ωH-sets and cardinal invariants

ALESSANDRO FEDELI

Abstract. A subset \(A\) of a Hausdorff space \(X\) is called an \(\omega H\)-set in \(X\) if for every open family \(U\) in \(X\) such that \(A \subset \bigcup U\) there exists a countable subfamily \(V\) of \(U\) such that \(A \subset \bigcup \{V : V \in V\}\). In this paper we introduce a new cardinal function \(t_{s\theta}\) and show that \(|A| \leq 2^{t_{s\theta}(X)\psi_c(X)}\) for every \(\omega H\)-set \(A\) of a Hausdorff space \(X\).

Keywords: cardinal function, \(\omega H\)-set
Classification: 54A25, 54D20

All spaces considered in this paper are Hausdorff. We refer the reader to [8], [10] and [11] for notations and details on cardinal functions. Our approach to elementary submodels follows that of [15] (see also [5], [6], [9]). As usual \(\psi(X)\) and \(\chi(X)\) denote the pseudocharacter and the character of the space \(X\). A Urysohn space is a space in which distinct points have disjoint closed neighbourhoods. Moreover, for any set \(S\), we denote by \(P_m(S)\) the collection of all subsets of \(S\) whose cardinality is not greater than \(m\).

A subset \(A\) of a space \(X\) is called an \(H\)-set (\(\omega H\)-set) in \(X\) if for every open family \(U\) in \(X\) such that \(A \subset \bigcup U\) there exists a finite (countable) family \(V \subset U\) satisfying \(A \subset \bigcup V\). A space \(X\) is said to be quasi Lindelöf if for every open cover \(U\) of \(X\) there is a countable subfamily \(V\) of \(U\) satisfying \(X = \bigcup \{U : U \in V\}\). It is clear that every \(H\)-set in a space \(X\) and every quasi Lindelöf space \(X\) is an \(\omega H\)-set in \(X\).

Let \(x \in X\), a closed pseudobase for \(x\) in \(X\) is a family \(V\) of open neighbourhoods of \(x\) in \(X\) such that \(\bigcap \{V : V \in V\} = \{x\}\). The closed pseudocharacter of \(X\), denoted by \(\psi_c(X)\), is the smallest infinite cardinal \(\kappa\) such that every point has a closed pseudobase of cardinality not greater than \(\kappa\).

The \(\theta\)-closure of a subset \(A\) of a space \(X\), denoted by \(cl_\theta(A)\), is the set of all points \(x \in X\) such that \(\overline{U} \cap A \neq \emptyset\) for every open neighbourhood \(U\) of \(x\) ([13]).

For any space \(X\), \(t_{s\theta}(X)\) will denote the smallest infinite cardinal \(\kappa\) such that for every \(C \subset X\) and any \(x \in cl_\theta(C)\) there is \(S \subset C\) with \(|S| \leq \kappa\), \(x \in cl_\theta(S)\) and \(|cl_\theta(S)| \leq 2^\kappa\) (see [12] for related concepts).

**Remark 1.** If \(X\) is a Urysohn space, then \(t_{s\theta}(X)\psi_c(X) \leq \chi(X)\). Set \(\chi(X) = \kappa\) and for every \(x \in X\) let \(B(x)\) be a base for \(X\) at the point \(x\) such that \(|B(x)| \leq \kappa\). Now let \(C \subset X\) and \(p \in cl_\theta(C)\), for every \(B \in B(p)\) take a point \(x_B \in \overline{B} \cap C\) and set \(S = \{x_B : B \in B(p)\}\). Clearly \(|S| \leq \kappa\) and \(p \in cl_\theta(S)\). Now let us show that \(|cl_\theta(S)| \leq 2^\kappa\). Since \(X\) is a Urysohn space, it follows that \(\bigcap \{cl_\theta(\overline{B} \cap S) :
Example 5. Let $\tau$ be the euclidean topology on $R$ and let $X$ be $R$ with the topology
$\sigma = \{V \setminus C : V \in \tau, C \in \mathcal{P}_\omega(R)\}$. $X$ is an Urysohn hereditarily
Lindel"of space (so, a fortiori, every subset of $X$ is an $\omega H$-set in $X$). Observe that $\psi_c(X) = \omega$
and $\chi(X) = c$. Now let us show that $t_{s\theta}(X) = \omega$. First note that $cl_\sigma(V \setminus C) = \emptyset$.
\( cl_\sigma (V) \) for every \( V \setminus C \in \sigma \). In fact, let \( x \in cl_\sigma (V) \) and take \( W \setminus K \in \sigma \) such that \( x \in W \setminus K \). Since \( (W \setminus K) \cap V \neq \emptyset \), it follows that \( |(W \cap V) \setminus K| = c \) (observe that \( W \cap V \) is a non-empty open set of the euclidean line). Therefore \( \emptyset \neq (W \setminus V) \setminus (K \cup C) = (W \setminus K) \cap (V \setminus C) \), and \( x \in cl_\sigma (V \setminus C) \). Now let \( B \subseteq R \) and \( x \in cl_\theta (B) \). Set \( V_n = (x - \frac{1}{n}, x + \frac{1}{n}) \) for every \( n \in \mathbb{N} \) and take a point \( x_n \in cl_\sigma (V_n) \cap B \). The set \( S = \{x_n : n \in \mathbb{N}\} \) is a countable subset of \( B \) such that \( |cl_\theta (S)| \leq c \). It remains to show that \( x \in cl_\theta (S) \). Let \( G = V \setminus C \in \sigma \) such that \( x \in G \) and let \( n \) be such that \( V_n \subset V \). Then \( x_n \in cl_\sigma (V_n) \subset cl_\sigma (V) = cl_\sigma (V \setminus C) = cl_\sigma (G) \) and \( cl_\sigma (G) \cap S \neq \emptyset \). So \( x \in cl_\theta (S) \). Therefore \( |X| = 2^{\sigma \chi (X)} < 2^{\chi (X)} \).

Recently A. Bella and I.V. Yaschenko have shown that “Urysohn” cannot be omitted in the above corollaries ([3]).

**Remark 6.** A space \( X \) is H-closed if every open cover of \( X \) has a finite subfamily whose union is dense in \( X \). It is worth noting that \( |X| \leq 2^{\psi_c (X)} \) for every H-closed space \( X \) ([7]). Moreover there is an H-closed space \( X \) such that \( |X| > 2^{\psi (X)} \) ([4]).

**Question 7.** Let \( A \) be an H-set in the Urysohn space \( X \). Is it true that \( |A| \leq 2^{\psi_c (X)} \) ?

**Remark 8.** Observe that it is not possible to obtain a bound for the cardinality of an \( \omega H \)-set in terms of its character, in fact there are discrete H-sets of any cardinality (see, e.g., [1] and [4]).

**References**


DEPARTMENT OF MATHEMATICS, UNIVERSITY OF L’AQUILA, VIA VETOIO (LOC. COPPITO), 67100 L’AQUILA, ITALY

E-mail: fedeli@axscaq.aquila.infn.it

*(Received June 20, 1997)*