

A note on Möbius inversion over power set lattices

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Abstract. In this paper, we establish a theorem on Möbius inversion over power set lattices which strongly generalizes an early result of Whitney on graph colouring.

Keywords: Möbius inversion, power set lattices, graphs, hypergraphs, colourings

Classification: 05C15, 05C65, 05A15, 06A07, 06E99

1. Introduction

An important technique in combinatorics is the principle of Möbius inversion over partially ordered sets (see [3, Chapter 25]). For power set lattices, the principle of Möbius inversion states the following:

Proposition. *Let S be a finite set, f and g mappings from the power set of S into an additive group such that $g(X) = \sum_{Y \in [X,S]} f(Y)$ for any $X \subseteq S$, where $[X, S]$ denotes the interval $\{Y \mid X \subseteq Y \subseteq S\}$. Then, for any $X \subseteq S$,*

$$(1) \quad f(X) = \sum_{Y \in [X,S]} (-1)^{|Y \setminus X|} g(Y).$$

PROOF: By the asserted relation between f and g , the sum in (1) equals

$$\sum_{Y \in [X,S]} (-1)^{|Y \setminus X|} \sum_{Z \in [Y,S]} f(Z) = \sum_{Z \in [X,S]} f(Z) \sum_{Y \in [X,Z]} (-1)^{|Y \setminus X|},$$

and this is $f(X)$ since the inner sum on the right is zero unless $X = Z$. □

2. A modified inversion formula

The following theorem states that under certain conditions not all terms have to be considered when evaluating the sum in (1). It can be thought of as a generalization of Whitney’s Broken-Circuits-Theorem on graph colouring.

Theorem. *Let S be a poset and f, g mappings from the power set of S into an additive group such that $g(X) = \sum_{Y \in [X,S]} f(Y)$ for any $X \subseteq S$. For fixed $X \subseteq S$, let \mathcal{C} be a set of non-empty subsets of S such that each $C \in \mathcal{C}$ is bounded*

from below by an element $\underline{C} \in S \setminus (C \cup X)$ and $f(Y) = 0$ for all Y including $C \cup X$ and not containing \underline{C} . Then

$$(2) \quad f(X) = \sum_{Y \in [X, S] \cap \mathcal{Y}_0} (-1)^{|Y \setminus X|} g(Y),$$

where

$$(3) \quad \mathcal{Y}_0 := \{Y \subseteq S \mid Y \not\supseteq C \text{ for all } C \in \mathcal{C}\}.$$

PROOF: Let \leq denote the partial ordering relation on S and \leq^* one of its linear extensions. For each subset Y of S , $\min^* Y$ denotes the minimum of Y with respect to \leq^* . Consider an enumeration C_1, \dots, C_n of \mathcal{C} such that $\min^* C_1 \leq^* \dots \leq^* \min^* C_n$, and define $\mathcal{Y}_m := \{Y \subseteq S \mid C_m \subseteq Y, C_{m+1} \not\subseteq Y, \dots, C_n \not\subseteq Y\}$ for $m = 1, \dots, n$. Obviously, the power set of S is the disjoint union of $\mathcal{Y}_0, \dots, \mathcal{Y}_n$. The proposition gives

$$f(X) = \sum_{m=0}^n \sum_{Y \in [X, S] \cap \mathcal{Y}_m} (-1)^{|Y \setminus X|} g(Y).$$

We claim that the inner sum on the right-hand side is zero for $m = 1, \dots, n$. The assertions force $\underline{C}_m < c$ and hence $\underline{C}_m <^* c$ for every $c \in C_m$. From the latter we conclude $\underline{C}_m <^* \min^* C_m \leq^* \min^* C_k$ and therefore $\underline{C}_m \notin C_k$ for $k = m, \dots, n$. For such k , $C_k \subseteq Y$ if and only if $C_k \subseteq Y_m$ where $Y_m := (Y \setminus \{\underline{C}_m\}) \cup (\{C_m\} \setminus Y)$. By this, $Y \in \mathcal{Y}_m$ if and only if $Y_m \in \mathcal{Y}_m$. In addition, $X \subseteq Y$ if and only if $X \subseteq Y_m$. Hence,

$$\sum_{Y \in [X, S] \cap \mathcal{Y}_m} (-1)^{|Y \setminus X|} g(Y) = \frac{1}{2} \sum_{Y \in [X, S] \cap \mathcal{Y}_m} \left((-1)^{|Y \setminus X|} g(Y) + (-1)^{|Y_m \setminus X|} g(Y_m) \right).$$

Since $|Y \setminus X| \not\equiv |Y_m \setminus X| \pmod{2}$, it suffices to check $g(Y) = g(Y_m)$ for all $Y \in [X, S] \cap \mathcal{Y}_m$. By the asserted relation between f and g ,

$$g(Y) = \sum_{\substack{Z \in [Y, S], \\ \underline{C}_m \notin Z}} f(Z) + \sum_{\substack{Z \in [Y, S], \\ \underline{C}_m \in Z}} f(Z).$$

It is easy to see that the right sum remains unchanged when Y is replaced by Y_m . The same holds for the left sum which, by the assertions of the theorem, equals zero. □

Remark. To compare the number of terms in (1) and (2), we define $\chi := |\mathcal{Y}_0 \cap [X, S]|/|[X, S]|$. Obviously, $0 \leq \chi \leq 1$. By the well-known principle of inclusion and exclusion (which is a particular case of the next corollary),

$$(4) \quad \chi = \sum_{C' \subseteq C} (-1)^{|C'|} 2^{|X| - |X \cup \bigcup_{C \in C'} C|}.$$

Hence, if C contains n pairwise disjoint sets of cardinality m ($n \in \mathbb{N}_0, m \in \mathbb{N}$) all of them being disjoint with X , then $\chi \leq (1 - 2^{-m})^n$, and this tends to zero as $n \rightarrow \infty$. \square

Corollary. Let \mathcal{A} be a boolean algebra of sets, P a mapping from \mathcal{A} into an additive group such that $P(\emptyset) = 0$ and $P(A \cup B) = P(A) + P(B)$ for all disjoint pairs $A, B \in \mathcal{A}$, S a finite poset, $\{A_s\}_{s \in S} \subseteq \mathcal{A}$, $X \subseteq S$ and \mathcal{C} a set of non-empty subsets of S such that each $C \in \mathcal{C}$ is bounded from below by an element $\underline{C} \in S \setminus (C \cup X)$ and $\bigcap_{c \in C} A_c \subseteq \underline{A}_{\underline{C}}$. Then

$$P \left(\bigcap_{x \in X} A_x \cap \bigcap_{s \in S \setminus X} \complement A_s \right) = \sum_{Y \in [X, S] \cap \mathcal{Y}_0} (-1)^{|Y \setminus X|} P \left(\bigcap_{y \in Y} A_y \right),$$

where \mathcal{Y}_0 is defined as in (3) and $\complement A_s$ denotes the complement of A_s in \mathcal{A} .

PROOF: For $Y \subseteq S$ define $f(Y) := P(\bigcap_{y \in Y} A_y \cap \bigcap_{s \in S \setminus Y} \complement A_s)$, $g(Y) := P(\bigcap_{y \in Y} A_y)$. For Y including C and not containing \underline{C} there is some $B \in \mathcal{A}$ such that $f(Y) = P(\bigcap_{c \in C} A_c \cap \complement A_{\underline{C}} \cap B)$, and hence $f(Y) = 0$. Therefore, the theorem can be applied. \square

Remark. Let X be empty and S_{\min} resp. S_{\max} denote the set of minimal resp. maximal elements of S . If the mapping $s \mapsto A_s$ is antitone, then it can be achieved that \mathcal{Y}_0 is the power set of S_{\min} (Proof: Set $\mathcal{C} := \{\{s\} \mid s \in S \setminus S_{\min}\}$, and for each $C \in \mathcal{C}$ choose a lower bound $\underline{C} \in S \setminus C$). By the duality principle for posets, ‘below’ can be replaced by ‘above’ both in the theorem and in the corollary. By this, if $s \mapsto A_s$ is isotone, then it can be achieved that \mathcal{Y}_0 becomes the power set of S_{\max} . \square

Example 1. In (4), \mathcal{C} can be replaced by the set of its minimal elements with respect to set inclusion. This is an immediate consequence of the corollary and the preceding remark since $C \mapsto [C, S]$ is an antitone mapping. \square

Example 2. A *hypergraph* is a set S of non-empty sets whose union $\bigcup S$ is finite. The elements of S resp. $\bigcup S$ are the *edges* resp. *vertices* of the hypergraph; their number is denoted by $m(S)$ resp. $n(S)$. Define $m^*(S) := \sum_{s \in S} (|s| - 1)$. For $k \in \mathbb{N}$, let $S^{(k)}$ consist of all k -element edges of S . The edges of $S^{(1)}$ are called *loops*. The subsets of S are called *partial hypergraphs* of S . A *cycle* in S is a sequence $(v_1, s_1, \dots, v_k, s_k)$ where $k > 1$ and v_1, \dots, v_k resp. s_1, \dots, s_k are

distinct vertices resp. edges, $v_i, v_{i+1} \in s_i$ for $i = 1, \dots, k-1$ and $v_k, v_1 \in s_k$. With respect to a linear ordering relation on S , a *broken circuit* of S is obtained from the edge-set of a cycle in S by removing the smallest edge. For any $\lambda \in \mathbb{N}$, a λ -colouring of S is a mapping $f : \bigcup S \longrightarrow \{1, \dots, \lambda\}$ (the set of *colours*). For $X \subseteq S$, $P_{S,X}(\lambda)$ stands for the number of λ -colourings of S such that X is the set of monochromatic edges. We now establish the following statement:

Let S be a loop-free, linearly ordered hypergraph, and let X be a partial hypergraph of S such that $S^{(2)} \setminus X$ is an initial segment of S and each cycle in S has an edge of $S^{(2)} \setminus X$. Then $P_{S,X}(\lambda) = \sum_{i,j} \rho_{ij} \lambda^{n(S)-i}$ where ρ_{ij} equals $(-1)^{j-|X|}$ times the number of partial hypergraphs Y of S including X but no broken circuits of S and satisfying $m^(Y) = i$ and $m(Y) = j$.*

PROOF: For $s \in S$ define A_s as the set of λ -colourings of S such that s is monochromatic. For any broken circuit C of S let \underline{C} be the unique edge such that $C \cup \{\underline{C}\}$ is the edge-set of a cycle in S . The assertions force $\underline{C} \in S^{(2)} \setminus (C \cup X)$. Obviously, $\underline{C} \in S^{(2)}$ entrains $\bigcap_{c \in C} A_c \subseteq A_{\underline{C}}$. By the corollary, $P_{S,X}(\lambda) = \sum_Y (-1)^{|Y \setminus X|} |\bigcap_{y \in Y} A_y|$ where the summation is extended over all partial hypergraphs Y of S including X but no broken circuits of S . By [1, Proposition], $|\bigcap_{y \in Y} A_y| = \lambda^{n(S)-m^*(Y)}$. The result now follows. \square

Note. A particular case of the previous example, namely where X is empty, is published in [2]. For simple graphs and empty X , the above statement is due to Whitney (see [4]) and called *Whitney's Broken-Circuits-Theorem*. \square

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(Received September 26, 1995)