A note on Möbius inversion over power set lattices

Klaus Dohmen

Abstract. In this paper, we establish a theorem on Möbius inversion over power set lattices which strongly generalizes an early result of Whitney on graph colouring.

Keywords: Möbius inversion, power set lattices, graphs, hypergraphs, colourings

Classification: 05C15, 05C65, 05A15, 06A07, 06E99

1. Introduction

An important technique in combinatorics is the principle of Möbius inversion over partially ordered sets (see [3, Chapter 25]). For power set lattices, the principle of Möbius inversion states the following:

Proposition. Let $S$ be a finite set, $f$ and $g$ mappings from the power set of $S$ into an additive group such that $g(X) = \sum_{Y \in [X,S]} f(Y)$ for any $X \subseteq S$, where $[X,S]$ denotes the interval $\{Y \mid X \subseteq Y \subseteq S\}$. Then, for any $X \subseteq S$,

\begin{equation}
    f(X) = \sum_{Y \in [X,S]} (-1)^{|Y \setminus X|} g(Y).
\end{equation}

Proof: By the asserted relation between $f$ and $g$, the sum in (1) equals

$$
\sum_{Y \in [X,S]} (-1)^{|Y \setminus X|} \sum_{Z \in [Y,S]} f(Z) = \sum_{Z \in [X,S]} f(Z) \sum_{Y \in [X,Z]} (-1)^{|Y \setminus X|},
$$

and this is $f(X)$ since the inner sum on the right is zero unless $X = Z$. \hfill \Box

2. A modified inversion formula

The following theorem states that under certain conditions not all terms have to be considered when evaluating the sum in (1). It can be thought of as a generalization of Whitney’s Broken-Circuits-Theorem on graph colouring.

Theorem. Let $S$ be a poset and $f, g$ mappings from the power set of $S$ into an additive group such that $g(X) = \sum_{Y \in [X,S]} f(Y)$ for any $X \subseteq S$. For fixed $X \subseteq S$, let $C$ be a set of non-empty subsets of $S$ such that each $C \in C$ is bounded
from below by an element $C \in S \setminus (C \cup X)$ and $f(Y) = 0$ for all $Y$ including $C \cup X$ and not containing $C$. Then

$$f(X) = \sum_{Y \in [X,S] \cap \mathcal{Y}_0} (-1)^{|Y \setminus X|} g(Y),$$

where

$$\mathcal{Y}_0 := \{ Y \subseteq S \mid Y \not\supseteq C \text{ for all } C \in \mathcal{C} \}.$$

**Proof:** Let $\leq$ denote the partial ordering relation on $S$ and $\leq^*$ one of its linear extensions. For each subset $Y$ of $S$, $\min^* Y$ denotes the minimum of $Y$ with respect to $\leq^*$. Consider an enumeration $C_1, \ldots, C_n$ of $\mathcal{C}$ such that $\min^* C_1 \leq^* \ldots \leq^* \min^* C_n$, and define $\mathcal{Y}_m := \{ Y \subseteq S \mid C_m \subseteq Y, C_{m+1} \not\subseteq Y, \ldots, C_n \not\subseteq Y \}$ for $m = 1, \ldots, n$. Obviously, the power set of $S$ is the disjoint union of $\mathcal{Y}_0, \ldots, \mathcal{Y}_n$. The proposition gives

$$f(X) = \sum_{m=0}^{n} \sum_{Y \in [X,S] \cap \mathcal{Y}_m} (-1)^{|Y \setminus X|} g(Y).$$

We claim that the inner sum on the right-hand side is zero for $m = 1, \ldots, n$. The assertions force $C_m < c$ and hence $C_m <^* c$ for every $c \in C_m$. From the latter we conclude $C_m <^* \min^* C_m \leq^* \min^* C_k$ and therefore $C_m \not\subseteq C_k$ for $k = m, \ldots, n$. For such $k$, $C_k \subseteq Y$ if and only if $C_k \subseteq Y_m$ where $Y_m := (Y \setminus \{C_m\}) \cup (\{C_m\} \setminus Y)$. By this, $Y \in \mathcal{Y}_m$ if and only if $Y_m \in \mathcal{Y}_m$. In addition, $X \subseteq Y$ if and only if $X \subseteq Y_m$. Hence,

$$\sum_{Y \in [X,S] \cap \mathcal{Y}_m} (-1)^{|Y \setminus X|} g(Y) = \frac{1}{2} \sum_{Y \in [X,S] \cap \mathcal{Y}_m} \left( (-1)^{|Y \setminus X|} g(Y) + (-1)^{|Y_m \setminus X|} g(Y_m) \right).$$

Since $|Y \setminus X| \not\equiv |Y_m \setminus X| (\text{mod } 2)$, it suffices to check $g(Y) = g(Y_m)$ for all $Y \in [X,S] \cap \mathcal{Y}_m$. By the asserted relation between $f$ and $g$,

$$g(Y) = \sum_{Z \in [Y,S], \ C_m \not\subseteq Z} f(Z) + \sum_{Z \in [Y,S], \ C_m \subseteq Z} f(Z).$$

It is easy to see that the right sum remains unchanged when $Y$ is replaced by $Y_m$. The same holds for the left sum which, by the assertions of the theorem, equals zero. \[\square\]
Remark. To compare the number of terms in (1) and (2), we define \( \chi := |Y_0 \cap [X, S]|/|X, S| \). Obviously, \( 0 \leq \chi \leq 1 \). By the well-known principle of inclusion and exclusion (which is a particular case of the next corollary),

\[
(4) \quad \chi = \sum_{C' \subseteq C} (-1)^{|C'|} 2^{|X|} - |X \cup \bigcup_{C \in C'} C|.
\]

Hence, if \( C \) contains \( n \) pairwise disjoint sets of cardinality \( m \) (\( n \in \mathbb{N}_0, m \in \mathbb{N} \)) all of them being disjoint with \( X \), then \( \chi \leq (1 - 2^{-m})^n \), and this tends to zero as \( n \to \infty \).

Corollary. Let \( A \) be a boolean algebra of sets, \( P \) a mapping from \( A \) into an additive group such that \( P(\emptyset) = 0 \) and \( P(A \cup B) = P(A) + P(B) \) for all disjoint pairs \( A, B \in A \), \( S \) a finite poset, \( \{A_s\}_{s \in S} \subseteq A \), \( X \subseteq S \) and \( C \) a set of non-empty subsets of \( S \) such that each \( C \in C \) is bounded from below by an element \( \bigcap \in S \setminus (C \cup X) \) and \( \bigcup_{C \in C'} A_c \subseteq A_C \). Then

\[
P\left(\bigcap_{x \in X} A_x \cap \bigcap_{s \in S \setminus X} C_A_s\right) = \sum_{Y \in [X, S] \cap Y_0} (-1)^{|Y \setminus X|} P\left(\bigcap_{y \in Y} A_y\right),
\]

where \( Y_0 \) is defined as in (3) and \( C_A_s \) denotes the complement of \( A_s \) in \( A \).

Proof: For \( Y \subseteq S \) define \( f(Y) := P(\bigcap_{y \in Y} A_y \cap \bigcap_{s \in S \setminus Y} \bigcup C_A_s) \), \( g(Y) := P(\bigcap_{y \in Y} A_y) \). For \( Y \) including \( C \) and not containing \( \bigcap \) there is some \( B \in A \) such that \( f(Y) = P(\bigcap_{C \in C} A_C \cap \bigcup C_A \cap B) \), and hence \( f(Y) = 0 \). Therefore, the theorem can be applied.

Remark. Let \( X \) be empty and \( S_{\min} \) resp. \( S_{\max} \) denote the set of minimal resp. maximal elements of \( S \). If the mapping \( s \mapsto A_s \) is antitone, then it can be achieved that \( Y_0 \) is the power set of \( S_{\min} \) (Proof: Set \( C := \{\{s\} \mid s \in S \setminus S_{\min}\} \), and for each \( C \in C \) choose a lower bound \( \bigcap \in S \setminus C \)). By the duality principle for posets, ‘below’ can be replaced by ‘above’ both in the theorem and in the corollary. By this, if \( s \mapsto A_s \) is isotone, then it can be achieved that \( Y_0 \) becomes the power set of \( S_{\max} \).

Example 1. In (4), \( C \) can be replaced by the set of its minimal elements with respect to set inclusion. This is an immediate consequence of the corollary and the preceding remark since \( C \mapsto [C, S] \) is an antitone mapping.

Example 2. A hypergraph is a set \( S \) of non-empty sets whose union \( \bigcup S \) is finite. The elements of \( S \) resp. \( \bigcup S \) are the edges resp. vertices of the hypergraph; their number is denoted by \( m(S) \) resp. \( n(S) \). Define \( m^*(S) := \sum_{s \in S} (|s| - 1) \). For \( k \in \mathbb{N} \), let \( S^{(k)} \) consist of all \( k \)-element edges of \( S \). The edges of \( S^{(1)} \) are called loops. The subsets of \( S \) are called partial hypergraphs of \( S \). A cycle in \( S \) is a sequence \((v_1, s_1, \ldots, v_k, s_k)\) where \( k > 1 \) and \( v_1, \ldots, v_k \) resp. \( s_1, \ldots, s_k \) are...
distinct vertices resp. edges, \( v_i, v_{i+1} \in s_i \) for \( i = 1, \ldots, k - 1 \) and \( v_k, v_1 \in s_k \). With respect to a linear ordering relation on \( S \), a broken circuit of \( S \) is obtained from the edge-set of a cycle in \( S \) by removing the smallest edge. For any \( \lambda \in \mathbb{N} \), a \( \lambda \)-colouring of \( S \) is a mapping \( f : \bigcup S \rightarrow \{1, \ldots, \lambda\} \) (the set of colours). For \( X \subseteq S \), \( P_{S,X}(\lambda) \) stands for the number of \( \lambda \)-colourings of \( S \) such that \( X \) is the set of monochromatic edges. We now establish the following statement:

Let \( S \) be a loop-free, linearly ordered hypergraph, and let \( X \) be a partial hypergraph of \( S \) such that \( S^{(2)} \setminus X \) is an initial segment of \( S \) and each cycle in \( S \) has an edge of \( S^{(2)} \setminus (C \cup X) \). Then \( P_{S,X}(\lambda) = \sum_{i,j} \rho_{ij} \lambda^{n(S) - i} \) where \( \rho_{ij} \) equals \((-1)^{j - |X|} \times \) the number of partial hypergraphs \( Y \) of \( S \) including \( X \) but no broken circuits of \( S \) and satisfying \( m^*(Y) = i \) and \( m(Y) = j \).

**Proof:** For \( s \in S \) define \( A_s \) as the set of \( \lambda \)-colourings of \( S \) such that \( s \) is monochromatic. For any broken circuit \( C \) of \( S \) let \( \overline{C} \) be the unique edge such that \( C \cup \{ \overline{C} \} \) is the edge-set of a cycle in \( S \). The assertions force \( \overline{C} \in S^{(2)} \setminus (C \cup X) \). Obviously, \( \overline{C} \in S^{(2)} \) entails \( \bigcap_{c \in C} A_c \subseteq A_{\overline{C}} \). By the corollary, \( P_{S,X}(\lambda) = \sum_Y (-1)^{|Y \setminus X|} |\bigcap_{y \in Y} A_y| \) where the summation is extended over all partial hypergraphs \( Y \) of \( S \) including \( X \) but no broken circuits of \( S \). By [1, Proposition], \( |\bigcap_{y \in Y} A_y| = \lambda^{n(S) - m^*(Y)} \). The result now follows.

**Note.** A particular case of the previous example, namely where \( X \) is empty, is published in [2]. For simple graphs and empty \( X \), the above statement is due to Whitney (see [4]) and called Whitney’s Broken-Circuits-Theorem.

**References**


_Humboldt-Universität zu Berlin, Unter den Linden 6, D-10099 Berlin, Germany_

*(Received September 26, 1995)*