On the Noetherian type of topological spaces

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Abstract. The Noetherian type of topological spaces is introduced. Connections between the Noetherian type and other cardinal functions of topological spaces are obtained.

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Let $X$ be a topological space and $\mathcal{B}$ be an open family in $X$. For a set $G$ let us denote by $\mathcal{B}_G$ the family $\{B \in \mathcal{B} : G \subset B\}$.

We define the Noetherian type of $\mathcal{B}$ as the cardinal
$$Nt(\mathcal{B}) = \min \{\alpha : \alpha \text{ is an infinite cardinal and } |\mathcal{B}_G| < \alpha \text{ for every nonempty open set } G \subset X\}.$$  

We define the lower Noetherian type of $\mathcal{B}$ as the cardinal
$$lnNt(\mathcal{B}) = \sup \{|\mathcal{B}_G| : G \text{ is a nonempty open subset of } X\}.\omega.$$  

The cardinal
$$\min \{\text{Nt}(\mathcal{B}) : \mathcal{B} \text{ is a base of the space } X\}$$

is called the Noetherian type of $X$ and is denoted by $Nt(X)$.

The cardinal
$$\min \{\text{lnNt}(\mathcal{B}) : \mathcal{B} \text{ is a base of the space } X\}$$

is called the lower Noetherian type of $X$ and is denoted by $lnNt(X)$.

Considering in the last definitions a $\pi$-base in place of a base we obtain definitions of the Noetherian $\pi$-type and the lower Noetherian $\pi$-type of $X$, which are denoted by $N\pi t(X)$ and $lnN\pi t(X)$ respectively.

Now let $\mathcal{B}$ be a family of sets. The cardinal
$$\text{rank}(\mathcal{B}) = \sup \{|\mathcal{B}'| : \mathcal{B}' \subset \mathcal{B}, \bigcap \mathcal{B}' \neq \emptyset \text{ and } \mathcal{B}' \text{ is an antichain (by the set theoretic inclusion)}\}$$

is called the rank of the family $\mathcal{B}$. We define the rank weight of a topological space $X$ as the cardinal
$$w_r(X) = \min \{\text{rank}(\mathcal{B}) : \mathcal{B} \text{ is a base of } X\}.\omega.$$  

Analogously the rank $\pi$-weight of $X$ is defined as the cardinal
$$\pi w_r(X) = \min \{\text{rank}(\mathcal{B}) : \mathcal{B} \text{ is a } \pi\text{-base of } X\}.\omega.$$
Lemma 1. Let $X$ be a topological space. Then
\[ w(X) = lNt(X).\pi w(X) \quad \text{and} \quad lNt(X).\chi(X) = lNt(X).\pi \chi(X). \]

Proof: Clearly $lNt(X).\pi w(X) \leq w(X)$. Let $B$ be a base of $X$ such that $|B| = w(X)$ and $lNt(B) = lNt(X)$. Let $\mathcal{H}$ be a $\pi$-base of $X$ such that $|\mathcal{H}| = \pi w(X)$. For every set $B \in \mathcal{H}$ choose a set $H \in \mathcal{H}$ such that $H \subset B$. We obtain a mapping $f : \mathcal{H} \to \mathcal{H}$. Since $|f^{-1}(H)| \leq lNt(X)$ and $\mathcal{B} = \bigcup \{f^{-1}(H) : H \in \text{rng}(f)\}$ it follows that $|\mathcal{B}| \leq lNt(X).\pi w(X)$. Consequently, $w(X) = lNt(X).\pi w(X)$. Clearly for every point $x \in X$ we have $x(x, X) \leq lNt(X).\pi \chi(X)$.

\[ \Pi \]

The inverse inequality is trivial.

Lemma 2. Let $X$ be a topological space. If $Nt(X) < \chi(X)$ then $w_r(X) = \chi(X)$.

Proof: Clearly for every base $B$ of the space $X$ and every point $x \in X$ we have $\text{ord}(x, B) \leq \chi(x, X).lNt(B)$. Consequently, provided that $Nt(X) < \chi(X)$ we have that $w_r(X) \leq \chi(X)$. Now let $B$ be an arbitrary base of $X$ and let $B'$ be a base of $X$ such that $Nt(B') = Nt(X)$. Take an arbitrary cardinal $\tau$ such that $Nt(X) \leq \tau < \chi(X)$. Choose a point $x \in X$ such that $\chi(x, X) \geq \tau^+$. Let $B'_x = \{B' : x \in B'\}$ and for every set $B' \in B'_x$ choose a set $B'' \subset B'$ such that $B'' \in B_x = \{B \in B : x \in B\}$. We obtain a family $B'' \subset B$ such that $Nt(B'') = Nt(X)$ and $x \in \bigcap B''$. Besides it is evident that $|B''| \geq \tau^+$. Choose a maximal antichain $B''_0$ out of $B''$. Consider the family of all sets $B \in B''$ such that $B$ is contained as a proper subset in some set belonging to $B''_0$. Choose a maximal antichain $B''_1$ out of this family. Continuing this process we obtain a $Nt(X)$-sequence $(B''_\xi : \xi < Nt(X))$ such that every its element $B''_\xi$ is a maximal antichain in the family \{ $B \in B'' : B$ is contained as a proper subset in some set belonging to $B''_\gamma$ for every $\gamma < \xi$. The family $B''_{Nt(X)} = \bigcup \{ B''_\xi : \xi < Nt(X) \}$ is dense in $B''$. Therefore it is a local base of $x$ in $X$. Then $|B''_{Nt(X)}| \geq \tau^+$ and there exists $\xi < Nt(X)$ such that $|B''_\xi| \geq \tau^+$. Hence, $\text{rank}(B) \geq \tau^+$. Since it is true for every cardinal $\tau$, such that $Nt(X) \leq \tau < \chi(X)$, we have that $\text{rank}(B) \geq \chi(X)$. Because $B$ is an arbitrary base of $X$ it follows that $w_r(X) \geq \chi(X)$.

Definitions. A topological space $X$ is called a Noetherian space provided that $Nt(X) = \omega$. A topological space $X$ is called a weakly Noetherian space provided that $lNt(X) = \omega$.

Corollary 1. If $X$ is a Noetherian space then $w_r(X) = \chi(X)$.

Example 1. Let $X = \omega_1 \cup \{\omega_1\}$. Introduce a topology on $X$ as the following. Let every point of $\omega_1$ be isolated. A base of neighborhoods of the point $\omega_1$ is defined as the family \{ $\{\xi, \omega_1) : \xi \leq \omega_1\}$ where $\{\eta, \omega_1) = \{\eta \leq \omega_1 : \xi \leq \eta\}$. Evidently $X$ is a regular Lindelöf space for which $lNt(X) = \omega$, $Nt(X) = \omega_1$, $\chi(X) = \omega_1$, and $w_r(X) = \omega$. 

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Lemma 3. Let $X$ be a compact Hausdorff space. If $Nt(X)$ is a regular cardinal and $w(X) = Nt(X)$ then $w_r(X) = w(X)$.

Proof: Let us assume that $w_r(X) < w(X)$. The space $X$ contains an everywhere dense subset $A$ such that $\pi\chi(x, X) \leq w_r(X)$ for every point $x \in A$ ([2]). Choose a base $B$ of the space $X$ such that $Nt(B) = Nt(X)$. Then $ord(x, B) < Nt(X)$ for every point $x \in A$. By induction in just the same way as in [6], construct a sequence $(S_n : n \in \omega)$ of subsets of the set $A$ such that the following conditions are fulfilled:

1. if $n, m \in \omega$ and $n < m$ then $S_n \subset S_m$;
2. $|S_n| < w(X)$ for every $n \in \omega$;
3. if $n \in \omega$, $B'$ is a finite subfamily of the family $\{B \in B : B \cap S_n \neq \emptyset\}$ and $A \setminus \bigcup B' \neq \emptyset$ then $S_{n+1} \cap (A \setminus \bigcup B') \neq \emptyset$.

The set $S = \bigcup \{S_n : n \in \omega\}$ by (3) is an everywhere dense subset of $X$. In addition $|S| < w(X)$ and $ord(x, B) < w(X)$ for every point $x \in S$. Then $w(X) < w(X)$, a contradiction. It follows that $w_r(X) = w(X)$. \(\square\)

Theorem 1. Let $X$ be a compact Hausdorff space. Then

$$w(X) = lNt(X).\pi w(X) = lNt(X).\pi \chi(X) = lNt(X).t(X) = lNt(X)\.\chi(X) =$$

$$lNt(X).w_r(X) = lNt(X).s(X) = lNt(X).hd(X) = lNt(X).hL(X).$$

Proof: Since $X$ is a compact Hausdorff space it follows from [7] that $\pi\chi(X) \leq t(X)$. By Lemma 1 it implies the first, the third, and the fourth equations. Further, if $B$ is a base of the space $X$ and $lNt(B) = lNt(X)$ then $ord(x, B) \leq lNt(X).\pi\chi(X)$ for every point $x \in X$. By Theorem Mishenko then $|B| \leq lNt(X).\pi\chi(X)$. It implies the second equation. Now let us assume that $lNt(X).w_r(X) < w(X)$. Then $w_r(X) < w(X)$ and $lNt(X) < w(X)$. By the fourth equation it implies that $w(X) = \chi(X)$ and by Lemma 2 we have $Nt(X) \geq w(X)$. But $w(X) > lNt(X)$, hence $Nt(X) = w(X) = lNt(X)^+$. Then by Lemma 3 we have that $w_r(X) = w(X)$. It is a contradiction, hence $w(X) = lNt(X).w_r(X)$. The other equations are consequences of the inequality $t(X) \leq s(X)$ for compact Hausdorff spaces ([1]). \(\square\)

Corollary 2. Let $X$ be a Hausdorff compact weakly Noetherian space. Then $w(X) = \pi w(X) = w_r(X) = \chi(X) = \pi\chi(X) = t(X) = s(X) = hd(X) = hL(X)$.

Corollary 3. Let $X$ be a Hausdorff locally compact weakly Noetherian space. Then $w_r(X) = \chi(X)$.

Corollary 4. Let $X$ be a compact Hausdorff space. If $Nt(X)$ is a weakly inaccessible cardinal then $w_r(X) = w(X)$.

Example 2. Let $X$ denote the “two arrows” space. It is known that $w_r(X) = \omega$ ([3]). By Theorem 1 it implies that $lNt(X) = w(X) = 2^{\omega}$ and because $cf(2^{\omega}) > \omega$ we get that $Nt(X) = (2^{\omega})^+$. 
**Example 3.** Let \( X = I^\tau \) where \( I \) is the unit segment. Because \( l\text{Nt}(X) = N\text{t}(X) = \omega \) ([5]), it follows from Theorem 1 that \( w_r(X) = \tau.\omega \).

**Example 4.** The space of ordinals \( X = \omega_1 \cup \{\omega_1\} \) is not weakly Noetherian because \( \chi(X) = \omega_1 \) and \( w_r(X) = \omega \) ([2]).

**Theorem 2.** For a compact Hausdorff space \( X \) the following conditions are equivalent:

1. \( w(X) = l\text{Nt}(X) \);
2. \( l\text{Nt}(Y) \leq l\text{Nt}(X) \) for every subspace \( Y \) of \( X \);
3. \( l\text{Nt}(Y) \leq l\text{Nt}(X) \) for every closed subspace \( Y \) of \( X \).

**Proof:** Evidently it is sufficient to prove the implication (3) \( \rightarrow \) (1). Let the condition (3) be fulfilled and suppose that \( w(X) > l\text{Nt}(X) \). Then by Theorem 1 there exists a discrete set \( Y \subset X \) such that \( |Y| > l\text{Nt}(X) \). By the assumption \( l\text{Nt}(clY) \leq l\text{Nt}(X) \). Choose a base \( B \) of the space \( clY \) such that \( l\text{Nt}(B) = l\text{Nt}(clY) \). Because \( \text{ord}(y,B) \leq l\text{Nt}(clY) \) for every point \( y \) belonging to \( Y \), we have by [6] that \( w(clY) \leq l\text{Nt}(clY) \). Hence \( |Y| \leq l\text{Nt}(X) \). It is a contradiction, consequently, \( w(X) = l\text{Nt}(X) \). \( \square \)

**Theorem 3.** If \( Y \) is an open or canonical closed subspace of a space \( X \) then \( l\text{Nt}(Y) \leq l\text{Nt}(X) \). If \( X \) is a Hausdorff space and \( \mathcal{F} \) is a family of compact subspaces of \( X \) having in \( X \) the character \( \leq l\text{Nt}(X) \), then if \( \bigcup \mathcal{F} \subset Y \subset cl \bigcup \mathcal{F} \) then \( l\text{Nt}(Y) \leq l\text{Nt}(X) \).

**Proof:** The first assertion is trivial. To prove the second assertion, take \( \mathcal{E}_F = \{A \subset F : A \neq \emptyset, \chi(A,X) \leq l\text{Nt}(X)\} \) for every \( F \in \mathcal{F} \) and put \( \mathcal{E} = \bigcup \{\mathcal{E}_F : F \in \mathcal{F}\} \). Let \( B \) be a base of \( X \) such that \( l\text{Nt}(B) = l\text{Nt}(X) \). If \( B \subset B \) and \( B \) contains a nonempty open in \( Y \) set \( P \), then there exists \( E \in \mathcal{E} \) such that \( E \subset P \). Because \( \chi(E,X) \leq l\text{Nt}(B) \), it follows that \( |\{B \in B : B \supset E\}| \) is not more than \( l\text{Nt}(B) \). This implies that \( l\text{Nt}(B|Y) \leq l\text{Nt}(B) \) and hence \( l\text{Nt}(Y) \leq l\text{Nt}(X) \). \( \square \)

**Theorem 4.** Let \( \{Z_\alpha : \alpha \in A\} \) be a family of topological spaces and \( \prod\{Z_\alpha : \alpha \in A\} \) is denoted by \( Z \). Then

\[
\text{Nt}(Z) \leq \sup\{\text{Nt}(Z_\alpha) : \alpha \in A\} \quad \text{and} \quad l\text{Nt}(Z) \leq \sup\{l\text{Nt}(Z_\alpha) : \alpha \in A\}.
\]

**Proof:** To prove this, choose a base \( B_\alpha \) of the space \( Z_\alpha \) for every \( \alpha \in A \) such that \( N\text{t}(B_\alpha) = N\text{t}(Z_\alpha) \). It is easy to see that a base of \( Z \) consisting of sets of the form \( B_{\alpha_1} \times \cdots \times B_{\alpha_n} \times \prod\{Z_\alpha : \alpha \in A \setminus \{\alpha_1, \ldots, \alpha_n\}\} \), where \( B_{\alpha_i} \in B_{\alpha_i} \) for \( i = 1, \ldots, n \), have the Noetherian type that is not greater than \( \sup\{N\text{t}(Z_\alpha) : \alpha \in A\} \). Analogously it can be proved that \( l\text{Nt}(Z) \leq \sup\{l\text{Nt}(Z_\alpha) : \alpha \in A\} \). \( \square \)

**Remark.** The following theorem, which has been proved in [4], is an essential supplement of Theorem 4:

\[
\text{if} \quad |A| \geq \sup\{w(Z_\alpha) : \alpha \in A\} \quad \text{then} \quad N\text{t}(Z) = \omega.
\]
Theorem 5. Let $X$ be a topological space such that $\pi w(X) > N\pi t(X)$ and let $\kappa$ be a cardinal such that $\kappa^+$ is a calibre of $X$. If $\pi w(X) > \kappa$ then $\pi w_r(X) > \kappa$.

Proof: Let $\mathcal{H}$ be a $\pi$-base of $X$. Choose a $\pi$-base $\mathcal{H}'$ of $X$ such that $Nt(\mathcal{H}') = N\pi t(X)$. Now choose a cardinal $\tau$ such that $N\pi t(X) \leq \tau < \pi w(X)$. Take a mapping $f : \mathcal{H}' \to \mathcal{H}$ such that $f(\mathcal{H}') \subset H'$ and put $\mathcal{H}'' = \text{rng}(f)$. It is evident that $\mathcal{H}''$ is a $\pi$-base of $X$. Also it is clear that $|\mathcal{H}''| \geq \pi w(X)$ and $Nt(\mathcal{H}'') = N\pi t(X)$. Consequently $|\mathcal{H}''| \geq \tau^+$. By induction construct a $N\pi t$-sequence $(\mathcal{H}''_{\xi} : \xi < N\pi t(X))$ of subsets of $\mathcal{H}''$ such that the following conditions are fulfilled:

1. $\mathcal{H}''_{\tau}$ is a maximal antichain (by inclusion) in $\mathcal{H}''$;
2. if $\xi > 0$ then $\mathcal{H}''_{\xi}$ is a maximal antichain in the family $\{H \in \mathcal{H}'' : \text{for every } \eta < \xi \text{ there exists } H' \in H''_{\eta} \text{ such that } H \subset H' \text{ and } H \neq H'\}$.

Put $\mathcal{H}''_{N\pi t(X)} = \bigcup \{\mathcal{H}''_{\xi} : \xi < N\pi t(X)\}$. It is easy to see that $\mathcal{H}''_{N\pi t(X)}$ is a $\pi$-base of $X$. This implies that $|\mathcal{H}''_{N\pi t(X)}| \geq \tau^+$. Because $N\pi t(X) \leq \tau$, there exists $\xi < N\pi t(X)$ such that $|\mathcal{H}''_{\xi}| \geq \tau^+$. Since $\kappa \leq \tau$, there exists $\hat{H}_{\xi} \subset \mathcal{H}''_{\xi}$ such that $|\hat{H}_{\xi}| > \kappa$ and $\bigcap \hat{H}_{\xi} \neq \emptyset$. Because $\mathcal{H}$ is an arbitrary $\pi$-base we get that $\pi w_r(X) > \kappa$.

Corollary 5. Let $X$ be a topological space such that $\pi w(X) > N\pi t(X)$. Then the following assertions are fulfilled:

(a) if $\pi w(X) > d(X)$ then $\pi w_r(X) = \pi w(X)$;
(b) if $\pi w(X) > sh(X)$ then $\pi w_r(X) > sh(X)$.

The following problem was raised by P. Bir"ukov:

Is there a compact Hausdorff space $X$ such that $|X| > 2^{w_r}(X)$?

In view of the above mentioned assertions the space may be encountered only where $Nt(X) = w(X)^+$ or $Nt(X) = w(X)$ is a singular cardinal.

Corollary 6 (MA). Let $X$ be a compact Hausdorff space. If $Nt(X) \leq 2^\omega$ then $|X| \leq 2^{w_r}(X)$.

References


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