On two results of Singhof

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Abstract. For a compact connected semisimple Lie group $G$ we shall prove two results (both related with Singhof’s paper [13]) on the Lusternik-Schnirelmann category of the adjoint orbits of $G$, respectively the 1-dimensional relative category of a maximal torus $T$ in $G$. The techniques will be classical, but we shall also apply some basic results concerning the so-called $A$-category (cf. [14]).

Keywords: Lusternik-Schnirelmann category, Lie groups, adjoint orbits

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The following results were proved in [13] by methods which combine in an ingenious manner the classical theories of Lie groups and of Lusternik-Schnirelmann-type categories.

Theorem A. Let $G$ be a compact connected Lie group and $T$ a maximal torus of $G$. Then

$$\text{cat } G/T = \frac{1}{2} \dim G/T + 1.$$ 

For an arbitrary finitely generated Abelian group $\pi$, denote by $\varphi(\pi)$ the smallest number $n$ such that $\pi$ is the direct sum of $n$ cyclic groups.

Theorem B. Let $G$ be a compact connected Lie group and $T$ a maximal torus of $G$. Then

$$\text{cat}_G T = \varphi(\pi_1 G) + 1.$$ 

Consider now $\mathfrak{g}$ the Lie algebra of $G$. Take $X \in \mathfrak{g}$ and denote by $G_X$ the Ad-stabilizer of $X$ (note that $X$ is regular iff $G_X$ is a maximal torus in $G$). The adjoint orbits $\text{Ad } G.X$ were during the last years frequently considered and studied, both from the topological point of view (mention only the detailed descriptions of the cohomology ring given in [1] or [2]) and from differential perspective (they represent fundamental examples of the so-called theory of isoparametric submanifolds, recently initiated by R. Palais and C.L. Terng). In connection with Theorem A we shall prove:

Theorem 1. Let $G$ be a compact connected semisimple Lie group and $X$ an element of its Lie algebra. Then

$$\text{cat } (\text{Ad } G.X) = \frac{1}{2} \dim(\text{Ad } G.X) + 1.$$
In [6] Fox considers for the first time the so-called $q$-dimensional relative (homotopical) category associated to an inclusion. Many other developments were obtained afterwards; among them, the notion of $\mathcal{A}$-category (cf. [4, Examples 1.2(3)]). The following result concerning the 1-dimensional category will be proved in the second section.

**Theorem 2.** Let $G$ be a compact connected semisimple Lie group and $T$ a maximal torus. Then

$$\pi_1 - \text{cat}_G T = \varphi(\pi_1 G) + 1.$$ 

1. The Lusternik-Schnirelmann category of $G/G_X$

Recall that the Lusternik-Schnirelmann category of a topological space $M$ is the number $\text{cat} M$ equal to the least number of sets in an open finite covering of $M$ with subsets contractible in $M$; if such a covering does not exist, take $\text{cat} M = \infty$. Both homotopical and differential aspects are concentrated in this notion; on the one hand, it is a homotopical invariant, and on the other hand, when $M$ is a compact differentiable manifold, the number of critical points of a real function on $M$ cannot be less than $\text{cat} M$.

Let us consider $G$ a compact connected Lie group, $T \subseteq G$ a maximal torus and $t \subseteq \mathfrak{g}$ their Lie algebras.

**Proposition 1.** For any $X$ belonging to $\mathfrak{g}$, the adjoint orbit $\text{Ad}_G X$ is simply connected. Equivalently the stabilizer $G_X$ is connected.

**Proof:** Let $X_0 \in t$ be regular. Its orbit $\text{Ad} G X_0$ is a full isoparametric submanifold of $\mathfrak{g}$, with uniform multiplicity 2. The orbit foliation $\{\text{Ad} G X \mid X \in t\}$ is just the parallel foliation of $\text{Ad} G X_0$ on $\mathfrak{g}$ (cf. [9, Example 6.5.6]). Since all multiplicities are greater than 1, by Theorem 5.7 of [8], any leaf $\text{Ad} G X$ is simply connected, and the proof is finished. \(\square\)

The following result is mentioned in A. Borel’s work [1]: the quotients of two locally isomorphic compact connected Lie groups $G$ and $G'$ by maximal tori $T$ and $T'$ are homeomorphic (see p. 188). We shall generalize it as follows:

**Proposition 2.** Let $p : \tilde{G} \to G$ be the universal group covering of the compact connected Lie group $G$ of Lie algebra $\mathfrak{g}$, $X$ an element of $\mathfrak{g}$, $\tilde{G}_X$ and $G_X$ the stabilizers of $X$. Then

(a) $p(\tilde{G}_X) = G_X$,

(b) the induced map $\varrho : \tilde{G}/\tilde{G}_X \to G/G_X$ is a homeomorphism.

**Proof:** (a) One can easily see that $p(\tilde{G}_X) \subseteq G_X$. It follows that $p \mid_{\tilde{G}_X} : \tilde{G}_X \to G_X$ is a local isomorphism and because $G_X$ is connected, it is generated by $p(\tilde{G}_X)$. So $p(\tilde{G}_X) = G_X$. 


(b) By the classical facts: \( \ker p \subseteq Z(\tilde{G}) \) (cf. \cite[Lemma 6, p. 195]{11}), \( Z(\tilde{G}) \subseteq T \) (cf. \cite[Theorem 2.3, Chapter IV]{3}) and \( T \subseteq \tilde{G}_X \), the injectivity of \( \varphi \) is clear. So \( \varphi \) is a homeomorphism. \( \square \)

Remark that the homogeneous space \( G/G_X \) depends only on \( g \) and \( X \), but not on the involved connected Lie group \( G \). This fact offers the possibility to deduce informations about the cohomology ring of \( G/G_X \) from Theorem III\'' of \cite{2}, even without the hypothesis \( G \) simply connected.

**Proposition 3.** Let \( G \) be a compact connected semisimple Lie group of Lie algebra \( g \) and \( X \) an element of \( G \). Then the ring \( H^*(G/G_X, \mathbb{Q}) \) is generated by 1 and \( H^2(G/G_X, \mathbb{Q}) \).

Notice that the above mentioned orbit \( G/G_X \) is of dimension \( n = \dim G - \text{rank } G - 2m \), where \( m \) is the number of hyperplanes of the infinitesimal diagram containing \( X \); it is also orientable (being simply connected) and so \( H^n(G/G_X, \mathbb{Q}) = \mathbb{Q} \). The \( \mathbb{Q} \)-cohomological length will be then cuplength \( (G/G_X) \geq \frac{n}{2} \), and so \( \text{cat } (G/G_X) \geq \frac{n}{2} + 1 \).

On the other hand, \( G/G_X \) being simply connected, by Corollary 3.3 of \cite{7} one obtains \( \text{cat}(G/G_X) \leq \frac{n}{2} + 1 \).

In the end of the section, let us take for instance the homogeneous space of the form \( G/G_X \) from \cite{12} and calculate their Lusternik-Schnirelmann category \((n, n_1, \ldots, n_k)\) will be positive integers, \( \sum n_j = n \).

(a) The complex flag manifold \( W(n_1, \ldots, n_k) = U(n)/U(n_1) \times \cdots \times U(n_k) \) has the Lusternik-Schnirelmann category equal to \( \frac{1}{2}(n^2 - \sum j n_j^2) + 1 \). Consequently, for the complex Grassmann manifold \( G_{k,n} = U(n)/U(k) \times U(n-k) \), we have \( \text{cat } G_{k,n} = k(n-k) + 1 \).

(b) \( \text{cat } \text{SO}(2n)/U(n_1) \times \cdots \times U(n_k) = \frac{1}{2}[n(2n-1) - \sum n_j^2] + 1 \) and so the symmetric space \( \text{SO}(2n)/U(n) \) will have \( \text{cat } \text{SO}(2n)/U(n) = \frac{1}{2}n(n-1) + 1 \).

(c) \( \text{cat } \text{SO}(2n+1)/U(n_1) \times \cdots \times U(n_k) \times 1 = \frac{1}{2}[n(2n+1) - \sum n_j^2] + 1 \).

(d) \( \text{cat } \text{Sp}(n)/U(n_1) \times \cdots \times U(n_k) = \frac{1}{2}[n(2n+1) - \sum n_j^2] + 1 \).

The symmetric space \( \text{Sp}(n)/U(n) \) will have \( \text{cat } \text{Sp}(n)/U(n) = \frac{n(n+1)}{2} + 1 \).

2. The 1-dimensional category of \( T \) in \( G \)

By technical reasons, we prefer to transpose the general definition of \( A \)-category and some basic results concerning it (cf. \cite{4}) to the older 1-dimensional category (see \cite{6} or \cite{5}).

Denote by \( \mathcal{C}_1 \) the class of 1-connected CW-complexes. Define the **\( \mathcal{C}_1 \)-category** of a map \( f : N \to M \) to be the number \( \mathcal{C}_1 - \text{cat}(f) \), the smallest cardinality \( k \) of a finite numerable covering \( \{N_1, \ldots, N_k\} \) of \( N \) such that for each \( j = 1, \ldots, k \) the restriction \( f|N_j : N_j \to M \) factors through some space in \( \mathcal{C}_1 \) up to homotopy (i.e. there exist \( C_j \in \mathcal{C}_1 \) and maps \( \alpha_j : N_j \to C_j, \beta_j : C_j \to M \) such that \( \beta_j \alpha_j \)}
is homotopic to $f \mid_{N_j}$). For a subspace $N$ of $M$, the relative **1-dimensional category** of $N$ in $M$ will be $\pi_1 - \text{cat}_M N = C_1 - \text{cat}(N \hookrightarrow M)$.

Let $G$ be again a compact connected Lie group and $T \subseteq G$ a maximal torus. Consider the decomposition of $\pi_1 G$ as $\pi_1 G = \mathcal{F} \oplus_q \oplus_{\text{prime}} T_q$, where $\mathcal{F}$ is the free part and $T_q$ the subgroup of all order $q^m$ ($m \geq 1$) elements; denote by $r = \text{rank} \mathcal{F}$, $r_q = \text{rank} T_q$. A classical result says that the inclusion $i : T \hookrightarrow G$ induces $i_\# : \pi_1 T \to \pi_1 G$ surjective. It then follows that $i^* : H^1(G, \mathbb{Z}_q) \to H^1(T, \mathbb{Z}_q)$ is injective, for any prime $q$. By the Hurewicz isomorphism, $H^1(G, \mathbb{Z}_q) \cong \text{Hom}(\pi_1 G, \mathbb{Z}_q)$ is isomorphic to a finite direct sum $\oplus \mathbb{Z}_q$ with $r + r_q$ terms. Since $H^*(T, \mathbb{Z}_q)$ is an exterior algebra, there exist in $H^1(G, \mathbb{Z}_q)$ a number of $r + r_q$ elements whose product does not go to zero under $i^*$. One can now use Proposition 3.1 of [4]: for any $C \in C_1$ and any $f : C \to G$, the map $f^* : H^1(G, \mathbb{Z}_q) \to H^1(C, \mathbb{Z}_q)$ is identically zero, and so

$$\pi_1 - \text{cat}_G T = C_1 - \text{cat}(T \hookrightarrow G) \geq r + r_q + 1.$$

But choosing $q$ with $r_q$ maximal, $r + r_q$ will be the minimal number of terms for a decomposition of $\pi_1 G$ in a direct sum of cyclic groups, the number that Singhof denotes by $\varphi(\pi_1 G)$. We have just proved:

**Lemma 1.** Let $G$ be a compact connected Lie group and $T \subseteq G$ a maximal torus. Then $\pi_1 - \text{cat}_G T \geq \varphi(\pi_1 G) + 1$.

It remains to show that:

**Lemma 2.** Let $G$ be a compact connected semisimple Lie group and $T \subseteq G$ a maximal torus. If $\pi_1 G$ admits a decomposition as a direct sum of $k$ cyclic groups, then $\pi_1 - \text{cat}_G T \leq k + 1$.

The proof is based on the relation between the 1-dimensional and sectional categories (see Section 4 of [4] for the definition and basic properties concerning the sectional category).

Let $\tilde{G}$ be the universal covering group of $G$. One can consider $G = \tilde{G}/C$, with $C \subseteq Z(\tilde{G})$ a finite central subgroup; moreover $\pi_1 G \cong C$ (cf. [3, Chapter V, Remark 7.13]). Any maximal torus of $G$ is of the form $\tilde{T}/C$, $\tilde{T}$ maximal torus in $\tilde{G}$.

The map $p : \tilde{G} \to G$ is $C_1$-universal (in the sense of [4]). Consequently $\pi_1 - \text{cat}_G \tilde{T}/C = \text{secat}(p')$, where $p' : U' \to \tilde{T}/C$ is the pullback over $i : \tilde{T}/C \hookrightarrow G$ of the Hurewicz fibration associated to $p$. Here $U' = \{(g, \alpha, tC) \in \tilde{G} \times \text{Top}(I, G) \times \tilde{T}/C \mid \alpha(0) = tC \}$ and $p'(g, \alpha, tC) = tC$. But considering $h : \tilde{T} \to U'$, $h(t) = (t, e_{tC}, tC)$, where $e_{tC}$ is the constant loop in $G$, we have $\text{secat}(p'h) \leq \text{secat}(p'h)$ (notice that $g = p'h : \tilde{T} \to \tilde{T}/C$ is the natural map). Because $C$ is a direct sum of $k$ cyclic subgroups of $\tilde{T}$, one can find a torus $\tilde{T}_C$, embedded as a subgroup of $\tilde{T}$, $\dim \tilde{T}_C \leq k$. There also exist an another toral subgroup $\tilde{T}' \subseteq \tilde{T}$, $\tilde{T} = \tilde{T}_C \times \tilde{T}'$. It follows that $\tilde{T}/C = \tilde{T}_C/C \times \tilde{T}'$ and $g' \times 1_{\tilde{T}'} : \tilde{T}_C \times \tilde{T}' \to \tilde{T}_C/C \times \tilde{T}'$, where $g' : \tilde{T}_C \to \tilde{T}_C/C$ is the quotient map.
$g' : \tilde{T}_C \to \tilde{T}_C/C$ the natural map. Conclude by $\secat(g' \times 1_{\tilde{T}_C}) = \secat(g') \leq 1 + \dim \tilde{T}_C/C \leq k + 1$ (cf. [4, Corollary 4.7]).

REFERENCES


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