On variations of functions of one real variable

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Abstract. We discuss variations of functions that provide conceptually similar descriptive definitions of the Lebesgue and Denjoy-Perron integrals.

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The conceptual affinity between the Denjoy-Perron and Lebesgue integrals was established vis-à-vis their Riemannian definitions more than twenty years ago in the works of Henstock [6], Kurzweil [8], and McShane [10]. Yet, until recently, the descriptive definitions of these integrals have little in common. Modifying the variational measures of Thomson [15] and elaborating on a new result of Bougiorno, Di Piazza, and Skvortsov [2], we shall elucidate the similarities between the contemporary descriptive definitions of the Lebesgue integral, Denjoy-Perron integral, and $F$-integral of [12, Chapter 11].

Our ambient space is the real line $\mathbb{R}$. The interior, diameter, and the Lebesgue measure of a set $E \subset \mathbb{R}$ are denoted by $\text{int } E$, $d(E)$, and $|E|$, respectively. A set $E \subset \mathbb{R}$ with $|E| = 0$ is called negligible. The terms “almost everywhere” and “absolutely continuous” always refer to the Lebesgue measure in $\mathbb{R}$. For $x \in \mathbb{R}$ and $\varepsilon \geq 0$, we let $U(x, \varepsilon) = (x - \varepsilon, x + \varepsilon)$.

A cell is a compact nondegenerate subinterval of $\mathbb{R}$, and a figure is a finite (possibly empty) union of cells. We say figures $A$ and $B$ overlap if their interiors meet. With each nonempty figure $A$, we associate two numbers: the perimeter $\|A\|$ equal to twice the number of connected components of $A$, and the regularity $r(A) = \frac{|A|}{d(A)\|A\|}$. For completeness, we let $\|A\| = r(A) = 0$ whenever $A$ is the empty figure. Note that a figure $A$ is a cell whenever $r(A) \geq 1/4$, in which case $r(A) = 1/2$.

Unless specified otherwise, all functions we shall consider are real-valued. If $F$ is a function defined on a cell $A$ and $B$ is a subfigure of $A$ whose connected components are the cells $[a_1, b_1], \ldots, [a_n, b_n]$, we let

$$F(B) = \sum_{i=1}^{n} [F(b_i) - F(a_i)].$$
Clearly, \( F(B \cup C) = F(B) + F(C) \) whenever \( B \) and \( C \) are nonoverlapping subfigures of \( A \). Denoting by the same symbol both the function of points and the associated function of figures will lead to no confusion.

A nonnegative function \( \delta \) on a set \( E \subseteq \mathbb{R} \) is called a *gage* on \( E \) whenever its null set \( N_\delta = \{ x \in E : \delta(x) = 0 \} \) is countable. A *partition* is a collection (possibly empty) \( P = \{(A_1, x_1), \ldots, (A_p, x_p)\} \) such that \( A_1, \ldots, A_p \) are nonoverlapping figures, and \( x_i \in A_i \) for \( i = 1, \ldots, p \). Given \( \varepsilon > 0, \ E \subseteq \mathbb{R}^m, \) and a gage \( \delta \) on \( E \), we say that \( P \) is

1. *cellular* if each \( A_i \) is a cell;
2. \( \varepsilon \)-regular if \( r(A_i) > \varepsilon \) for \( i = 1, \ldots, p \);
3. in \( E \) if \( \bigcup_{i=1}^p A_i \subseteq E \);
4. anchored in \( E \) if \( \{ x_1, \ldots, x_p \} \subseteq E \);
5. \( \delta \)-fine if it is anchored in \( E \) and \( d(A_i) < \delta(x_i) \) for \( i = 1, \ldots, p \).

Given a positive gage \( \delta \) on \( A \), a collection \( Q = \{(B_1, y_1), \ldots, (B_q, y_q)\} \) is called a \( \delta \)-fine *McShane partition* in \( A \) if \( B_1, \ldots, B_q \) are nonoverlapping subcells of \( A \), each \( y_i \) is a point in \( A \), and \( d(B_i \cup \{ y_i \}) < \delta(y_i) \) for \( i = 1, \ldots, q \). If each \( y_i \) belongs to a set \( E \subset A \), we say \( Q \) is anchored in \( E \).

**Proposition 1.** A function \( f \) on a cell \( A \) is Lebesgue integrable in \( A \) if and only if there is a function \( F \) on \( A \) satisfying the following condition: given \( \varepsilon > 0 \), we can find a positive gage \( \delta \) on \( A \) so that

\[
\sum_{i=1}^p |f(x_i)|A_i - F(A_i)| < \varepsilon
\]

for each \( \delta \)-fine partition \( \{(A_1, x_1), \ldots, (A_p, x_p)\} \) in \( A \). The function \( F \) is the indefinite Lebesgue integral of \( f \) in \( A \); in particular, \( F \) is continuous.

**Proof:** The continuity of \( F \) at \( x \in A \) is easily established by choosing a sufficiently small positive gage \( \delta \) on \( A \) and considering a \( \delta \)-fine partition

\[
\{(A \cap [x-\eta, x+\eta], x)\}
\]

(see [12, Corollary 2.3.2] for details).

Suppose the condition of the proposition is satisfied, and select a \( \delta \)-fine McShane partition \( \{(B_1, y_1), \ldots, (B_q, y_q)\} \) in \( A \). Denote by \( x_1, \ldots, x_p \) the distinct points among \( y_1, \ldots, y_q \), and let \( C_i = \bigcup\{ B_j : y_j = x_i \} \). As \( F \) is continuous, there is a \( \delta \)-fine cellular partition \( \{(D_1, x_1), \ldots, (D_p, x_p)\} \) in \( A \) such that

\[
\sum_{i=1}^p \left[ |f(x_i)| \cdot |D_i| + |F(D_i)| \right] < \varepsilon
\]

and

\[
\sum_{i,k=1}^p \left[ |f(x_i)| \cdot |C_i \cap D_k| + |F(C_i \cap D_k)| \right] < \varepsilon.
\]
If \( A_i = D_i \cup (C_i - \bigcup_{k=1}^{p} D_k) \), then \( \{(A_1, x_1), \ldots, (A_p, x_p)\} \) is a \( \delta \)-fine partition in \( A \), and we have

\[
\varepsilon > \sum_{i=1}^{p} \left[ f(x_i)|A_i| - F(A_i) \right] = \sum_{i=1}^{p} \left[ f(x_i)|D_i| - F(D_i) \right] \\
+ \sum_{i=1}^{p} \left[ f(x_i)|C_i| - F(C_i) \right] - \sum_{i,k=1}^{p} \left[ f(x_i)|C_i \cap D_k| - F(C_i \cap D_k) \right] \\
> \sum_{j=1}^{q} \left[ f(y_j)|B_j| - F(B_j) \right] - 2\varepsilon.
\]

From this inequality we deduce \( \sum_{j=1}^{q} \left| f(y_j)|B_j| - F(B_j) \right| < 6\varepsilon \).

Conversely, suppose we can find a positive gage \( \delta \) on \( A \) so that

\[
\sum_{j=1}^{q} \left| f(y_j)|B_j| - F(B_j) \right| < \varepsilon
\]

for each \( \delta \)-fine McShane partition in \( A \), and select a \( \delta \)-fine partition \( \{(A_1, x_1), \ldots, (A_p, x_p)\} \) in \( A \). If \( A_{i,1}, \ldots, A_{i,n_i} \) are the connected components of \( A_i \), then

\[
\{(A_{i,j}, x_i) : j = 1, \ldots, n_i \text{ and } i = 1, \ldots, p\}
\]

is a \( \delta \)-fine McShane partition in \( A \), and we have

\[
\sum_{i=1}^{p} \left| f(x_i)|A_i| - F(A_i) \right| \leq \sum_{i=1}^{p} \sum_{j=1}^{n_i} \left| f(x_i)|A_{i,j}| - F(A_{i,j}) \right| < \varepsilon.
\]

Thus the condition of the theorem is equivalent to \( f \) being McShane integrable in \( A \), and the proposition follows from [5, Theorem 10.9]. \( \square \)

In Proposition 1, a positive gage is needed to assure the continuity of \( F \). If \( F \) is assumed continuous and a positive gage is replaced by an arbitrary gage, the condition of Proposition 1 defines an integral that is closed with respect to the formation of improper integrals, and thus slightly more general than the Lebesgue integral.

**Proposition 2.** A function \( f \) on a cell \( A \) is Denjoy-Perron integrable in \( A \) if and only if there is a continuous function \( F \) on \( A \) satisfying the following condition: given \( \varepsilon > 0 \), we can find a gage \( \delta \) on \( A \) so that

\[
\sum_{i=1}^{p} \left| f(x_i)|A_i| - F(A_i) \right| < \varepsilon
\]
for each \( \delta \)-fine cellular partition \( \{(A_1, x_1), \ldots, (A_p, x_p)\} \) in \( A \). The function \( F \) is the indefinite Denjoy-Perron integral of \( f \) in \( A \).

**Proof:** In view of [5, Chapter 11], it suffices to show that if the condition of the proposition holds, it holds already for a positive gage \( \delta_+ \). To this end, enumerate the null set \( N_\delta \) of \( \delta \) as \( z_1, z_2, \ldots \), and find \( \theta_n > 0 \) so that
\[
|f(z_n) \cdot |C| + |F(C)| < 2^{-n} \varepsilon
\]
for each cell \( C \subset U(z_n, \theta_n) \) and \( n = 1, 2, \ldots \). Now let
\[
\delta_+(x) = \begin{cases} 
\theta_n & \text{if } x = z_n \text{ for an integer } n \geq 1, \\
\delta(x) & \text{if } x \in A - N_\delta.
\end{cases}
\]
Given a \( \delta_+ \)-fine cellular partition \( \{(A_1, x_1), \ldots, (A_p, x_p)\} \), observe that
\[
\sum_{i=1}^{p} |f(x_i)|A_i| - F(A_i)| < \sum_{\delta(x_i)>0} |f(x_i)|A_i| - F(A_i)| + \sum_{n=1}^{\infty} 2^{-n} \varepsilon < 2\varepsilon,
\]
which establishes the proposition. \( \square \)

According to [5, Chapter 11], a gage in Proposition 2 can be replaced by a positive gage, in which case the continuity of \( F \) can be deduced as in Proposition 1. However, a slight modification of [12, Example 12.3.5] shows that Proposition 2 is false when cellular partitions, which are \((1/4)\)-regular partitions, are replaced by \( \alpha \)-regular partitions with \( \alpha < 1/4 \).

Propositions 1 and 2 lead to the definition of the \( \mathcal{F} \)-integral, which lies properly in between the Lebesgue and Denjoy-Perron integrals. It was introduced in [13] as a coordinate free multidimensional integral that integrates partial derivatives of differentiable functions (cf. [11]).

**Definition 3.** A function \( f \) on a cell \( A \) is called \( \mathcal{F} \)-integrable in \( A \) whenever there is a continuous function \( F \) on \( A \) satisfying the following condition: given \( \varepsilon > 0 \), we can find a gage \( \delta \) on \( A \) so that
\[
\sum_{i=1}^{p} |f(x_i)|A_i| - F(A_i)| < \varepsilon
\]
for each \( \delta \)-fine \( \varepsilon \)-regular partition \( \{(A_1, x_1), \ldots, (A_p, x_p)\} \) in \( A \). The function \( F \), uniquely determined by \( f \), is called the indefinite \( \mathcal{F} \)-integral of \( f \) in \( A \).

We note that the additivity properties of the \( \mathcal{F} \)-integral depend on the use of arbitrary, not necessarily positive, gages.
Remark 4. One may also consider the integrals defined by means of $\alpha$-regular partitions, where $0 < \alpha < 1/4$ is a fixed number. Whether different $\alpha$’s produce different integrals is unclear, however, the work of Jarník and Kurzweil [9] suggests this may be the case. We do not study these integrals, since they may not be invariant with respect to diffeomorphisms (a diffeomorphic image of an $\alpha$-regular figure need not be $\alpha$-regular).

Let $F$ be a function defined on a cell $A$, and let $E \subset A$ be an arbitrary set. Elaborating on the ideas of B.S. Thomson [15, Chapter 3], we define variations of $F$ corresponding to the integrals discussed earlier.

**Lebesgue variation:**

$$V^L F(E) = \inf_\delta \sup_P \sum_{i=1}^{p} |F(A_i)|$$

where $\delta$ is a positive gage on $E$ and $P = \{(A_1, x_1), \ldots, (A_p, x_p)\}$ is a $\delta$-fine partition in $A$ anchored in $E$.

**Denjoy-Perron variation:**

$$V^{DP} F(E) = \inf_\delta \sup_P \sum_{i=1}^{p} |F(A_i)|$$

where $\delta$ is a gage on $E$ and $P = \{(A_1, x_1), \ldots, (A_p, x_p)\}$ is a $\delta$-fine cellular partition in $A$ anchored in $E$.

**$\mathcal{F}$-variation:**

$$V^{\mathcal{F}} F(E) = \sup_\alpha \inf_\delta \sup_P \sum_{i=1}^{p} |F(A_i)|$$

where $\alpha > 0$, $\delta$ is a gage on $E$, and $P = \{(A_1, x_1), \ldots, (A_p, x_p)\}$ is a $\delta$-fine $\alpha$-regular partition in $A$ anchored in $E$.

Arguments analogous to those of [15, Theorems 3.7 and 3.15] reveal that the extended real-valued functions $V^L F$, $V^{DP} F$, and $V^{\mathcal{F}} F$ are Borel regular measures in $A$ (cf. [12, Lemma 3.3.14] and [3, Lemma 4.6]). We shall use this important fact in the proof of Proposition 6 below. The inequalities

$$V^{DP} F \leq V^{\mathcal{F}} F \leq V^L F$$

follow directly from the definitions.

Remark 5. Let $F$ be a continuous function on a cell $A$. Employing ideas which proved Proposition 1, it is easy to show that in defining $V^L F(E)$ we can use $\delta$-fine McShane partitions. Similarly, $V^{DP} F(E)$ can be defined by positive gages (cf. [2, Proposition 6] and the proof of Proposition 2).

If $F$ is a function on a cell $A$, we denote by $VF(B)$ the usual variation of $F$ over a figure $B \subset A$ [5, Chapter 4].
Proposition 6. If $F$ is a continuous function in a cell $A$, then

\[ V^{DP} F(B) = V^F F(B) = VF(B) \]

for each figure $B \subset A$, and $V^L F(A) = VF(A)$. Moreover, $V^{DP} F = V^F F$ whenever $V^F F$ is $\sigma$-finite, and $V^F F = V^L F$ whenever $V^L F$ is $\sigma$-finite.

Proof: Equality (2), which is an easy consequence of generalized Cousin’s lemma [7, Lemma 6], was established in [1, Proposition 4.8].

If $V^F F$ is $\sigma$-finite, then $V^{DP} F$ and $V^F F$ vanish on all but countably many singletons. Thus it is not difficult to deduce from (2) that $V^{DP} F(U) = V^F F(U)$ for each relatively open set $U \subset A$ (see [12, Lemma 3.4.4] for details). As $V^{DP} F$ and $V^F F$ are $\sigma$-finite Borel regular measures in $A$, they coincide.

Let $B$ be a subfigure of $A$, and let $\text{int}_A B$ be the relative interior of $B$ in $A$. Choose a positive gage $\delta$ on $\text{int}_A B$ so that $A \cap U(x, \delta(x)) \subset B$ for each $x \in \text{int}_A B$, and let $\{(A_1, x_1), \ldots, (A_p, x_p)\}$ be a $\delta$-fine partition in $A$ anchored in $\text{int}_A B$. By the choice of $\delta$, each $A_i$ is contained in $B$, and so if $A_{i,1}, \ldots, A_{i,k_i}$ are the connected components of $A_i$, then

\[ \sum_{i=1}^{p} |F(A_i)| \leq \sum_{i=1}^{p} \sum_{j=1}^{k_i} |F(A_{i,j})| \leq VF(B). \]

From this and (1), we obtain

\[ V^F F(\text{int}_A B) \leq V^L F(\text{int}_A B) \leq VF(B); \]

in particular, $V^L F(A) = VF(A)$ by (2). Using (3), the proof is completed by the argument employed in the previous paragraph.

\[ \Box \]

Lemma 7. Let $F$ be a function on a cell $A$. If $VF^L(\{x\}) = 0$ for each $x \in A$, then $VF^L(A) < +\infty$.

Proof: Observe first $F$ is continuous at $x \in A$ whenever $V^L F(\{x\}) = 0$. According to Remark 5, for each $y \in A$, there is an $\eta_y > 0$ such that $\sum_{j=1}^q |F(B_j)| < 1$ for every $\eta_y$-fine McShane partition $\{(B_1, y_1), \ldots, (B_q, y_q)\}$ in $A$ anchored in $\{y\}$, i.e., with $y_1 = \cdots = y_q = y$. Since $A$ is compact, we can find points $z_1, \ldots, z_n$ in $A$ so that $A$ is covered by $U(z_1, \eta_{z_1}), \ldots, U(z_n, \eta_{z_n})$. Define a positive gage $\delta$ on $A$ as follows: given $x \in A$, select a $\delta(x) > 0$ so that $U(x, \delta(x))$ is contained in some $U(z_k, \eta_{z_k})$. Now each $\delta$-fine McShane partition $\{(A_1, x_1), \ldots, (A_p, x_p)\}$ in $A$ is the disjoint union of families $P_1, \ldots, P_n$ such that $A_i \subset U(z_k, \eta_{z_k})$ whenever $(A_i, x_i) \in P_k$. It follows that $\{(A_i, z_k) : (A_i, x_i) \in P_k\}$ is an $\eta_{z_k}$-fine McShane partition in $A$ anchored in $\{z_k\}$, and so

\[ \sum_{i=1}^{p} |F(A_i)| = \sum_{k=1}^{n} \sum_{(A_i, x_i) \in P_k} |F(A_i)| < n. \]

In view of this and Remark 5, we have $VF^L(A) \leq n$. \[ \Box \]
Proposition 8. A function $F$ in a cell $A$ is absolutely continuous if and only if $V^L F$ is absolutely continuous.

Proof: Let $F$ be absolutely continuous, and choose an $\eta > 0$ and a negligible set $E \subset A$. There is a $\delta > 0$ such that $\sum_{j=1}^n |F(B_j)| < \varepsilon$ for each collection $B_1, \ldots, B_n$ of nonoverlapping subcells of $A$ with $\sum_{j=1}^n |B_j| < \eta$. Find an open set $U$ containing $E$ so that $|U| < \eta$, and select a positive gage $\delta$ on $E$ such that $U(x, \delta(x)) \subset U$ for each $x \in E$. Now if $\{(A_1, x_1), \ldots, (A_p, x_p)\}$ is a $\delta$-fine partition in $A$ anchored in $E$, then it is a partition in $U$. If $A_{i,1}, \ldots, A_{i,n_i}$ are the connected components of $A_i$, then

$$\sum_{i=1}^p |F(A_i)| \leq \sum_{i=1}^p \sum_{j=1}^{n_i} |F(A_{i,j})| < \varepsilon,$$

and $V^L F(E) = 0$ by the arbitrariness of $\varepsilon$.

Conversely, assume that $V^L F$ is absolutely continuous, and choose an $\varepsilon > 0$. In view of Lemma 7, there is an $\eta > 0$ such that $V^L F(E) < \varepsilon$ whenever $E \subset A$ and $|E| < \eta$ [14, Theorem 6.11]. If $B \subset A$ is the union of nonoverlapping cells $B_1, \ldots, B_n$ and $|B| < \eta$, then Proposition 6 implies

$$\sum_{j=1}^n |F(B_j)| \leq \sum_{j=1}^n V^F(B_j) = V^F(B) = V^{DP} F(B) \leq V^L F(B) < \varepsilon,$$

establishing the absolutely continuous of $F$.  

We shall use the expression “$F$ is the indefinite integral of its derivative,” which has the following usual meaning: the function $F$ is differentiable almost everywhere in its domain, and it is the indefinite integral of $F'$ extended arbitrarily to the domain of $F$.

Theorem 9. A function $F$ on a cell $A$ is the indefinite Lebesgue integral of its derivative if and only if $V^L F$ is absolutely continuous.

Proof: The theorem follows from Proposition 8 and [5, Theorem 4.15].  

Corollary 10. A function $F$ on a cell $A$ is the indefinite Lebesgue integral of its derivative whenever $V^{DP} F$ is absolutely continuous and $V^L F$ is $\sigma$-finite.

Proof: If $V^L F$ is $\sigma$-finite, then $V^L F = V^{DP} F$ by Proposition 6, and the corollary follows from Theorem 9.  

Proposition 11. Let $F$ be a continuous function on a cell $A$. If $V^{DP} F$ is absolutely continuous it is $\sigma$-finite.

Proof: In a roundabout way the proposition was proved in [2, Theorem 5]. We present a direct proof, which is virtually identical to that of [2, Theorem 1].
Suppose \( V^{DP}F \) is absolutely continuous but not \( \sigma \)-finite, and denote by \( U_0 \) the union of all open sets \( U \) with \( V^{DP}F(A \cap U) < +\infty \). Since \( U_0 \) is Lindelöf, the \( V^{DP}F \) measure of \( A \cap U_0 \) is \( \sigma \)-finite. The set \( K = A - U_0 \) is compact, and it is easy to verify that \( V^{DP}F(K \cap U) = +\infty \) for each open set \( U \) which meets \( K \). As \( V^{DP}F(\{x\}) = 0 \) for every \( x \in A \), the set \( K \) is perfect.

\textit{Claim.} If \( U \) is an open set which meets \( K \), then \( A \cap U \) contains a disjoint collection \( A_1, \ldots, A_p \) of at least two cells such that the interior of each \( A_i \) meets \( K \), and

\[
(4) \quad \sum_{i=1}^{p} |F(A_i)| > 1.
\]

\textbf{Proof:} Select a gage \( \delta \) on \( K \cap U \) so that \( U(x, \delta(x)) \subset U \) for each \( x \in K \cap U \). There is a \( \delta \)-fine cellular partition \( \{(A_1, x_1), \ldots, (A_p, x_p)\} \) in \( A \) anchored in \( K \cap U \) such that (4) holds. By the choice of \( \delta \), each \( A_i \) is contained in \( A \cap U \). Since \( F \) is continuous and \( K \) is perfect, we can modify the cells \( A_i \) so that they become disjoint, their interiors meet \( K \), and they are still contained in \( A \cap U \) and satisfy (4). If \( p = 1 \) and \( A_1 = [a, b] \), find points \( c \) and \( d \) so that \( a < c < d < b \) and both \( (a, c) \) and \( (d, b) \) meet \( K \). As \( F \) is continuous and

\[
1 < |F(A_1)| \leq |F([a, c])| + |F([c, d])| + |F([d, b])|,
\]

the points \( c \) and \( d \) can be selected so that \( 1 < |F([a, c])| + |F([d, b])| \). Thus we may assume \( p \geq 2 \), and the claim is established.

Using the claim, construct inductively disjoint families \( \{A_{k,1}, \ldots, A_{k,p_k}\} \) of subcells of \( A \) so that the following conditions are satisfied for \( k = 1, 2, \ldots \).

1. \( K \cap \text{int} A_{k,i} \neq \emptyset \) for \( i = 1, \ldots, p_k \).
2. Each \( A_{k+1,j} \) is contained in some \( A_{k,i} \).
3. Each \( A_{k,i} \) contains at least two cells \( A_{k+1,j} \).
4. \( \bigcup_{i=1}^{p_k} A_{k,i} < 1/k \).
5. \( \sum_{A_{k+1,j} \subset A_{k,i}} |F(A_{k+1,j})| > 1 \) for \( i = 1, \ldots, p_k \).

It follows from conditions 3 and 4 that \( N = \bigcap_{k=1}^{\infty} \bigcup_{i=1}^{p_k} A_{k,i} \) is a negligible perfect subset of \( A \). We obtain a contradiction by showing that \( V^{DP}F(N) \geq 1 \).

To this end, choose a gage \( \delta \) on \( N \), and for \( k = 1, 2, \ldots \), let

\[
N_k = \{x \in N : \delta(x) > 1/k\}.
\]

Since the set \( \bigcup_{k=1}^{\infty} N_k = N - N_\delta \) is completely metrizable [4, Theorem 4.3.23]. By the Baire category theorem some \( N_s \) is dense in \( (N - N_\delta) \cap U \), where \( U \) is an open set which meets \( N - N_\delta \). There is an integer \( k > s \) such that some \( A_{k-1,j} \) is contained in \( U \). Condition 4 implies that \( d(A_{k,i}) < 1/s \) for \( i = 1, \ldots, p_k \). Hence choosing \( x_i \in A_{k,i} \cap N_s \), we obtain a \( \delta \)-fine cellular partition \( \{(A_{k,1}, x_1), \ldots, (A_{k,p_k}, x_{p_k})\} \) in \( A \) anchored in \( N \). The desired contradiction follows from condition 5.

\[\square\]
Theorem 12. A continuous function \( F \) on a cell \( A \) is the indefinite Denjoy-Perron integral of its derivative if and only if \( V^{DP} F \) is absolutely continuous.

Proof: The theorem follows from Proposition 11 and [1, Theorem 4.4], which asserts that \( F \) is the indefinite Denjoy-Perron integral of its derivative if and only if \( V^{DP} F \) is absolutely continuous and \( \sigma \)-finite. \( \square \)

Theorem 13. A continuous function \( F \) on a cell \( A \) is the indefinite integral of its derivative if and only if \( V^{F} F \) is absolutely continuous.

Proof: As the converse follows from [3, Theorem 5.3], assume \( V^{F} F \) is absolutely continuous. Then \( V^{DP} F \) is absolutely continuous by (1), and Theorem 12 implies that \( F \) is differentiable at each \( x \in A - N \), where \( N \) is a negligible subset of \( A \). We show that \( F \) is the indefinite integral of the function \( f \) defined by the formula

\[
f(x) = \begin{cases} 
  F'(x) & \text{if } x \in A - N, \\
  0 & \text{if } x \in N.
\end{cases}
\]

To this end, choose an \( \varepsilon > 0 \), and for each \( x \in A - N \), find an \( \eta_x > 0 \) so that

\[
|F'(x)|B - F(B)| < \varepsilon^2 d(B)\|B\|
\]

for each figure \( B \subset A \cap U(x, \eta_x) \); the existence of \( \eta_x \) is a readily verifiable consequence of the differentiability of \( F \) at \( x \). By our assumption, there is a gage \( \beta \) on \( N \) such that \( \sum_{i=1}^{p}|F(A_i)| < \varepsilon \) for each \( \beta \)-fine \( \varepsilon \)-regular partition \( \{(A_1, x_1), \ldots, (A_p, x_p)\} \) in \( A \) anchored in \( N \). Let

\[
\delta(x) = \begin{cases} 
  \eta_x & \text{if } x \in A - N, \\
  \beta(x) & \text{if } x \in N,
\end{cases}
\]

and select a \( \delta \)-fine \( \varepsilon \)-regular partition \( \{(A_1, x_1), \ldots, (A_p, x_p)\} \) in \( A \). Then

\[
\sum_{i=1}^{p}|f(x)|A_i - F(A_i)| = \sum_{x_i \in N} |F(A_i)| + \varepsilon^2 \sum_{x_i \notin N} d(B)\|B\|
\]

\[
< \varepsilon + \varepsilon \sum_{x_i \notin N} |A| \leq \varepsilon (1 + |A|),
\]

and the theorem is proved. \( \square \)

Corollary 14. Let \( F \) be a continuous function on a cell \( A \). If \( V^{F} F \) is absolutely continuous it is \( \sigma \)-finite.

Proof: In view of Theorem 13, the function \( F \) is the indefinite integral of a function \( f \) on \( A \). Fix an integer \( n \geq 1 \) and let \( E = \{x \in A : |f(x)| < n\} \). Since

\[
A = \bigcup_{k=1}^{\infty} \{x \in A : |f(x)| < k\},
\]
it suffices to show that $V^F F(E) < +\infty$. To this end, choose a positive $\varepsilon \leq 1$, and find a gage $\delta$ on $A$ so that

$$\sum_{i=1}^{p} |f(x)|A_i| - F(A_i)| < \varepsilon$$

for each $\delta$-fine $\varepsilon$-regular partition in $A$. If such a partition is anchored in $E$, then

$$\sum_{i=1}^{p} |F(A_i)| \leq \sum_{i=1}^{p} |f(x)|A_i| - F(A_i)| + \sum_{i=1}^{p} |f(x)| \cdot |A_i|$$

$$< \varepsilon + n \sum_{i=1}^{p} |A_i| \leq 1 + n|A|,$$

and we conclude that $V^F F(E) \leq 1 + n|A|$. \hfill \Box

**Corollary 15.** A continuous function $F$ on a cell $A$ is the indefinite $\mathcal{F}$-integral of its derivative whenever $V^{DP} F$ is absolutely continuous and $V^F F$ is $\sigma$-finite.

**Proof:** If $V^F F$ is $\sigma$-finite, then $V^F F = V^{DP} F$ by Proposition 6, and the corollary follows from Theorem 13. \hfill \Box

**References**


On variations of functions of one real variable


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