On linear functorial operators extending pseudometrics

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Abstract. For a functor $F \supset Id$ on the category of metrizable compacta, we introduce a conception of a linear functorial operator $T = \{T_X : Pc(X) \to Pc(FX)\}$ extending (for each $X$) pseudometrics from $X$ onto $FX \supset X$ (briefly LFOEP for $F$). The main result states that the functor $SP^n_G$ of $G$-symmetric power admits a LFOEP if and only if the action of $G$ on $\{1, \ldots, n\}$ has a one-point orbit. Since both the hyperspace functor exp and the probability measure functor $P$ contain $SP^2$ as a subfunctor, this implies that both exp and $P$ do not admit LFOEP.

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The results of this note are related to recent authors’ results [Ba] and [Pi] stating that every metrizable compact pair $X \subset Y$ admits a linear operator $T : Pc(X) \to Pc(Y)$ extending continuous pseudometrics from $X$ onto $Y$. In the light of this result the question arises naturally: given a functor $F$ putting in correspondence to each metrizable compactum $X$ a space $FX \supset X$ is it possible for every $X$ to define in some natural way a linear operator $T_X : Pc(X) \to Pc(FX)$ extending pseudometrics from $X$ onto $FX$? This question is of interest because for many classical constructions such as the hyperspace functor exp or the functor $P$ of probability measures all known operators extending (pseudo)metrics (e.g. the Hausdorff extension of metrics onto exp $X$ or Kantorovich extension of metrics onto $PX$) are not linear. In this note we show that it is not occasionally and these functors do not admit any natural (or functorial) linear operator extending pseudometrics from $X$ onto $FX$. This will be shown by proving that for $n > 1$ the symmetric power functor $SP^n$ does not admit such a linear functorial extension operator, and noticing that both exp and $P$ contain $SP^2$ as a subfunctor.

Now let us give precise definitions. For a topological space $X$ by $Pc(X)$ the set of all continuous pseudometrics on $X$ is denoted. The set $Pc(X)$ has the cone structure, i.e. given $t \in [0, \infty)$ and $p, p' \in Pc(X)$ we have $tp \in Pc(X)$ and $p + p' \in Pc(X)$.

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Let $X, Y$ be two topological spaces. We say that a map $T : P_c(X) \to P_c(Y)$ is a linear operator if for every $t \geq 0$ and $p, p' \in P_c(X)$ we have $T(tp) = tT(p)$ and $T(p+p') = T(p) + T(p')$. In case $X \subseteq Y$ we call $T : P_c(X) \to P_c(Y)$ an extension operator if for every $p \in P_c(X)$ the pseudometric $T_p$ extends $p$. Notice that any continuous map $f : X \to Y$ induces a linear operator $f^* : P_c(Y) \to P_c(X)$ acting by $f^*(p) = p(f \times f)$ for $p \in P_c(Y)$.

By $\text{Top}$ we denote the category of all topological spaces and their continuous maps and by $\mathcal{M}\text{Comp}$ its full subcategory consisting of all metrizable compacta. A natural transformation $\eta : F \to G$ between two functors $F, G : \mathcal{M}\text{Comp} \to \text{Top}$ is a family of morphisms ( = continuous maps) $\eta = \{\eta_X : FX \to GX\}$ such that for every morphism $f : X \to Y$ in $\mathcal{M}\text{Comp}$ we get $Gf \circ \eta_X = \eta_Y \circ Ff$. A natural transformation $\eta = \{\eta_X\} : F \to G$ with all components $\eta_X$ being embeddings is called an embedding of functors. This is denoted by $F \subset G$ and $F$ is called a subfunctor of $G$. In this note we consider only functors $F$ containing the identity functor $Id$ as a subfunctor. Note that if $F$ preserves one-point spaces then $F$ admits at most one natural transformation $\eta : Id \to F$, see [Fe1] or [FF].

Now we introduce the conception of a functorial operator extending pseudometrics, the central conception in this paper. Let $F : \mathcal{M}\text{Comp} \to \text{Top}$ be a functor with $Id \subset F$. A collection $T = \{T_X : P_c(X) \to P_c(FX)\}$ of extension operators is called a functorial operator extending pseudometrics (briefly $\text{FOEP}$) for the functor $F$ if for every morphism $f : X \to Y$ in $\mathcal{M}\text{Comp}$ the following diagram is commutative

$$
\begin{array}{ccc}
P_c(Y) & \xrightarrow{T_Y} & P_c(FY) \\
\downarrow{f^*} & & \downarrow{(Ff)^*} \\
P_c(X) & \xrightarrow{T_X} & P_c(FX).
\end{array}
$$

If, moreover, all $T_X$’s are linear operators, then $T = \{T_X\}$ is called a linear functorial operator extending pseudometrics (briefly LFOEP) for $F$.

Notice that the introduced conceptions are near to the notion of a metrizable functor [Fe2].

Classical examples of $\text{FOEP}$ are the Hausdorff extension of (pseudo)metrics from a compactum $X$ onto the hyperspace $\exp X$ of all non-empty compact sets in $X$ and Kantorovich extension of (pseudo)metrics from $X$ onto the space $PX$ of probability measures on $X$, see [FF] or [Fe2]. These operators are not linear (and as we will see later they cannot be linear). An important example of a functor admitting a linear $\text{FOEP}$ is the functor $M$ putting in corresponding to a compactum $X$ the space $M(X)$ of all Borel-measurable functions $[0, 1] \to X$ [BP]. A linear $\text{FOEP}$ for the functor $M$ can be defined by the formula

$$
T_X(d)(f, g) = \int_0^1 d(f(t), g(t)) \, dt, \quad \text{where } f, g \in M(X) \text{ and } d \in P_c(X).
$$

The functor $M(X)$ and defined above LFOEP play a crucial role in the construction of linear extension operators in [Za].
Therefore, the question is: which functors admit and which do not admit linear FOEP’s? It turns out that depends much on relationships between \( F \) and the functors \( SP^n_G \) of \( G \)-symmetric power which definitions we are going to recall now.

Let \( G \subset S_n \) be a subgroup of the symmetric group \( S_n \) (i.e. the group of all bijections of the set \( n = \{1, \ldots, n\} \)). For a compactum \( X \) let \( SP^n_G(X) \) be the quotient space of \( X^n \) with respect to the equivalence relation \( \sim: (x_1, \ldots, x_n) \sim (y_1, \ldots, y_n) \) iff \( (x_1, \ldots, x_n) = (y_{\sigma(1)}, \ldots, y_{\sigma(n)}) \) for some \( \sigma \in G \). Further by \( [x_1, \ldots, x_n] \in SP^n_G(X) \) the equivalence class of an element \( (x_1, \ldots, x_n) \in X^n \) is denoted. It is easily seen that the construction of \( SP^n_G \) determines a functor on the category \( \text{MComp} \).

The principal result of this note is the following

**Theorem.** The functor \( SP^n_G \) admits a linear functorial operator extending pseudometrics if and only if the action of \( G \) on \( \{1, \ldots, n\} \) has a one-element orbit (i.e. \( G \cdot k = \{\sigma(k) \mid \sigma \in G\} = \{k\} \) for some \( k \in \{1, \ldots, n\} \)).

Applications of this theorem rely on the following simple

**Proposition.** Let \( F_1, F_2 : \text{MComp} \to \text{Top} \) be two functors such that each \( F_i \), \( i = 1, 2 \), preserves point and contains the identity functor \( \text{Id} \). If there is a natural transformation \( \varphi = \{\varphi_X\} : F_1 \to F_2 \) and the functor \( F_2 \) admits LFOEP then \( F_1 \) admits LFOEP either.

**Proof:** For \( i = 1, 2 \) denote by \( \eta_i : \text{Id} \to F_i \) the functorial embedding. Since \( F_i \) preserves point, the transformation \( \eta_i \) is unique. Hence \( \varphi \circ \eta_1 = \eta_2 \).

If \( T_2 = \{T_{2,X} : P_c(X) \to P_c(F_2X)\} \) is a LFOEP for \( F_2 \) then letting \( T_{1,X}(d) = T_{2,X}(d)(\varphi_X \times \varphi_X) \) for \( X \in \text{MComp} \) and \( d \in P_c(X) \), we obtain a LFOEP \( T_1 = \{T_{1,X}\} \) for \( F_1 \).

Since both functors \( \text{exp} \) and \( P \) contain the symmetric square functor \( SP^2 = SP^2_{S_2} \) as a subfunctor, Theorem and Proposition imply

**Corollary.** The functors \( \text{exp} \) and \( P \) on \( \text{MComp} \) do not admit any linear functorial operator extending pseudometrics.

**Proof of Theorem**

To prove the theorem we will need two simple lemmas first.

**Lemma 1.** Suppose for a finite space \( X = \{x_1, \ldots, x_m\} \) and reals \( a_{ij}, 1 \leq i < j \leq m \), the equality

\[
\sum_{i<j} a_{ij} d(x_i, x_j) = 0,
\]

holds for every metric \( d \) on \( X \). Then all \( a_{ij} \) are equal to 0.

**Proof:** Choose two different metrics on \( X, d_1 \) and \( d_2 \): in the first metric all distances between different points are equal to 1, the second is the same, except
the distance between $x_i$ and $x_j$ is equal to 2. Subtracting the corresponding equalities (1), we obtain $a_{ij} = 0$. □

Lemma 2. Any pseudometric $d$ on a finite $X = \{x_1, \ldots, x_m\}$, $m > 2$, may be expressed as a linear combination of $E_{ij}$ ($E_{ij}$ is defined as a pseudometric on $X$ gluing together points $x_i$ and $x_j$, while all other non-zero distances are equal to 1), i.e. there exist real $e_{ij}$ such that

\[ d = \sum_{i<j} e_{ij}E_{ij}. \tag{2} \]

Proof: Evaluating both sides of (2) on the pair $(x_k, x_l)$ we receive the following linear system of equations (in terms of $e$’s):

\[ d(x_k, x_l) = \sum_{i<j} e_{ij}E_{ij}(x_k, x_l) = -e_{kl} + \sum_{i<j} e_{ij}. \tag{3} \]

Summing the above equality over all pairs $(x_k, x_l)$ we have $\sum_{i<j} d(x_i, x_j) = (m^2-m-2)\sum_{i<j} e_{ij}$ and finally (taking into the account (3)):

\[ e_{kl} = \frac{2\sum_{i<j} d(x_i, x_j)}{m^2 - m - 2} - d(x_k, x_l). \tag{4} \]

Proof of the Theorem: Suppose that there is a one-element orbit: for some $k \forall g \in G$ $g(k) = k$. We may define $T = (Pr_k)^*$, where $Pr_k : SP^n_G \to Id$ is natural transformation of functors, taking $[x_1, \ldots, x_n]$ to $x_k$. The explicit formula looks as (here and further on we omit sometimes subscripts for the clarity of language):

\[ T(d)([x_1, \ldots, x_n], [y_1, \ldots, y_n]) = d(x_k, y_k). \]

The routine verification will show that so defined $T$ is a desired LFOEP.

Conversely, suppose that such operator $T$ exists and there is no stationary elements in $n$ with respect to $G$. Consider some finite $X$, $|X| \geq 2n$ and calculate $T(d)$ on elements $[x_1, \ldots, x_n]$ and $[y_1, \ldots, y_n]$ where all $x_i$ and $y_i$ are different. Taking into the account (2) and (4) and using the linearity of $T$, we have:

\[ T(d)([x_1, \ldots, x_n], [y_1, \ldots, y_n]) = \sum_{i<j} e_{ij}T(E_{ij})([x_1, \ldots, x_n], [y_1, \ldots, y_n]) \]

\[ = \sum_{i,j} a_{ij}d(x_i, y_j) + \sum_{i<j} b_{ij}d(x_i, x_j) + \sum_{i<j} c_{ij}d(y_i, y_j) \tag{5} \]

for some real constant $a_{ij}, b_{ij}, c_{ij}$. Note, that is general all coefficients $e_{ij}$ are not necessarily nonnegative, but formula (5) still holds. Really, if for pseudometrics
$d_1$ and $d_2$ the function $d_1 - d_2$ (pointwise subtraction) is a pseudometric, then $T(d_1) = T(d_2 + (d_1 - d_2)) = T(d_2) + T(d_1 - d_2)$, so $T(d_1 - d_2) = T(d_1) - T(d_2)$, for any linear $T$.

From functoriality of $T$ we can read that formula (5) is true for all $X$, $d$ and distinct $x_i, y_i \in X$: just consider embeddings of some fixed space with $2n$ points mapping it onto $\{x_1, \ldots, x_n, y_1, \ldots y_n\}$. It must be true for all (not necessarily distinct) $x_i, y_i$ as $T(d)$ is continuous function on $X^2$: take appropriate connected metric space, and consider limits of both sides of (5) when some of $x$’s and $y$’s approach each other.

Now, $T(d)$ as a pseudometric is symmetric. So, swap $y$ and $x$ in (5) and compare. We obtain:

$$
\sum_{i<j} d(x_i, x_j)(b_{ij} - c_{ij}) + \sum_{i<j} d(y_i, y_j)(c_{ij} - b_{ij}) + \sum_{i,j} d(x_i, y_j)(a_{ij} - a_{ji}) = 0
$$

and, according to Lemma 1,

$$
b_{ij} = c_{ij} \text{ and } a_{ij} = a_{ji}.
$$

Next, $T(d)([x_1, \ldots, x_n], [x_1, \ldots, x_n]) = 0$. After simple transformations we obtain: $\sum_{i<j} d(x_i, x_j)(a_{ij} + a_{ji} + b_{ij} + c_{ij}) = 0$. Therefore (applying (6)):

$$
a_{ij} = a_{ji} = -b_{ij} = -c_{ij}.
$$

Suppose that we have $g \in G$ which moves $k$ to $l$. Then, the two elements $[x, \ldots, x, z, x, \ldots, x]$ with one $z$ at $k$-th and $l$-th positions respectively are equivalent, and therefore, for every $[y_1, \ldots, y_n]$ formula (5) should yield the same values. After routine transformations we obtain: $\sum_i d(z, y_i)(a_{ki} - a_{li}) + (\text{other terms}) = 0$. Therefore for all $i$ $a_{ki} = a_{li}$. So, assuming (6) $a_{ij} = a_{kl}$, if $i$ and $k$ are $G$-related and $j$ and $l$ are $G$-related. The same is true for $b$’s and $c$’s.

If we have a 2-element orbit (let it be $\{1, 2\}$) then consider the following three points $[x, x, z, \ldots, z]$, $[y, y, z, \ldots, z]$ and $[x, y, z, \ldots, z]$ and use all that we know about the coefficients:

$$
T(d)([x, x, z, \ldots, z], [y, y, z, \ldots, z]) = 4a_{11}d(x, y),
$$

$$
T(d)([x, x, z, \ldots, z], [x, y, z, \ldots, z]) = a_{11}d(x, y),
$$

$$
T(d)([x, y, z, \ldots, z], [y, y, z, \ldots, z]) = a_{11}d(x, y).
$$

To satisfy the triangular inequality we must put $a_{11} = 0$.

If we have a $k$-element ($k > 2$) orbit (let it be $\{1, \ldots, k\}$) then consider the following two points in $SP^n_G(X)$: $[x_1, \ldots, x_k, z, \ldots, z]$ and $[y_1, \ldots, y_k, z, \ldots, z]$ with following original distances in $X$: all nonzero distances are 1 except $d(x_i, y_j) = 2$, all $i, j$. Calculate:

$$
T(d)([x_1, \ldots, x_k, z, \ldots, z], [y_1, \ldots, y_k, z, \ldots, z]) = (2k - k^2)a_{11}.
$$
Since, $2k - k^2 < 0$ when $k > 2$, $a_{11} \leq 0$.

So, if all orbits are non-degenerated then for all $i$ $a_{ii} \leq 0$. Finally, let us for some $x, y$ with $d(x, y) > 0$ find:

$$d(x, y) = T(d)([x, \ldots, x], [y, \ldots, y]) = \sum_i a_{ii}d(x, y) \leq 0.$$ 

Contradiction. \hfill \Box

References


