \text{ is a path between} \]
the points \(a_{i+1}\) and \(a_i\) in \(1/i\)-neighborhood of \(Y, i = 1, 2, \ldots\).

The mapping \(f\) satisfies the following conditions:

(a) for every open neighborhood \(O\) of the continuum \(Y\) in \(I^\infty\) there exists a natural number \(n\) such that \(f(0, 1/n] \subset O\);

(b) for every non-empty open subset \(U \subset Y\) and every natural number \(n\) there exists a number \(m \geq n\) such that \(f(1/m) \in U\).

2.a Particular case
Define a mapping \(g : [-1, 1] \setminus \{0\} \to I^\infty\) by \(g(x) = f(|x|)\) and mappings
\[ g_t : [-1 + t, 1 + t] \setminus \{t\} \to I^\infty \text{ by } g_t(x) = g(x - t), \text{ where } t \in R. \]
Consider the disjoint union \(B = \bigcup\{Y_t : t \in R\}\), where \(Y_t\) is a point, if \(t \in R \setminus Q\), and \(Y_t \simeq Y\) if \(t \in Q\).
Let \(p_t : Y \to Y_t\) be the homeomorphism above, where \(t \in Q\).
Define the mapping \(\pi : B \to R\) as follows, \(\pi(y) = t\), if \(y \in Y_t\).
Let \(\{V_n\}_{n=1}^\infty\) be a base in \(R\), and \(\{U_k\}_{k=1}^\infty\) be a base in \(I^\infty\).
The topology \(\tau\) on the set \(B\) we define as follows.
We take all sets \(\pi^{-1}V_n, n = 1, 2, \ldots\), and \(O(U_k, t, V_n) = p_t(U_k \cap Y) \cup \pi^{-1}(g_t^{-1}U_k \cap V_n)\), where \(t \in Q \cap V_n\) and \(m, n = 1, 2, \ldots\), as the basis sets of the topology on \(B\).

Note that in the case the mapping \(\pi\) is fully closed, ring-like, irreducible and monotone.

Denote the subspace \(\pi^{-1}[0, 1]\) of \(B\) via \(F(Y)\).
Some properties of \(F(Y)\).

(a) \(FY\) is a continuum which is the disjoint union of continua \(Y_t, t \in [0, 1]\).

(b) \(F(Y) \setminus \bigcup\{Y_t : t \in Q\} \simeq Irr \cap I\).

(c) every non-empty open subset of \(F(Y)\) contains a copy of \(Y\).

(d) every subcontinuum of \(F(Y)\) either embeds in \(Y\) or is equal to \(\pi^{-1}[a, b]\), where \(0 \leq a < b \leq 1\).

(e) \(F(Y)\) is c.d. (w.i.d., h.s.i.d.) if \(Y\) is c.d. (w.i.d., h.s.i.d.).

Example of c.d. Cantor \(trInd\)-manifold of class \((\alpha + 1), \alpha < \omega_1\).
Consider the path-connected compactum \(Z = F(H^\alpha) \times I / F(H^\alpha) \times \{0\}\).
Denote the compactum \(Z^2\) via \(A(H^\alpha)\). It is clear that \(A(H^\alpha)\) is c.d. and every non-empty open subset of \(Z\) contains \(H^{\alpha+1}\). One can prove (see [Ch1]) that for every partition \(L\) in \(A(H^\alpha)\) we have \(trIndL \geq \alpha + 1\). Hence the continuum \(A(H^\alpha)\) is a Cantor \(trInd\)-manifold of class \((\alpha + 1)\).
2.6 General case

Consider a continuum \( X \) and a countable subset \( L \) of \( X \). Fix a point \( x \in L \) and a sequence \( \{L_i^x\}_{i=1}^\infty \) of partitions in \( X \) such that

(a) \( L_i^x = X \setminus (U_i^x \cup V_i^x) \), where \( U_i^x \), \( V_i^x \) are disjoint non-empty open subsets of the continuum \( X \) and \( x \in U_i^x \) for every \( i \);

(b) \( U_i^x \cup L_i^x \subset U_{i-1}^x \), \( i = 2, 3, \ldots \);

(c) \( \{U_i^x\}_{i=1}^\infty \) is a base in the point \( x \).

Note that all partitions \( L_i^x \), \( i = 1, 2, \ldots \) are non-empty.

Define a mapping \( h_x : V_i^x \cup \bigcup_{i=1}^\infty L_i^x \to (0, 1] \) as follows

(a) \( h_x(X \setminus U_i^x) = 1 \);

(b) \( h_x(L_i^x) = 1/i, i = 2, 3, \ldots \).

By \( q_x : X \setminus \{x\} \to (0, 1] \) we denote an extension of \( h_x \) on \( X \setminus \{x\} \) such that
\[ q_x((U_i^x \cup L_i^x) \setminus U_{i+1}^x) \subset [1/(i+1), 1/i], \quad i = 1, 2, \ldots \, . \]

Put \( g_x = f \circ q_x \). The mapping \( g_x \) satisfies the following conditions:

(a) for every open neighborhood \( O \) of the continuum \( Y \) in \( I^\infty \) there exists a natural number \( n \) such that \( g_x U_n^x \subset O \);

(b) for every non-empty open subset \( U \subset Y \) and every natural number \( n \) there exists a number \( m \geq n \) such that \( g_x(L_m^x) \subset U \).

Consider the disjoint union \( B(X, Y, L) = \bigcup\{Y_x : x \in X\} \), where \( Y_x \) is a point if \( x \in X \setminus L \) and \( Y_x \simeq Y \) if \( x \in L \).

Let \( p_x : Y \to Y_x \) be the homeomorphism above, where \( x \in L \).

Define the mapping \( \pi : B(X, Y, L) \to X \) by \( \pi(y) = x \) if \( y \in Y_x \).

Let \( \{V_n\}_{n=1}^\infty \) be a base in \( X \), and \( \{U_k\}_{k=1}^\infty \) be a base in \( I^\infty \).

We define the topology \( \tau \) on the set \( B(X, Y, L) \) as follows.
We take all sets \( \pi^{-1}V_n, n = 1, 2, \ldots \), and \( O(U_k, x, V_n) = p_x(U_k \cap Y) \cup \pi^{-1}(g_x^{-1}U_k \cap V_n) \), where \( x \in L \cap V_n \) and \( m, n = 1, 2, \ldots \), as the basis sets of the topology on \( B(X, Y, L) \).

Note that in this case the mapping \( \pi \) is fully closed, ring-like, irreducible and monotone.

Note some properties of \( B(X, Y, L) \).

**Proposition 2.1.** (a) \( B(X, Y, L) \) is a continuum which is the disjoint union of continua \( Y_x, x \in X \).

(b) \( B(X, Y, L) \setminus \bigcup\{Y_x : x \in L\} \simeq X \setminus L \).

(c) Every non-empty open subset of \( B(X, Y, L) \) contains a copy of \( Y \) if \( L \) is an everywhere dense subset of \( X \).
(d) Every subcontinuum $C$ of $B(X,Y,L)$ either embeds in $Y$ or is equal to $\pi^{-1}\pi C = B(\pi C, Y, L \cap \pi C)$. Moreover in the last case either $C$ lies in $X \setminus L$ if $L \cap \pi C = \emptyset$ or $C$ contains a copy of $Y$ if $L \cap \pi C \neq \emptyset$.

(e) $B(X,Y,L)$ is c.d. (w.i.d., h.s.i.d.) if $X,Y$ are c.d. (w.i.d., h.s.i.d.).

(f) Let $C$ be a partition in $B(X,Y,L)$. Then there exists a partition $C_1$ in $X$ such that for each subspace $Z$ of $C_1$ the subspace $Z \setminus L$ embeds into $C$. In particular, if $X$ is an infinite-dimensional Cantor manifold then $B(X,Y,L)$ is the same.

**Proof:** (a)–(d) follow from the construction and the properties of $\pi$.

(e) We shall prove only that the continuum $B(X,Y,L)$ is w.i.d. if the continua $X,Y$ are w.i.d. Consider a countable family $\{(A^i_j,B^j_i) : i = 0,1,\ldots; j = 1,2,\ldots\}$ of pairs of disjoint closed subsets of $B(X,Y,L)$. Let $L = \{l_1,l_2,\ldots\}$. For every $i = 1,2,\ldots$ there exist partitions $L^i_j$ between $A^i_j$ and $B^i_j$ in $B(X,Y,L)$ such that $\bigcap_{j=1}^\infty L^i_j \cap Y_{i_j} = \emptyset$. Denote the compactum $\bigcap_{i=1}^\infty (\bigcap_{j=1}^\infty L^i_j)$ via $A$. Note that $A \subset B(X,Y,L) \cup \{Y_x : t \in L\}$ $\simeq X \setminus L$ and hence $A$ is w.i.d. There exist partitions $L^0_j$ between $A^0_j$ and $B^0_j$ in $B(X,Y,L)$ such that $\bigcap_{j=1}^\infty L^0_j \cap A = \bigcap_{i=1}^\infty (\bigcap_{j=1}^\infty L^i_j) = \emptyset$. Hence the compactum $B(X,Y,L)$ is w.i.d.

(f) Let $C = B(X,Y,L) \setminus (U \cup V)$ where $U,V$ are disjoint non-empty open subsets of $B(X,Y,L)$. Note that the subsets $\pi U, \pi V$ of $X$ are disjoint non-empty open and the subset $C_1 = X \setminus (\pi U \cup \pi V)$ is a partition in $X$. It is clear that for each subspace $Z$ of $C_1$ the subspace $Z \setminus L$ embeds into $C$. Suppose that $X$ is an infinite-dimensional Cantor manifold and the partition $C$ is finite-dimensional. Therefore the subspace $C_1 \setminus L$ is finite-dimensional and hence the partition $C_1$ is finite-dimensional too. It is a contradiction.

**Proposition 2.2.** Let $L$ be an everywhere dense subset of $X$ and $Y_1,Y_2$ be injectively different continua, which do not embed into $X$. Then continua $B(X,Y_1,L)$, $B(X,Y_2,L)$ are locally $r$-incomparable.

**Proof:** Let $U$ be an open non-empty subset of $B(X,Y_1,L)$. Suppose that $g : U \to B(X,Y_2,L)$ is an embedding. By Proposition 2.1 (c) $U$ contains a copy of $Y_1$. By Proposition 2.1 (d) the image $g(Y_1)$ of the copy of $Y_1$ either embeds into $Y_2$ (it is a contradiction) or is equal to $\pi^{-1}\pi g(Y_1)$. In the last case $g(Y_1)$ either lies in $X \setminus L \subset X$ or contains a copy of $Y_2$. It is a contradiction too.

### 3. On E. Pol’s proposition

The following statement in fact was proved in [EP].

**Proposition 3.1.** Let $A,B$ be two c.d. continua which cannot be separated by a point and which are injectively different. Then there exists an injectively different family $\{L_a : a \in \{0,1\}^\infty\}$ of c.d continua such that for every $a \in \{0,1\}^\infty$, $L_a$ contains copies of $A$ and $B$.

We repeat here the description from [EP].
Choose two pairs of different points $a_1, a_2 \in A$ and $b_1, b_2 \in B$. Let $X_1, X_2, \ldots$ be a sequence of spaces such that $X_i$ is a copy of $A$ or $B$ and $x_i^j = a_j$ if $X_i = A$ and $x_i^j = b_j$ if $X_i = B$, for $j = 1, 2$. Consider the equivalence relation $E$ on the free sum $X = \bigoplus_{i=1}^{\infty} X_i$ such that $xEy$ iff $x = y$ or $x = x_i^1$ and $y = x_i^{i+1}$ for some $i \in \mathbb{N}$.

Let $Y = X/E$ be the quotient space and $Z = Z(X_1, X_2, \ldots)$ be the one-point compactification of $Y$. Then $Z$ is a c.d. continuum. Let $K$ be the class of all spaces $Z(X_1, X_2, \ldots)$ obtained in this way.

It was shown in [EP] that $K$ contains an injectively different uncountable family $\{L_a : a \in \{0, 1\}^\infty\}$. Namely, for $a = \{a_k\}_{k=1}^{\infty} \in \{0, 1\}^\infty$, $L_a = Z(X_1^a, X_2^a, \ldots)$, where $X_1^a = A, X_2^a = B$ and for $k = 1, 2, \ldots$:

- if $a_k = 0$ then $X_{5k-3+l}^a$ is $A$, for $l = 1, 2$; and it is $B$, for $l = 3, 4, 5$;
- if $a_k = 1$ then $X_{5k-3+l}^a$ is $A$, for $l = 1, 2, 3$; and it is $B$, for $l = 4, 5$;

4. Two c.d. injectively different infinite-dimensional continua which cannot be separated by a point

Let $\gamma$ be an infinite ordinal with $n(\gamma) \geq 1$. Remind that the compactum $A(H^\gamma)$ is a c.d. Cantor trInd-manifold of class $(\gamma + 1)$. Put $\beta = trIndA(H^\gamma) + 1 < \omega_1$. Note that $n(\beta) \geq 1$. Continua $A(H^\gamma)$ and $H^\beta$ cannot be separated by a point. Since $trIndH^\beta = \beta > trIndA(H^\gamma)$, $H^\beta$ does not embed into $A(H^\gamma)$.

We shall prove that $A(H^\gamma)$ does not embed into $H^\beta$. Remind that $H^\beta$ is the union of countably many finite-dimensional compacta. Assume that $A(H^\gamma)$ embeds into $H^\beta$. Hence $A(H^\gamma)$ is the union of countably many finite-dimensional compacta at least one of which contains a non-empty open subset of $A(H^\gamma)$. But every non-empty open subset of $A(H^\gamma)$ contains a copy of $H^\gamma$ with $trIndH^\gamma = \gamma \geq \omega$. It is a contradiction. Hence $A(H^\gamma)$ does not embed into $H^\beta$.

Note that both compacta $A(H^\gamma)$ and $H^\beta$ contain $H^\gamma$. Now with help of Proposition 3.1 the following statement is evident.

**Proposition 4.1.** For every ordinal $\gamma < \omega_1$ there exists an injectively different family $\{L_a : a \in \{0, 1\}^\infty\}$ of c.d continua such that for every $a \in \{0, 1\}^\infty$, $L_a$ contains a copy of $H^\gamma$.

5. Main results

Here we shall construct uncountable locally $r$-incomparable families of named in the introduction types of infinite-dimensional Cantor manifolds.

First we need the following evident (see the separation theorem for dimension 0 ([E, p.11])

**Lemma 5.1.** Let $A$ be a 0-dimensional subset of a compactum $Z$. Assume that $trIndQ < \alpha$ for every compactum $Q \subset Z \setminus A$. Then $trIndZ \leq \alpha$. 

In particular, if \( \text{trInd} Z \geq \beta + 1 \), then there exists a compactum \( Q \subset Z \setminus A \) such that \( \text{trInd} Q = \beta \).

**Theorem 5.1.** For every \( \alpha < \omega_1 \) there exists a locally \( r \)-incomparable family \( A (|A| = 2^{\aleph_0}) \) of c.d. Cantor \( \text{trInd} \)-manifolds of class \( \alpha \).

**Proof:** Fix an ordinal \( \alpha < \omega_1 \). Denote \( A(H^\alpha) \) via \( X \). Note that \( X \) is a c.d. Cantor \( \text{trInd} \)-manifold of class \( (\alpha + 1) \). Let \( \gamma = \text{trInd} X + 1 \). By Proposition 4.1 there exists an injectively different family \( \{ L_a : a \in \{0,1\}^\infty \} \) of c.d continua such that for every \( a \in \{0,1\}^\infty \), \( L_a \) contains a copy of \( H^\gamma \). Remind that \( \text{trInd} H^\gamma = \gamma \) ([H2]) and the dimension \( \text{trInd} \) is monotone. Hence for every \( a \in \{0,1\}^\infty \), \( L_a \) does not embed into \( X \).

Let \( L \) be an everywhere dense countable subset of \( X \).

By Propositions 2.1(e), (f), 2.2 and Lemma 5.1 the family \( \{ B(X, L_a, L) : a \in \{0,1\}^\infty \} \) is locally \( r \)-incomparable and it consists of c.d. Cantor \( \text{trInd} \)-manifolds of class \( \alpha \). \( \square \)

Now we need the following evident

**Lemma 5.2.** Let \( X \) be a \( A \)-s.i.d. space and \( Y \) be a 0-dimensional subspace of \( X \). Then the subspace \( X \setminus Y \) is \( A \)-s.i.d.

**Theorem 5.2.** There exists a locally \( r \)-incomparable family \( A (|A| = 2^{\aleph_0}) \) of w.i.d. Cantor manifolds which cannot be separated by any hereditarily \( A \)-w.i.d. subspace.

**Proof:** Denote the w.i.d. compactum \( P \) from part 1 via \( X \). Let \( \text{dim}_w X = \alpha < \omega_1 \), where \( \text{dim}_w \) is Borst’s transfinite extension of the covering dimension \( \text{dim} \) ([PB]). Put \( \gamma = \alpha + 1 \). By Proposition 4.1 there exists an injectively different family \( \{ L_a : a \in \{0,1\}^\infty \} \) of c.d continua such that for every \( a \in \{0,1\}^\infty \), \( L_a \) contains a copy of \( H^\gamma \). Remind that \( \text{dim}_w H^\gamma = \gamma \) ([PB]) and the dimension \( \text{dim}_w \) is monotone. Hence for every \( a \in \{0,1\}^\infty \), \( L_a \) does not embed into \( X \). Let \( L \) be an everywhere dense countable subset of \( X \). By Propositions 2.1(e), (f), 2.2 and Lemma 5.2, the family \( \{ B(X, L_a, L) : a \in \{0,1\}^\infty \} \) is locally \( r \)-incomparable and it consists of w.i.d. Cantor manifolds which cannot be separated by any hereditarily \( A \)-w.i.d. subspace. \( \square \)

**Theorem 5.3.** There exists a locally \( r \)-incomparable family \( A (|A| = 2^{\aleph_0}) \) of h.s.i.d. Cantor manifolds.

**Proof:** By Theorem 0.1 (see also Remark 0.1) there exists an injectively different family \( \{ L_a : a \in \{0,1\}^\infty \} \) of h.s.i.d. Cantor manifolds. Put \( X = L_{(0,0,...)} \) and \( M_{(b_1,b_2,...)} = L_{(1,b_1,b_2,...)} \) for every \( (b_1,b_2,...) \in \{0,1\}^\infty \). Note that for every \( b \in \{0,1\}^\infty \), \( M_b \) does not embed into \( X \). Let \( L \) be an everywhere dense countable subset of \( X \). By Propositions 2.1(e), (f) and 2.2 the family \( \{ B(X, M_b, L) : b \in \{0,1\}^\infty \} \) is locally \( r \)-incomparable and it consists of h.s.i.d. Cantor manifolds. \( \square \)
On locally $r$-incomparable families of infinite-dimensional Cantor manifolds

References


Department of Mathematics, Linköping University, 581 83 Linköping, Sweden
E-mail: vitja@mai.liu.se

(Received September 16, 1997)