On the cardinality of Hausdorff spaces

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Abstract. The aim of this paper is to show, using the reflection principle, three new cardinal inequalities. These results improve some well-known bounds on the cardinality of Hausdorff spaces.

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Two of the most known inequalities in the theory of cardinal functions are the Hajnal-Juhász’s inequality [7]: “For $X \in T_2$, $|X| \leq 2^c(X)\chi(X)$” and the Arhangel’skii’s inequality [5]: “For $X \in T_2$, $|X| \leq 2^L(X)t(X)\psi(X)$”.

In this paper we will use the language of elementary submodels (see [4], [10], [1] and [2]) to establish three new cardinal inequalities which generalize the results mentioned above. We refer the reader to [3], [5], [7] for notations and terminology not explicitly given. $\chi$, $c$, $\psi$, $t$, $L$ and $\pi_\chi$ denote character, cellularity, pseudocharacter, tightness, Lindelöf degree and $\pi$-character respectively.

Definitions. (i) Let $X$ be a Hausdorff space.

The closed pseudocharacter of $X$, denoted $\psi_c(X)$, is the smallest infinite cardinal $\kappa$ such that for every $x \in X$ there is a collection $U_x$ of open neighbourhoods of $x$ such that $\bigcap \{U : U \in U_x\} = \{x\}$ and $|U_x| \leq \kappa$ ([7]).

The Hausdorff pseudocharacter of $X$, denoted $H\psi(X)$, is the smallest infinite cardinal $\kappa$ such that for every $x \in X$ there is a collection $U_x$ of open neighbourhoods of $x$ with $|U_x| \leq \kappa$ such that if $x \neq y$, there exist $U \in U_x$, $V \in U_y$ with $U \cap V = \emptyset$ ([6]).

Clearly $\psi_c(X) \leq H\psi(X) \leq \chi(X)$ for every Hausdorff space $X$.

(ii) Let $X$ be a topological space, $ac(X)$ is the smallest infinite cardinal $\kappa$ such that there is a subset $S$ of $X$ such that $|S| \leq 2^\kappa$ and for every open collection $U$ in $X$ there is a $V \in [U]^{\leq \kappa}$ with $\bigcup U \subset S \cup \bigcup \{V : V \in V\}$.

Observe that $ac(X) \leq c(X)$ for every space $X$.

Theorem 1. If $X$ is a $T_2$-space then $|X| \leq 2^{ac(X)H\psi(X)}$.

Proof: Let $\lambda = ac(X)H\psi(X)$, $\kappa = 2^\lambda$, let $\tau$ be the topology on $X$ and let $S$ be an element of $[X]^{\leq \kappa}$ witnessing that $ac(X) \leq \lambda$. For every $x \in X$ let $B_x$ be a collection of open neighbourhoods of $x$ with $|B_x| \leq \lambda$ such that if $x \neq y$ then
there exist \( U \in \mathcal{B}_x, V \in \mathcal{B}_y \) such that \( U \cap V = \emptyset \), and let \( f : X \to \mathcal{P}(\tau) \) be the map defined by \( f(x) = \mathcal{B}_x \) for every \( x \in X \).

Let \( A = \kappa \cup \{S, X, \tau, \kappa, f\} \) and take a set \( \mathcal{M} \) such that \( \mathcal{M} \supset A, |\mathcal{M}| = \kappa \) and which reflects enough formulas to carry out our argument. To be more precise we ask that \( \mathcal{M} \) reflects enough formulas so that the following conditions are satisfied:

(i) \( C \in \mathcal{M} \) for every \( C \in [\mathcal{M}]^{\leq \kappa} \);
(ii) \( \mathcal{B}_x \in \mathcal{M} \) for every \( x \in X \cap \mathcal{M} \);
(iii) if \( B \subset X \) and \( B \in \mathcal{M} \) then \( \overline{B} \in \mathcal{M} \);
(iv) if \( A \in \mathcal{M} \) then \( \bigcup A \in \mathcal{M} \);
(v) if \( B \) is a subset of \( X \) such that \( X \cap \mathcal{M} \subset B \) and \( B \in \mathcal{M} \) then \( X = B \);
(vi) if \( E \in \mathcal{M} \) and \( |E| \leq \kappa \) then \( E \subset \mathcal{M} \).

Observe that by (ii) and (vi) \( \mathcal{B}_y \subset \mathcal{M} \) for every \( y \in X \cap \mathcal{M} \).

Claim: \( X \subset \mathcal{M} \) (and hence \( |X| \leq 2^{ac(X)H\psi(X)} \)). Suppose not and take \( p \in X \setminus \mathcal{M} \). Let \( \mathcal{B}_p = \{B_\alpha\}_{\alpha < \lambda} \), clearly \( \bigcap \{\overline{B}_\alpha : \alpha < \lambda\} = \{p\} \). Now for every \( \alpha < \lambda \) let \( \langle X \cap \mathcal{M}\rangle_\alpha = \{y \in X \cap \mathcal{M}: \exists B \in \mathcal{B}_y \text{ for which } B \cap B_\alpha = \emptyset\} \).

For every \( y \in \langle X \cap \mathcal{M}\rangle_\alpha \) choose a \( B_{y, \alpha} \in \mathcal{B}_y \) such that \( B_{y, \alpha} \cap B_\alpha = \emptyset \), clearly \( \mathcal{U}_\alpha = \{B_{y, \alpha} : y \in \langle X \cap \mathcal{M}\rangle_\alpha\} \) covers \( \langle X \cap \mathcal{M}\rangle_\alpha \). Since \( ac(X) \leq \lambda \) it follows that there is a \( \mathcal{V}_\alpha \in [\mathcal{U}_\alpha]^{\leq \lambda} \) such that \( \langle X \cap \mathcal{M}\rangle_\alpha \subset S \cup \{\overline{\mathcal{V} : \mathcal{V} \in \mathcal{V}_\alpha}\} \). Observe that \( p \notin S \cup \{\overline{\mathcal{V} : \mathcal{V} \in \mathcal{V}_\alpha}\} \) (\( S \in \mathcal{M} \) and \( |S| \leq \kappa \) so by (vi) \( S \subset \mathcal{M} \), moreover \( \{\overline{\mathcal{V} : \mathcal{V} \in \mathcal{V}_\alpha}\} \subset X \setminus B_\alpha \)). We have also \( \bigcup \{\overline{\mathcal{V} : \mathcal{V} \in \mathcal{V}_\alpha}\} \subset \mathcal{M} \) \( \forall \mathcal{V} \in \mathcal{M} \) for every \( \mathcal{V} \in \mathcal{V}_\alpha \), so by (iii) \( \overline{\mathcal{V} : \mathcal{V} \in \mathcal{V}_\alpha} \subset \mathcal{M} \) and \( \{\overline{\mathcal{V} : \mathcal{V} \in \mathcal{V}_\alpha}\} \subset \mathcal{M} \) by (i), hence by (iv) \( \bigcup \{\overline{\mathcal{V} : \mathcal{V} \in \mathcal{V}_\alpha}\} \subset \mathcal{M} \), so \( \bigcup \{\overline{\mathcal{V} : \mathcal{V} \in \mathcal{V}_\alpha}\} \subset \mathcal{M} \) by (iii)).

Set \( C_\alpha = S \cup \{\overline{\mathcal{V} : \mathcal{V} \in \mathcal{V}_\alpha}\} \) for every \( \alpha < \lambda \) and observe that \( C_\alpha \in \mathcal{M} \) (recall that \( S, \{\overline{\mathcal{V} : \mathcal{V} \in \mathcal{V}_\alpha}\} \in \mathcal{M} \)). Now \( X \cap \mathcal{M} \subset \{C_\alpha : \alpha < \lambda\} \), since \( \{C_\alpha : \alpha < \lambda\} \in \mathcal{M} \), so \( \{C_\alpha : \alpha < \lambda\} \in \mathcal{M} \), hence by (iv) \( \{C_\alpha : \alpha < \lambda\} \in \mathcal{M} \) it follows by (v) that \( \{C_\alpha : \alpha < \lambda\} = X \). This is a contradiction (\( p \notin \{C_\alpha : \alpha < \lambda\}\)). \( \square \)

**Corollary 2** ([7]). If \( X \) is a \( T_2 \)-space then \( |X| \leq 2^{ac(X)\chi(X)} \).

**Remark 3.** The above result of Hajnal and Juhász has been improved also by Hodel, in fact in [6] it is shown that \( |X| \leq 2^{c(X)H\psi(X)} \) for every Hausdorff space \( X \). It is clear that Theorem 1 generalizes also this result of Hodel.

Now let \( X \) be the Michael line, i.e. let \( X \) be \( \mathbb{R} \) topologized by isolating the points of \( \mathbb{R} \setminus \mathbb{Q} \) and leaving the points of \( \mathbb{Q} \) with their usual neighbourhoods. Then \( X \) is a normal space such that \( |X| = 2^{ac(X)H\psi(X)} < 2^{c(X)H\psi(X)} \).

Observe that in Theorem 1 \( H\psi(X) \) cannot be replaced by \( \psi_c(X) \), in fact for every infinite cardinal \( \kappa \) there is a \( T_3 \)-space \( X \) with \( |X| = \kappa \) and \( \psi(X) = c(X) = ac(X) = \omega \) (see e.g. [5]).

**Definition 4.** Let \( X \) be a topological space, \( lc(X) \) is the smallest infinite cardinal \( \kappa \) such that there is a closed subset \( F \) of \( X \) such that \( |F| \leq 2^\kappa \) and for every open collection \( \mathcal{U} \) in \( X \) there is a \( \mathcal{V} \in [\mathcal{U}]^{\leq \kappa} \) with \( \bigcup \mathcal{U} \subset F \cup \bigcup \{\overline{\mathcal{V} : \mathcal{V} \in \mathcal{V}}\} \).
Clearly \( ac(X) \leq lc(X) \leq c(X) \) for every space \( X \).

**Theorem 5.** If \( X \) is a Hausdorff space then \(|X| \leq 2^{lc(X)\pi_X(X)\psi_c(X)}\).

**Proof:** Let \( \lambda = lc(X)\pi_X(X)\psi_c(X) \) and let \( \kappa = 2^\lambda \) be the topology on \( X \) and let \( \tau \) be the map defined by \( f(x) = B_x \) for every \( x \in X \). Let \( A = \kappa \cup \{F, X, \tau, \kappa, f\} \) and take a set \( M \supset A \) such that \(|M| = \kappa \) and which reflects enough formulas so that the conditions (i)-(vi) listed in Theorem 1 are satisfied.

Claim: \( X \subset M \) (and hence \(|X| \leq 2^{lc(X)\pi_X(X)\psi_c(X)}\)). Suppose not and take \( p \in X \setminus M \). Let \( \{G_\alpha : \alpha \in \lambda\} \) be a family of open neighbourhoods of \( p \) such that \( \bigcap\{G_\alpha : \alpha \in \lambda\} = \{p\} \). Set \( V_\alpha = X \setminus \overline{G_\alpha} \) and \( S_\alpha = X \cap M \cap V_\alpha \) for every \( \alpha \in \lambda \). Now let \( W_\alpha = \{B : B \in B_y, y \in S_\alpha \cap B \subset V_\alpha\} \), since \( lc(X) \leq \lambda \) it follows that there is a \( \nu_\alpha \in |W_\alpha|^{\leq \lambda} \) such that \( \bigcup W_\alpha \subset F \cup \bigcup\{V : V \in V_\alpha\} \).

Since \( S_\alpha \subset \bigcup W_\alpha \) (let \( y \in S_\alpha \) and \( U \) be an open neighbourhood of \( y \), \( y \notin \overline{G_\alpha} \)) so there is an open neighbourhood \( V \) of \( y \) such that \( V \cap G_\alpha = \emptyset \), then \( B \subset U \cap V \), \( \emptyset \neq B \subset (\bigcup W_\alpha) \cap U \) and \( y \in \bigcup W_\alpha \) it follows that \( S_\alpha \subset F \cup \bigcup\{V : V \in V_\alpha\} \); moreover \( \bigcup\{V : V \in V_\alpha\} \in M \) and \( p \notin F \cup \bigcup\{V : V \in V_\alpha\} \).

Set \( C_\alpha = \bigcup\{V : V \in V_\alpha\} \), since \( X \cap M \subset F \cup \bigcup\{C_\alpha : \alpha < \lambda\} \) and \( F \cup \bigcup\{C_\alpha : \alpha < \lambda\} \in M \) it follows that \( F \cup \bigcup\{C_\alpha : \alpha < \lambda\} = X \), a contradiction. \( \square \)

By Theorem 5 it follows again that \(|X| \leq 2^{c(X)\chi(X)}\) for every \( T_2 \)-space \( X \).

Moreover we have the following

**Corollary 6 ([6]).** If \( X \) is a \( T_3 \)-space then \(|X| \leq 2^{c(X)\pi_X(X)\psi_c(X)}\).

**Remark 7.** A generalization of the inequality in corollary 6 has also been obtained by Sun in [8]: "\(|X| \leq 2^{c(X)\pi_X(X)\psi_c(X)}\) for every Hausdorff space \( X \)." Note that even this result is a corollary of Theorem 5. Moreover if \( X \) is the Michael line then \(|X| = 2^{lc(X)\pi_X(X)\psi_c(X)} < 2^{c(X)\pi_X(X)\psi_c(X)}\). Observe also that the \( \pi \)-character cannot be omitted in Theorem 5 (see the comment at the end of Remark 3).

Now let us turn our attention to the Arhangel’skii’s inequality: "For \( X \in T_2 \), \(|X| \leq 2^{L(X)\chi(X)\psi_c(X)}\)."
Theorem 8. If $X$ is a Hausdorff space then $|X| \leq 2^{aqL(X)t(X)\psi_c(X)}$.

Proof: Let $\lambda = aqL(X)t(X)\psi_c(X)$, $\kappa = 2^\lambda$, let $\tau$ be the topology on $X$ and let $S$ be an element of $[X]^{<\kappa}$ witnessing that $aqL(X) \leq \lambda$. For every $x \in X$ let $B_x$ be a family of open neighbourhoods of $x$ with $|B_x| \leq \lambda$ and $\bigcap \{B : B \in B_x\} = \{x\}$, and let $f : X \to \mathcal{P}(\tau)$ be the map defined by $f(x) = B_x$ for every $x \in X$. Let $A = \kappa \cup \{S, X, \tau, \kappa, f\}$ and take a set $\mathcal{M} \supset A$ such that $|\mathcal{M}| = \kappa$ and which reflects enough formulas so that the conditions (i)–(vi) listed in Theorem 1 are satisfied. First observe that $X \cap \mathcal{M}$ is a closed subset of $X$, although this fact follows from a general result which can be found in [4] we give a proof of it for the sake of completeness: let $x \in X \cap \mathcal{M}$, since $t(X) \leq \lambda$ there is a $C \in [X \cap \mathcal{M}]^{\leq \lambda}$ such that $x \in \overline{C}$. Since $C \in \mathcal{M}$ (by (i)), it follows that $\overline{C} \in \mathcal{M}$ (by (iii)). Now it remains to observe that $|\overline{C}| \leq \kappa$ (recall that $t(X)\psi_c(X) \leq \lambda$) and hence by (vi) $x \in \overline{C} \subset X \cap \mathcal{M}$.

We have done if we show that $X \subset \mathcal{M}$. Suppose there is a $p \in X \setminus \mathcal{M}$, for every $y \in X \cap \mathcal{M}$ let $B_y \in B_y$ such that $p \notin B_y$. Since $\mathcal{U} = \{B_y : y \in X \cap \mathcal{M}\} \cup \{X \setminus \mathcal{M}\}$ is an open cover of $X$ and $aqL(X) \leq \lambda$ there is a $\mathcal{V} \in [\mathcal{U}]^{<\lambda}$ such that $X = \mathcal{S} \cup (\bigcup \mathcal{V})$. Let $\mathcal{W} = \{B_y : B_y \in \mathcal{V}\}$, since $X \cap \mathcal{M} \subset \mathcal{S} \cup (\bigcup \mathcal{W})$ and $\mathcal{S} \cup (\bigcup \mathcal{W}) \in \mathcal{M}$ it follows that $X = \mathcal{S} \cup (\bigcup \mathcal{W})$, a contradiction ($p \notin \mathcal{S} \cup (\bigcup \mathcal{W})$).

A consequence of Theorem 8 is the following generalization of the Arhangel’skii’s inequality.

Corollary 9 ([8]). If $X$ is a Hausdorff space then $|X| \leq 2^{aqL(X)t(X)\psi_c(X)}$.

Proof: It is enough to note that $aqL(X) \leq qL(X)t(X)\psi_c(X)$. $\square$

Remark 10. Let $\kappa$ be an infinite cardinal number and let $X$ be the discrete space of cardinality $2^\kappa$. This space shows that Theorem 8 can give a better estimation than the one in Corollary 9.

References


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