Sequential continuity on dyadic compacta and topological groups

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Abstract. We study conditions under which sequentially continuous functions on topological spaces and sequentially continuous homomorphisms of topological groups are continuous.

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§1. Introduction

It is well known that the sequential continuity of a real-valued function defined on a topological space is in general far too weak to imply its continuity. The problem whether sequentially continuous functions on a product of separable metric spaces are continuous is much more delicate.

An investigation originated by Mazur [M] and continued by Noble [N] and Antonovskiǐ and Chudnovskiǐ [AC] brought out several interesting results. For instance, the question whether the product of $\kappa$ separable metric spaces is a space in which sequential continuity suffices for continuity appears to be equivalent to the question whether the Cantor cube $D^\tau$ has that property. Moreover, unless certain large cardinals exist, $D^\tau$ does have the property for every cardinal number $\tau$. On the other hand, if $\tau$ is a real-valued measurable cardinal, then a universal measure on $\tau$ provides an example of a sequentially continuous function on $D^\tau$ which is not continuous. This suggests that one should not expect the above problem to be decidable within the usual axioms of set theory (see also [P1] and [P2]).

In Section 1 of this article we establish certain conditions under which a sequentially continuous mapping is continuous. We also construct an example destroying some overly optimistic expectations in this respect: It turns out that being a homomorphism of topological groups is not among such conditions, even if the domain is compact. This example sheds new light on a classical result of Varopoulos [Vs], and also leads to a partial solution of a problem posed by Comfort and Remus [CR].

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Section 2 contains a discussion of the role of large cardinals and of compactness-like properties of the sequential leader of $D^\tau$.

§2. Preliminaries

We follow notation and terminology in [E] and [A1]. A space means a topological space. Symbols $X$, $Y$, $Z$ always stand for topological spaces. Topological groups are assumed to be $T_1$-spaces, which implies that they are Tychonoff spaces (see [Pn], [RD]).

A space of point-countable type is a Hausdorff space $X$ such that for each point $x \in X$ there is a compact subspace $F$ of $X$ satisfying the conditions: $x \in F$, and $\chi(F, X) \leq \omega$, where the last condition means that there is a countable base of neighbourhoods for $F$ in $X$.

A Baire subset of a space $X$ is any member of the $\sigma$-algebra generated by the zero-subsets of $X$.

A space of countable pseudocharacter is a space in which all points are $G_\delta$’s, and the tightness of $X$ is countable if every point in the closure of any subset $A$ of $X$ is in the closure of some countable subset of $A$.

A cardinal is identified with the smallest ordinal corresponding to it, and an ordinal is considered as the set of all smaller ordinals; $P(\tau)$ stands for the set of all subsets of $\tau$.

A topological group $G$ is said to be $\omega$-bounded (see [A3]), if for each nonempty open subset $U$ of $G$, there is a countable subset $A \subset G$ such that the set $U \circ A = \{b \ast a : b \in U, a \in A\}$ coincides with $G$.

A dyadic compactum is a compact Hausdorff space which can be represented as a continuous image of some Cantor cube $D^\tau$. Every compact topological group is a dyadic compactum, according to a famous theorem proved independently by Ivanovski [Iv] and Kuz’minov [Ku].

We call a (not necessarily Tychonoff) space $X$ pseudocompact, if every discrete in $X$ family of nonempty open subsets of $X$ is finite. For Tychonoff spaces this is equivalent to the requirement that every continuous real-valued function on $X$ is bounded.

Recall that a subset $A$ of a topological space $X$ is called sequentially closed if it contains a limit of every converging sequence $(x_n)_{n \in \omega}$ of its elements. For every $A \subseteq X$ we denote by $A^{seq}$ the set of all points that are limits of converging sequences from $A$. (Note that $A^{seq}$ need not be sequentially closed!).

A real-valued function $g$ defined on $X$ is said to be sequentially continuous if $\lim_{n \to \infty} g(t_n) = g(t)$ for every (ordinary) sequence $(t_n)_{n \in \omega} \subseteq X$ that converges to $t$. If $g$ is such a function, then $g^{-1}(F)$ is sequentially closed whenever $F$ is a closed set in the real line.

The space $D^\kappa$ will sometimes be identified with the power set of $\kappa$ via the mapping $\chi_a \longleftrightarrow a$, $a \subseteq \kappa$. Thus subsets of $\kappa$ are elements of $D^\kappa$, and are denoted by lower case letters.

The subset $A \subseteq D^\kappa$ is said to depend on a set $i \subseteq \kappa$ (of coordinates) provided that $a \in A$ and $b \cap i = a \cap i$ imply $b \in A$. It is well known that every zero-
set in $D^\kappa$ depends on a countable set (see [M]). Every subset of $D^\kappa$ of the form 
\{a : i \subseteq a \subseteq j^c\}, where $i, j \subseteq \kappa$ are finite, is called an elementary set (here and 
below $Y^c$ stands for the complement of a set $Y$). Elementary subsets of $D^\kappa$ form 
a standard base for the usual topology of $D^\kappa$.

Note that $A^{seq} = \overline{A}$ for every subset of $D^\kappa$ depending on a countable set $i \subseteq \kappa$. 
Indeed, let $a \in \overline{A}$. Write $i$ as an increasing union of finite sets $i_n$ and choose, for 
every $n$,
\[ a_n \in A \cap \{x : i_n \cap a \subseteq x \subseteq (i_n \setminus a)^c\}. \]
Put $s_n = (a_n \cap i) \cup (a \setminus i)$; it follows that $s_n$’s form a sequence in $A$ that converges 
to $a$.

A measure $\mu$ defined on the Borel $\sigma$-algebra of a topological space $X$ is said to be strictly positive 
provided $\mu(V) > 0$ for every nonempty open subset $V$ of $X$. A measure $\mu$ is completion-regular 
if for every Borel set $B$ there are Baire sets $A_1, A_2$ with $A_1 \subseteq B \subseteq A_2$ and $\mu(A_1) = \mu(A_2)$. Since every finite measure is 
inner-regular on the $\sigma$-algebra of Baire sets, the completion-regularity of $\mu$ means that for every Borel set $B$ with $\mu(B) > 0$ there exists a zero-set $Z \subseteq B$ such that 
$\mu(Z) > 0$ (see [Wh]).

The usual product measure on $D^\kappa$ is strictly positive and completion-regular. This is a particular case of a theorem due to Kakutani stating that a product 
of any family of strictly positive Borel measures defined on metric spaces is completion-regular. Recall also that the Haar measure on a compact topological group has that property (see [Wh] for references, and [Gr] for more recent results 
on completion-regularity).

§3. Sufficient and insufficient conditions for continuity of sequentially 
continuous maps

1. Theorem. Let $X$ be a dyadic compactum, and let $f$ be a sequentially contin-
uous mapping of $X$ onto a space $Y$ of countable pseudocharacter. Let us also 
assume that, for each $y$ in $Y$, the preimage of $y$ under $f$ is a space of countable 
pseudocharacter. Then the mapping $f$ is continuous, and the spaces $X$ and $Y$ 
are metrizable.

Proof: We only have to show that $X$ is metrizable. This already implies that $f$ 
is continuous, and that $Y$ is a metrizable compact space. Let us assume that $X$ 
is not metrizable. Then $X$ contains a topological copy of the space $D^{\omega_1}$ (see, e.g. 
Theorem 1 (iv) of [H]). It follows that there exists a nonempty countably compact 
sequential subspace $Z$ of $X$ such that none of the points in $Z$ is a $G_\delta$-point in $Z$. 
Indeed, take a copy of any $\Sigma$-product in $D^{\omega_1}$. Let $g$ be the restriction of $f$ to $Z$. 
Since the space $Z$ is sequential, $g$ is a continuous mapping of $Z$ onto a subspace $M$ 
of $Y$. Take any point $y$ of $M$. Since $\{y\}$ is a $G_\delta$-subset of $M$, and $g$ is continuous, 
g$^{-1}\{y\}$ is a $G_\delta$-set in $Z$. Clearly, $g^{-1}\{y\}$ is contained in $f^{-1}\{y\}$. Therefore, each 
point of the space $g^{-1}\{y\}$ is a $G_\delta$-subset of it. It follows that every point of the 
nonempty set $g^{-1}\{y\}$ is a $G_\delta$-subset of $Z$ — a contradiction with the choice of $Z$. 
\[ \square \]
2. **Corollary.** Every sequentially continuous one-to-one mapping of a dyadic compactum $X$ onto a first countable $T_1$-space $Y$ is continuous, $X$ is metrizable, and $Y$ is either not Hausdorff, or also metrizable.

3. **Theorem.** Let $f$ be a sequentially continuous homomorphism of a topological group $G$ of point-countable type into a countably tight topological group $F$ such that the kernel $A = f^{-1}\{e\}$ is closed. Then $f$ is continuous.

**Proof:** Let $H = G/A$ be the topological quotient group of $G$ with respect to $A$, and let $g$ be the quotient homomorphism of $G$ onto $H$. Then $g$ is continuous and there is a unique isomorphism $h$ of $H$ onto $F$ such that $f = h \circ g$.

Let $\{y_n : n \in \omega\}$ be any sequence in $H$, converging to a point $y \in H$. Then there is a sequence $\{x_n : n \in \omega\}$ in $G$ converging to a point $x$ in $G$, such that $g(x_n) = y_n$ for each $n \in \omega$ (see Theorem 3 of [Pas]). It follows that the mapping $h$ of $H$ into $F$ is sequentially continuous. Clearly, $H$ is a topological group of point-countable type. Therefore, there is a compact subgroup $P$ of $H$ such that the set $P$ has a countable base of neighbourhoods in $H$. The restriction of $h$ to $P$ is a continuous mapping by the following lemma:

4. **Lemma.** Let $f$ be a sequentially continuous mapping of a dyadic compactum $X$ onto a Hausdorff space $Y$ such that the tightness of every compact subspace of $Y$ is countable. Furthermore, assume that $t(f^{-1}\{y\}) = \omega$ for each $y$ in $Y$. Then $f$ is continuous, and the spaces $X$ and $Y$ are metrizable.

**Proof:** It is enough to prove that $X$ is metrizable. Let us assume the contrary. Then the space $X$ contains a topological copy $Z$ of the space $D^{\omega_1}$. By a theorem of Mazur [M], the restriction of the mapping $f$ to $Z$ is continuous. Then $F = f(Z)$ is compact; therefore $t(F) = \omega$. Also, $t(g^{-1}\{y\}) = \omega$ for every $y \in F$. Thus by Proposition 4.5 in [A1], $t(Z) = \omega$. Since every dyadic compactum of countable tightness is metrizable (see [E]), we get a contradiction. 

By Lemma 4, $P$ is metrizable. Since $P$ has a countable base of neighbourhoods in $H$, it follows that $H$ is first countable. Thus $f$ is continuous.

5. **Theorem.** If $f$ is a sequentially continuous homomorphism of a topological group $G$ of point-countable type into a countably tight group $F$ such that the kernel $K = f^{-1}\{e\}$ contains a nonempty Baire subset of $G$, then $f$ is continuous.

**Proof:** Fix a compact subspace $T$ of $G$ which contains $e$ and has a countable base of neighbourhoods in $G$. Since $K$ contains a nonempty Baire subset $G$, there is a nonempty $G_\delta$-subset $P$ of $G$ contained in $K$. Clearly, we can assume that $P$ is also contained in $T$. Moreover, we can assume that $P$ is closed and is a subgroup of $G$ (see [Pn], [RD]). Take the quotient space $G/P$ (which is not necessarily a topological group). It is first countable, since $P$ has a countable base of neighbourhoods in $G$ (by transitivity of character in Tychonoff spaces, see [A3]). The natural mapping of $G/P$ into $F$ is well-defined (since $P \subset K$), is sequentially continuous by Theorem 3 of [Pas], and thus is continuous, since $G/P$
is first countable. Therefore $f$ is a composition of two continuous mappings and is continuous.

The following example shows that some assumptions on the kernel are necessary in Theorems 3 and 5.  

6. **Theorem.** Let $\kappa \leq 2^{\aleph_0}$ be a real-valued measurable cardinal. Then there exists a sequentially continuous homomorphism $g$ of $D^\kappa$ onto a metrizable topological group $G$ that is not pseudocompact.

Theorem 6 can be proved quite easily using seminorms and the result of Varopoulos [Vs]. The reader interested in this approach and in related results is referred to [U] and [Hu]. Here we present a direct proof that does not depend on Varopoulos’s result. The construction of topological groups used in this argument will also be needed in the proof of Theorem 16.

Let $\kappa$ be an uncountable cardinal, and let $\lambda$ be an ordinal. Consider a collection $\mathcal{M} = \{\mu_\xi : \xi < \lambda\}$ of probability measures on $\mathcal{P}(\kappa)$. For $\xi < \lambda$, $\varepsilon > 0$ and $X \subseteq \kappa$, let $U(X, \xi, \varepsilon) = \{Y \subseteq \kappa : \mu_\xi(X \Delta Y) < \varepsilon\}$. Let $\mathcal{T}_M$ (or just $T$ if no confusion can arise) be the topology on $\mathcal{P}(\kappa)$ generated by the subbase $\{U(X, \xi, \varepsilon) : X \subseteq \kappa, \xi < \lambda, \varepsilon > 0\}$, and let $S$ be the usual topology on $\mathcal{P}(\kappa)$ (i.e. $S$ is obtained by identifying $\mathcal{P}(\kappa)$ and $D^\kappa$ with the product topology).

Define an equivalence relation $\sim$ on $\mathcal{P}(\kappa)$ by: $X \sim Y$ iff $\forall \xi < \lambda \mu_\xi(X \Delta Y) = 0$. Denote $G = \mathcal{P}(\kappa)/\sim$, and let $h : \mathcal{P}(\kappa) \to G$ be the quotient mapping. We shall use the same symbol $\Delta$ for operations on $\mathcal{P}(\kappa)$ and $G$; if $X/\sim, Y/\sim \in G$, then $X/\sim \Delta Y/\sim = (X \Delta Y)/\sim$. Clearly, $\Delta$ is a well-defined operation on $G$. We also shall use the same symbol $T$ for the quotient topology on $G$ and for the topology on $\mathcal{P}(\kappa)$.

7. **Lemma.** (a) The map $id : (\mathcal{P}(\kappa), S) \to (\mathcal{P}(\kappa), T)$ is sequentially continuous.

(b) $((\mathcal{P}(\kappa), \Delta), T)$ and $((G, \Delta), T)$ are topological groups.

(c) $h \circ id$ is a sequentially continuous group homomorphism from $((\mathcal{P}(\kappa), \Delta), S)$ onto $((G, \Delta), T)$.

(d) If $\lambda < \aleph_1$, then $G$ is metrizable.

**Proof:** Note that if $X = \lim_{n \to \infty} X_n$ (in the sense of $S$), and $\mu$ is a probability measure on $\mathcal{P}(\kappa)$, then $\lim_{n \to \infty} \mu(X_n) = \mu(X)$ [Hal]. Using this fact, the verification of (a) is straightforward.

In order to prove that $((\mathcal{P}(\kappa), \Delta), T)$ is a topological group, first note that $X^{-1} = \kappa \Delta X$ for every $X \subseteq \kappa$. Therefore, it suffices to show that the function $\Delta : (\mathcal{P}(\kappa))^2 \to \mathcal{P}(\kappa)$ is continuous with respect to $T$. Now it suffices to note that if $\mu$ is a measure on $\kappa$, then the function $\varrho_\mu : (\mathcal{P}(\kappa))^2 \to \mathcal{P}(\kappa)$ defined by $\varrho_\mu(X, Y) = \mu(X \Delta Y)$ is a pseudometric on $\mathcal{P}(\kappa)$ (see [Hal, §40]), $T$ is the topology generated by the collection of pseudometrics $\{\varrho_{\mu_\xi} : \xi < \lambda\}$, and the function $\Delta : (\mathcal{P}(\kappa))^2 \to \mathcal{P}(\kappa)$ is continuous with respect to every pseudometric $\varrho_{\mu_\xi}$. For a similar reason, $((G, \Delta), T)$ is a topological group.

The verification of (c) is left to the reader.
The result of (d) will only be used if \( \lambda = 1 \). In this case it suffices to note that \( \rho_{\mu_0} \) is a metric on \( G \), and \( T \) is induced by this metric. □

**Proof of Theorem 6:** Let \( \kappa \) be as in the assumption. For convenience of notation, we shall identify \( D_\kappa \) with \( P(\kappa) \).

Let \( \mu : \mathcal{P}(\kappa) \to [0,1] \) be a countably additive probability measure such that \( \mu(\{\alpha\}) = 0 \) for each \( \alpha < \kappa \). Let \( T = T_{\mu} \) be the topology on \( \mathcal{P}(\kappa) \) defined as in the discussion preceding Lemma 7, let \( G \) be the corresponding topological group, and \( h \) the quotient mapping. Let \( g = h \circ id \).

Then \( g \) is a sequentially continuous group homomorphism by Lemma 7(c), and \( G \) is metrizable by Lemma 7(d). So we will be done if we prove the following:

**8. Claim.** \((G, T)\) is not pseudocompact.

**Proof:** Since \((G, T)\) is metrizable, it is enough to show that it is not countably compact. There exists a sequence \((X_n)_{n \in \omega}\) of subsets of \( \kappa \) such that \( \mu(X_n) = 0.5 \) and

\[
\mu(X_n \Delta X_m) = 0.5 \quad \text{for all} \quad n < m < \omega.
\]

This is true in general, and can be very easily observed if we assume that \( \mu \) extends Lebesgue measure (this does not lead to loss of generality; see [J, p. 302]): In this case, let \((X_n)_{n \in \omega}\) be a standard Rademacher sequence.

But note that the \( X_n \)'s form an infinite collection \( A \) of points whose distance from each other is 0.5 in the metric \( \rho_\mu \). Since \( T \) is induced by this metric, \( A \) is a closed discrete subspace of \((G, T)\). Therefore, the space \((G, T)\) is not countably compact. □

The group \( G \) constructed in our proof of Theorem 6 is not separable. The density of the space \((G, T)\) is equal to the Maharam type of the measure \( \mu \) (see [F1, 2.20]). By a theorem of Gitik and Shelah, the Maharam type of \( \mu \) is at least as big as \( \min\{\kappa^+, 2^\kappa\} \) (see [F2] for an elementary proof). Hence the following question arises:

**Problem 1:** What is the smallest possible density of a group \( G \) as in Theorem 6? In particular, can one construct an example as in Theorem 6 such that \( G \) is separable (but still not compact)?

**9. Remark.** The map \( g \) in the above example is open. Indeed, since

\[
\forall x, y \in \mathcal{P}(\kappa) \ (|x \Delta y| < \aleph_0 \Rightarrow g(x) = g(y)),
\]

the preimage of any point of \( G \) under \( g \) is dense in \( \mathcal{P}(\kappa) \). This implies that \( g \) maps every nonempty open subset of \( \mathcal{P}(\kappa) \) onto \( G \). Thus \( g \) is open.

**10. Theorem.** Let \( f \) be a sequentially continuous mapping of a dyadic compactum \( X \) onto a space \( Y \) of countable pseudocharacter such that the preimages of all points are closed. Then \( f \) is continuous.

**Proof:** The space \( X \) is a quotient space of some Cantor cube \( D^\tau \) under a (continuous) mapping \( h \), and it suffices to show that the composition mapping \( g = f \circ h \)…. 
is continuous. Clearly, the images of points under $g$ are closed subsets of $D^\tau$, and $g$ is sequentially continuous. Therefore, we may assume that $f$ maps $D^\tau$ onto $Y$, where $Y$ has countable pseudocharacter. If $f$ is sequentially continuous, then $f$ is continuous on the sigma-product $\Sigma = \{f \in D^\tau : |f^{-1}\{1\}| \leq \aleph_0\}$, and hence $f|\Sigma$ depends only on a countable set of coordinates $A$ (see [E]). Now suppose that the fibres of $f$ are closed, and let $y \in Y$. Let $f(x) = y$, and let $p = x|A$. Then $x \in cl(Z)$, where $Z = \{z \in \Sigma : z|A = p\}$. But $cl(Z)$ is all contained in one fibre, by the choice of $A$. Thus, the whole function $f$ depends only on the coordinates from the countable set $A$; hence $f$ is continuous. □

We conclude this section with a fairly general measurability condition that implies continuity of a sequentially continuous mapping.

Let $X$ be a compact (Hausdorff) topological space. We denote by $Z(X)$ the family of all zero subsets of $X$; we also put $Z^+(X) = Z(X) \setminus \{\emptyset\}$.

We define two families of subsets of $X$ in the following way:

$$S_0(X) = \{A \subseteq X : \forall V \ (V \cap A \neq \emptyset \Rightarrow \exists Z \in Z^+(X) \ Z \subseteq V \cap A)\},$$

$$S(X) = \{A \subseteq X : \forall V \exists Z \in Z^+(X) \ (Z \subseteq V \cap A \cup Z \subseteq V \cap A^c)\},$$

where $V$ denotes an open subset of $X$.

We shall list some basic properties of the families $S(X)$ and $S_0(X)$ thus defined.

(i) $A \in S_0(X)$ if and only if there exists a family $\mathcal{Z} \subseteq Z(X)$ such that $\bigcup \mathcal{Z} \subseteq A \subseteq \bigcup \mathcal{Z}$;

(ii) $S_0(X)$ contains all zero-sets and all open sets in $X$, and is closed under arbitrary unions;

(iii) $S(X)$ is closed under complements;

(iv) every subset of $X$ having the Baire property belongs to $S(X)$;

(v) $S(X)$ contains $\mu$-measurable sets for every completion-regular and strictly positive measure $\mu$ on $X$.

Indeed, it is immediate from the definition of $S_0(X)$ that if $A \in S_0(X)$, then the union of all zero-sets contained in $A$ is dense in $A$. On the other hand, if $\bigcup \mathcal{Z} \subseteq A \subseteq \bigcup \mathcal{Z}$, then every open set $V$ with $A \cap V \neq \emptyset$ meets some $Z \in \mathcal{Z}$; since $Z \cap V$ contains a non-empty zero-set, so does $V \cap A$.

Properties (ii) and (iii) are clear. To check (iv) it suffices to notice that if $V$ is open and non-empty, and $N$ is a set of the first category in $X$, then $V \setminus N$ contains some $Z \in Z^+(X)$.

Let $\mu$ be a measure as in (v) and let $A$ be measurable. If $V$ is open and non-empty, then either $\mu(V \cap A) > 0$ or $\mu(V \setminus A) > 0$, so by completion-regularity we can find a non-empty zero-set either in $V \cap A$ or in $V \setminus A$.

Let us remark that $S_0(X)$ is the whole power set of $X$ whenever $X$ is first-countable (since then singletons are zero-sets). It is also worth noticing that $S(X)$ may be fairly large in other cases.
For instance, every subset $A$ of $D^c$ (where $c$ is the cardinality of the real line) can be written as $A = B \cup C$ with $B, C \in S(D^c)$. Indeed, there is a decomposition $D^c = M \cup N$, where $N$ is of first category and $M$ is a null-set with respect to the usual product measure $\lambda$ (which is strictly positive and completion-regular). It follows from the remarks above that $A = (A \cap M) \cup (A \cap N)$ is the decomposition as required. On the other hand, if $P$ is the family of all countable subsets of $c$, then $P \notin S(D^c)$. This shows that $S(X)$ need not be closed under finite unions.

In the sequel, we shall say that a real-valued function $g$ defined on a space $X$ is $S(X)$-measurable provided $g^{-1}(H) \in S(X)$ for every open subset $H$ of the reals (equivalently: for every closed $H$). It is clear from the above observations that every function having the Baire property is $S(X)$-measurable, and so is every function which is measurable with respect to some strictly positive and completion-regular measure.

11. Proposition. If $X$ is a dyadic compactum and $A \in S_0(X)$, then the sequential closure and the closure of $A$ coincide.

Proof: The substantial part of the argument is contained in the following well-known fact (see [E1], [EK]):

12. Lemma. Let $\mathcal{D}$ be a family of subsets of $D^\kappa$ such that every $D \in \mathcal{D}$ depends on a countable set of coordinates. Then there exists a countable subfamily $\mathcal{D}_0 \subseteq \mathcal{D}$ such that $\bigcup \mathcal{D}_0 = \bigcup \mathcal{D}$.

In order to prove Proposition 11, suppose that we have a non-empty set $A \in S_0(X)$, and let us fix a continuous surjection $\theta$ from $D^\kappa$ onto $X$. We let $\mathcal{Z}$ to be the family of all zero-sets in $X$ that are contained in $A$, and put $\mathcal{D} = \{\theta^{-1}(Z) : Z \in \mathcal{Z}\}$. Since every set of the form $\theta^{-1}(Z)$, where $Z \in \mathcal{Z}(X)$, is a zero subset of $D^\kappa$, it depends on a countable subset of $\kappa$. Now it follows from Lemma 12 and the remark from Section 2 that $\bigcup \mathcal{D} = (\bigcup \mathcal{D})^{seq}$. Since the union of $\mathcal{Z}$ is dense in $A$ (see (i) above), we get

$$\overline{A} = \overline{\bigcup \mathcal{Z}} = \theta\left(\bigcup \mathcal{D}\right) = \theta\left(\bigcup \mathcal{D}\right) = \theta\left((\bigcup \mathcal{D})^{seq}\right) \subseteq \theta\left(\bigcup \mathcal{D}\right)^{seq} \subseteq A^{seq},$$

and we are done. \qed

13. Theorem. Let $g$ be a sequentially continuous real-valued function defined on a dyadic compactum $X$. If $g$ is $S(X)$-measurable, then $g$ is continuous.

Proof: We shall check first that

$$\quad (1) \quad g^{-1}(H) \in S_0(X) \text{ whenever } H \subseteq \mathbb{R} \text{ is open.}$$

Let $V$ be open subset of $X$; suppose that $g^{-1}(H) \cap V$ contains no elements of $\mathcal{Z}^+(X)$. It follows that $g^{-1}(H^c) \cap V \in S_0(X)$, and $g^{-1}(H^c) \cap V$ is dense in $V$. Thus, applying Proposition 11 and the fact that $g^{-1}(H^c)$ is sequentially closed, we have

$$V \subseteq g^{-1}(H^c) \cap V = (g^{-1}(H^c) \cap V)^{seq} \subseteq g^{-1}(H^c),$$
so $V \cap g^{-1}(H) = \emptyset$.

Now we shall check that

$$g^{-1}(H) \subseteq g^{-1}(\overline{H})$$

for every open $H$.

Indeed, using (1) and Proposition 11 once again, we can write

$$g^{-1}(H) = (g^{-1}(H))^{seq} \subseteq (g^{-1}(\overline{H}))^{seq} = g^{-1}(\overline{H}).$$

Property (2) is equivalent to continuity of $g$.

Indeed, for a closed set $F \subseteq \mathbb{R}$ we can find open sets $H_n$ such that $F = \bigcap_{n \in \omega} H_n = \bigcap_{n \in \omega} \overline{H_n}$. Thus

$$g^{-1}(F) = \bigcap_{n \in \omega} g^{-1}(\overline{H_n}) \supseteq \bigcap_{n \in \omega} g^{-1}(H_n) \supseteq \bigcap_{n \in \omega} g^{-1}(H_n) \supseteq g^{-1}(F),$$

so the set $g^{-1}(F)$ is closed.

14. Corollary. If $g$ is a sequentially continuous function on a compact space $X$, then $g$ is continuous provided any of the following is satisfied:

(i) $X$ is a dyadic compactum and $g$ has the Baire property;

(ii) $X$ is a compact group and $g$ is measurable with respect to its Haar measure.

Proof: If (i) holds, the assertion follows from Theorem 13, since functions having the Baire property are measurable with respect to $S(X)$.

Recall again that every compact topological group is a dyadic compactum ([Iv], [Ku]). As we already noticed, functions that are measurable with respect to a strictly positive and completion-regular measure on $X$ are $S(X)$-measurable. Moreover, the Haar measure is such a one. Thus, by Theorem 13 again, if (ii) holds, then $g$ is continuous.

§4. The role of large cardinals and compactness-like properties of the sequential leader

Some large cardinal assumption is needed in Theorem 6. This was first observed by Mazur in [M], where it is shown that unless $\kappa$ is at least as big as the first weakly inaccessible cardinal, all sequentially continuous functions from $D^\kappa$ into metric (and thus, into Tychonoff) spaces are continuous. The lower bound for $\kappa$ was later improved by Antonovski˘ı and Chudnovski˘ı [AC] and Ciesielski [C]. It is still unknown whether a real-valued measurable cardinal is needed.

The proof of Theorem 6 can be modified to yield further interesting results. The following theorem answers Question 8 in [A2].
15. **Theorem.** If \( \kappa \leq 2^{\aleph_0} \) is a real-valued measurable cardinal, then there exists a sequentially continuous isomorphism of the topological group \( D^\kappa \) onto a topological group \( G \) that is not \( \omega \)-bounded.

**Proof:** Let \( f \) be a sequentially continuous mapping of a (Tychonoff) space \( X \) onto a (Tychonoff) space \( Y \). Let \( T_1 \) be the smallest (Tychonoff) topology on the set \( X \), containing the topology \( T \) of the space \( X \) and all inverse images of open subsets of \( Y \). The identity mapping \( i : (X, T) \rightarrow (X, T_1) \), defined by the rule: \( i(x) = x \) for each \( x \) in \( X \), is sequentially continuous and one-to-one. Obviously, \( i \) is continuous if and only if \( f \) is continuous. If \( X \) and \( Y \) are topological groups, and \( f \) is a homomorphism, then \( (X, T_1) \) is also a topological group, and \( i \) is an algebraic isomorphism. One can prove Theorem 15 by applying this technique to the example of Theorem 6. Of course, the resulting isomorphism is not continuous. \( \square \)

W. Comfort and D. Remus asked whether every compact group \( H \) of measurable cardinality admits a strictly stronger countably compact group topology of weight \( 2^{\|H\|} \) (see Question 5.4(b) of [CR]). The following theorem gives a positive answer for the group \( (\mathcal{P}(\kappa), \Delta) \) if \( \kappa \) is strongly compact (note that strongly compact cardinals are measurable, but not vice versa).

16. **Theorem.** Let \( \kappa \) be a strongly compact cardinal, and let \( S \) be the usual (product) topology on \( \mathcal{P}(\kappa) \). Then there exists an extension \( \Theta \supset S \) such that

1. \( ((\mathcal{P}(\kappa), \Delta), \Theta) \) is a topological group,
2. \( (\mathcal{P}(\kappa), \Theta) \) is countably compact,
3. \( w((\mathcal{P}(\kappa), \Theta)) = 2^{2\kappa} \).

**Proof:** Let \( \kappa \) be a cardinal, and let \( \mathcal{M} = \{ \mu_\xi : \xi < \lambda \} \) be a family of two-valued \( \kappa \)-additive probability measures on \( \kappa \) such that \( \lambda > \kappa \) and for \( \alpha < \kappa \) we have \( \mu_\alpha(X) = 1 \) iff \( \alpha \in X \). Let \( \Theta \) be the topology on \( \mathcal{P}(\kappa) \) induced by the subbasis

\[ \{ \{x \subseteq \kappa : \mu_\xi(x) = 0\}, \{x \subseteq \kappa : \mu_\xi(x) = 1\} : \xi < \lambda\}. \]

It is not hard to see that \( \Theta = T_M \). Thus, by Lemma 7, \( ((\mathcal{P}(\kappa), \Delta), \Theta) \) is a topological group. \( \square \)

17. **Lemma.** \( (\mathcal{P}(\kappa), \Theta) \) is countably compact.

**Proof:** Let \( \{x_n : n \in \omega\} \) be a countable infinite subspace of \( (\mathcal{P}(\kappa), \Theta) \). We show that the set \( \{x_n : n \in \omega\} \) has a cluster point. Let \( \mathcal{F} \) be a nonprincipal ultrafilter on \( \omega \), and let \( e = \lim_{\mathcal{F}} x_n \) (i.e. \( \alpha \in e \) if and only if \( \{n \in \omega : \alpha \in x_n\} \in \mathcal{F} \)). Consider an arbitrary basic open neighborhood \( U \) of \( e \). We may assume that \( U = U_0 \cap \ldots \cap U_k \) for some \( k \in \omega \), and for each \( i \leq k \) there exist \( \xi_i < \lambda \) and \( \varepsilon_i \in \{0, 1\} \) such that \( U_i = \{X \subseteq \kappa : \mu_{\xi_i} = \varepsilon_i\} \). For \( A \in \mathcal{F} \), let \( e_A = \{\alpha \in \kappa : \forall n \in A (\alpha \in e \leftrightarrow \alpha \in x_n)\} \). Clearly, \( \bigcup_{A \in \mathcal{F}} e_A = \kappa \). Since \( 2^{\aleph_0} < \kappa \), by \( \kappa \)-completeness of the measures \( \mu_\xi \) there exists an \( A \in \mathcal{F} \) such that \( \mu_{\xi_i}(e_A) = 1 \) for all \( i \leq k \). It is not hard to see that the latter implies that \( x_n \in U \) for all
Let $\kappa$ be strongly inaccessible. Then there exists a $\kappa$-independent family $\mathcal{A}$ of size $2^{2^\kappa}$ in $\mathcal{P}(\kappa)$.

This lemma is a slight variation on the theme of a well-known result of Fichtenholf, Kantorovich and Hausdorff. Before we prove it, we show how it can be used to prove Theorem 16. Let $\{g_\alpha : \alpha < 2^\kappa\}$ be a set of functions from $2^{2^\kappa}$ into $\{0,1\}$ that is dense in $2^{2^\kappa}D$ with the product topology. Denote $\lambda = 2^{2^\kappa}$. Let $\{h_\xi : \xi < \lambda\}$ be the set of functions from $2^\kappa$ into $\{0,1\}$ defined as follows:

$$h_\xi(\alpha) = g_\alpha(\xi).$$

Note that the functions $h_\xi$ are pairwise different: If $\eta \neq \xi$, then there is some $\alpha$ such that $g_\alpha(\eta) = 1 - g_\alpha(\xi)$.

Let $\mathcal{A} = \{A_\alpha : \alpha < 2^\kappa\}$ be a family as in Lemma 18. For each $\xi < \lambda$, let $F_\xi$ be the filter on $\kappa$ generated by the family $\{A_\alpha : h_\xi(\alpha) = 1\} \cup \{\kappa \setminus A_\alpha : h_\xi(\alpha) = 0\}$. The family $\mathcal{A}$ was chosen in such a way that $F_\xi$ can be extended to a $\kappa$-complete filter $G_\xi$ on $\kappa$ that is uniform (i.e. all elements of $G_\xi$ have cardinality $\kappa$). Since $\kappa$ was assumed to be a strongly compact cardinal, each $G_\xi$ can be extended to a $\kappa$-complete nonprincipal ultrafilter $H_\xi$ on $\kappa$. For each $\xi < \lambda$, let $\mu_\xi$ be the two-valued probability measure associated with $H_\xi$, let $\mathcal{M} = \{\mu_\xi : \xi < \lambda\}$, and let $\Theta = T_\mathcal{M}$.

We show that $w(\mathcal{P}(\kappa), \Theta) = \lambda$. The inequality $w(\mathcal{P}(\kappa), \Theta) \leq \lambda$ follows immediately from the way the topology was introduced; so we need only prove the inequality $w(\mathcal{P}(\kappa), \Theta) \geq \lambda$. Suppose toward a contradiction that $\mathcal{B}$ is a base for this space of cardinality less than $\lambda$. Since the space $\mathcal{P}(\kappa)$ has cardinality less than $\lambda$, we can without loss of generality assume that each $B \in \mathcal{B}$ is of the form $B = \{x \subseteq \kappa : \{\alpha_0^B, \ldots, \alpha_i^B\} \subset x \& \{\alpha_{i+1}^B, \ldots, \alpha_j^B\} \cap x = \emptyset \& \mu_{x}^B(x) = \ldots = \mu_{\xi_\ell}^B(x) = 0 \& \mu_{\xi_{k+1}^B}^B(x) = \ldots = \mu_{\xi_k}^B(x) = 1\}$ for some $\alpha_0^B, \ldots, \alpha_j^B \in \kappa$ and $\xi_0^B, \ldots, \xi_\ell^B \in \lambda$. Now let $\xi \in \lambda$ be such that $\xi \neq \xi_n^B$ for any $B \in \mathcal{B}$ and $n \in \omega$. Let $V = \{x \subseteq \kappa : \mu_\xi(x) = 1\}$. There must be some $B \in \mathcal{B}$ with $B \subseteq V$. Assume $B = \{x \subseteq \kappa : \{\alpha_0^B, \ldots, \alpha_i^B\} \subset x \& \{\alpha_{i+1}^B, \ldots, \alpha_j^B\} \cap x = \emptyset \& \mu_\xi^B(x) = \ldots = \mu_{\xi_k}^B(x) = 0 \& \mu_{\xi_k}^B(x) = \ldots = \mu_{\xi_{k+1}^B}^B(x) = \ldots = \mu_{\xi_\ell}^B(x) = 1\}$. By the choice of the functions $g_\alpha$, there exists some $\alpha \neq \alpha_0^B, \ldots, \alpha_j^B$ such that $g_\alpha(\xi) = 0$, $g_\alpha(\xi_n^B) = \cdots = g_\alpha(\xi_k^B) = 0$.
and \( g_\alpha(\xi^B_{k+1}) = \cdots = g_\alpha(\xi^B_{\ell}) = 1 \). Let \( a = (A_\alpha \cup \{\alpha_0^B, \ldots, \alpha^B_{i}\}) \setminus \{\alpha^B_{i+1}, \ldots, \alpha^B_{j}\} \).

A straightforward verification of the relevant definitions shows that \( a \in B \). On the other hand, \( a \notin H_\xi \), and therefore, \( a \notin V \). This contradicts the assumption that \( B \subseteq V \). \( \square \)

It remains to prove Lemma 18.

**Proof of Lemma 18:** It will be convenient to enumerate the elements of the family \( \mathcal{A} \) that we are going to construct by functions \( g : \kappa \to \{0, 1\} \). Also since \( \kappa \) is strongly inaccessible (in fact, it suffices to assume that \( \kappa \) is strongly limit), we can enumerate the family \([\kappa]^{< \kappa} = \{d_\alpha : \alpha < \kappa\} \) of all subsets of \( \kappa \) of cardinality less than \( \kappa \) in such a way that each \( d \in [\kappa]^{< \kappa} \) appears cofinally often in the enumeration. Also, we can partition \( \kappa = \{b_\alpha : \alpha < \kappa\} \) into pairwise disjoint sets such that \( |b_\alpha| = 2^{2^{|d_\alpha|}} \). We enumerate: \( b_\alpha = \{\beta^\alpha_H : H : \{0, 1\}^{d_\alpha} \to \{0, 1\}\} \).

After all these preliminaries we are ready to construct \( \mathcal{A} = \{A_g : g \in \{0, 1\}^\kappa\} \).

We set:

\[
\beta^\alpha_H \in A_g \iff H(g|d_\alpha) = 1.
\]

We show that this family is \( \kappa \)-independent. Let \( h : C \to \{0, 1\} \) be a function defined on a subset \( C \) of \( \{0, 1\}^\kappa \) of cardinality \( < \kappa \). Let \( d \subset \kappa \) be such that for each pair of different \( g_0, g_1 \in C \) there is some \( \delta \in d \) with \( g_0(\delta) \neq g_1(\delta) \). Let \( H : \{0, 1\}^d \to \{0, 1\} \) be any function such that \( H(g|d) = h(g) \) for each \( g \in C \). Let \( \alpha \) be such that \( d = d_\alpha \). Then \( \beta^\alpha_H \in \bigcap_{g \in D} A^h(g) \). Since there are \( \kappa \) many such \( \alpha \)'s, we are done. \( \square \)

Let \( \tau \) be a cardinal. By \( D^\tau_{\text{seq}} \) we denote \( D^\tau \) with the strongest topology extending the Tychonoff product topology and still preserving convergence of sequences in \( D^\tau \). We shall distinguish the two different closure operations in \( D^\tau \) by writing \( cl(X) \) for the closure in the product topology, and \( cl_{\text{seq}}(X) \) for the closure operation in \( D^\tau_{\text{seq}} \). The space \( D^\tau_{\text{seq}} \) is called the sequential leader of \( D^\tau \). This space is Hausdorff, but it need not be a Tychonoff space. In this section we show that unless some large cardinals exist, the space \( D^\tau_{\text{seq}} \) is pseudocompact. Before proving it, let us compare this result with some other compactness-like properties of \( D^\tau \).

**19. Theorem.** Let \( \tau \) be a cardinal. The following are equivalent:

(a) \( D^\tau_{\text{seq}} \) is countably compact;
(b) \( D^\tau_{\text{seq}} \) is sequentially compact;
(c) \( \tau \) is less than the splitting number \( s \).

Theorem 19 is a minor variation on the theme of Theorem 1 in [B], Theorem 5.12 in [Vn], respectively Theorem 6.1 in [vD]. The definition of the splitting number \( s \) can be found in [vD] or [Vn]. For our purposes it suffices to know that \( s \) is an uncountable cardinal not exceeding \( 2^{\aleph_0} \). In particular, it follows that \( D^\tau_{\text{seq}} \) is never countably compact, even if \( 2^{\aleph_0} \) is as small as \( \omega_1 \). For pseudocompactness, we get a very different picture.
20. **Theorem.**  
(a) If \( \tau \) is at least as big as the first real-valued measurable cardinal, then \( D_{\text{seq}}^\tau \) is not pseudocompact.

(b) If \( \tau \) is smaller than the first weakly inaccessible cardinal, then \( D_{\text{seq}}^\tau \) is pseudocompact.

**Proof:** Note that (a) follows from Theorem 6: If we consider the function \( g \) of this theorem as a function with domain \( D_{\text{seq}}^\tau \), then \( g \) is continuous.

We prove (b) by an argument closely resembling the reasoning in [M]. As in the proof of Theorem 6, it will be convenient to identify elements of \( D^\tau \) with subsets of \( \tau \). Call a subset \( U \) of \( D_{\text{seq}}^\tau \) *special* if all finite subsets of \( \tau \) are in \( U \), and \( \tau \notin \text{cl}_{\text{seq}} U \). The following lemma was first proved in a different terminology by Mazur [M].

21. **Lemma.** Suppose \( \kappa \) is such that there exists a sequentially continuous map \( f : D^\kappa \to \mathbb{R} \) that is not continuous. Then, for some \( \tau \leq \kappa \), there exists a special subset \( U \) of \( D_{\text{seq}}^\tau \) that is both closed and \( \mathcal{G}_\delta \) in \( D_{\text{seq}}^\tau \).

We shall not prove Lemma 21 here. For our purposes, a very similar lemma will be relevant.

22. **Lemma.** Suppose \( \kappa \) is such that \( D_{\text{seq}}^\kappa \) is not pseudocompact. Then, for some \( \tau \leq \kappa \), there exists a special open subset \( U \) of \( D_{\text{seq}}^\tau \).

**Proof:** Let \( \kappa \) be as in the assumption, and let \( \{ U_n : n \in \omega \} \) be a countable discrete family of nonempty open subsets of \( D_{\text{seq}}^\kappa \). By compactness of \( D^\kappa \) (with the usual product topology), there must be an \( n \) such that \( \text{cl}(U_n) \) has empty interior in the product topology on \( D^\kappa \). To simplify notation, assume \( n = 0 \) is such.

Let \( x \in U_0 \). There is a finite set \( F \subset \kappa \) such that if \( G \) is a finite subset of \( \kappa \setminus F \) and \( y \in D^\kappa \) is such that \( x \Delta y \subset G \), then \( y \in U_0 \): If not, then we could inductively construct a sequence \( (y_k)_{k \in \omega} \) of elements of \( D^\kappa \setminus U_0 \) such that the sequence \( (x \Delta y_k)_{k \in \omega} \) consists of pairwise disjoint finite subsets of \( \kappa \), and thus \( \lim_{k \to \omega} y_k = x \). The latter is impossible though, since \( U_0 \) is a neighborhood of \( x \) and no \( y_k \in U_0 \).

Next observe that there is some \( y \in D^\kappa \setminus \text{cl}(U_0) \) with \( y \cap F = x \cap F \) (otherwise \( U_0 \) would contain the basic open neighborhood of \( x \) in \( D^\kappa \) determined by \( F \)). Fix such \( y \). Then \( x \Delta y \) is an infinite set. To simplify notation we may assume that \( x \Delta y = \tau \) for some infinite cardinal \( \tau \). Consider the set

\[
V = \{ x \Delta z : z \in U_0 \land x \Delta z \subseteq \tau \}.
\]

This set is open in \( D_{\text{seq}}^\tau \), since if \( w_n \notin V \) for every \( n \in \omega \) and \( \lim_{n \to \infty} w_n = w_\omega \) (in \( D_{\text{seq}}^\tau \)), then for each \( n \leq \omega \) there exists exactly one \( z_n \in D^\kappa \) such that \( x \Delta z_n = w_n \). Moreover, \( \lim_{n \to \infty} z_n = z_\omega \), and \( z_\omega \notin U_0 \) since \( U_0 \) is open in \( D_{\text{seq}}^\kappa \). It follows that \( w_\omega \notin V \). A similar reasoning shows that \( \tau = x \Delta y \notin \text{cl}_{\text{seq}}(V) \). Moreover, if \( G \) is a finite subset of \( \tau \), and \( z \Delta x \subset G \), then \( z \in U_0 \) (by the choice of \( F \)), and thus \( G \in V \). We conclude that \( V \) is a special open subset of \( D_{\text{seq}}^\tau \). □
Now let $\tau_0$ be the smallest cardinal $\tau$ such that there exists a special open subset of $D^\tau_{seq}$. In view of Lemma 22, point (b) of Theorem 20 is a consequence of the following.

23. **Lemma.** $\tau_0$ is a weakly inaccessible cardinal.

**Proof:** The proof is based on the following observations.

24. **Claim.** Suppose $U$ is a special subset of some $D^\tau_{seq}$, and $X \subset \tau$ is such that $X \notin \text{cl}_{seq}U$. Then the family $U \cap \mathcal{P}(X)$ is a special subset of $D^X_{seq}$. Moreover, $U \cap \mathcal{P}(X)$ is open in $D^X_{seq}$ whenever $U$ is open in $D^\tau_{seq}$.

**Proof:** Follows from the definition of a special subset and the fact that $\mathcal{P}(X)$ can be treated as a subspace of $D^\tau_{seq}$.

25. **Claim.** If $U$ is a special open subset of $D^\tau_{seq}$, and $X$ is a subset of $\tau$ of cardinality smaller than $\tau_0$, then $X \in U$.

**Proof:** Immediate from Claim 24 and the choice of $\tau_0$.

26. **Claim.** If $U$ is a special open subset of $D^\tau_{seq}$, and $X$ is a family of pairwise disjoint nonempty subsets of $\tau_0$ such that $|X| < \tau_0$ and $\bigcup X = \tau_0$, then there is a finite subfamily $Y \subseteq X$ such that $\bigcup Y \notin \text{cl}_{seq}U$.

**Proof:** Suppose $X = \{X_\alpha : \alpha < \lambda < \tau_0\}$ is a counterexample. Then the function $h : \mathcal{P}(\lambda) \to \mathcal{P}(\tau_0)$ defined by: $h(Y) = \bigcup\{X_\alpha : \alpha \in Y\}$ is a homeomorphism between $\mathcal{P}(\lambda)_{seq}$ and a closed subspace of $\mathcal{P}(\tau_0)_{seq}$. Let $V = h^{-1}U$. By the assumptions on $X$, $\lambda \notin \text{cl}_{seq}V$, each finite subset of $\lambda$ is in $V$, and $V$ is open in $\mathcal{P}(\lambda)_{seq}$. Thus $\lambda$ contradicts the choice of $\tau_0$.

27. **Corollary.** $\tau_0$ is a regular uncountable cardinal.

**Proof:** Note that every subset of $\omega$ is the limit of a sequence of finite subsets of $\omega$. Therefore, $\tau_0 > \omega$. Now suppose $\tau_0$ is singular, and consider a partition $\mathcal{X}$ of $\tau_0$ into pairwise disjoint subsets of smaller cardinality. Let $\mathcal{Y}$ be as in Claim 26. Then $\bigcup \mathcal{Y}$ has cardinality less than $\tau_0$ which contradicts Claim 25.

Since $D^\omega_{seq}$ is the same as $D^\omega_{seq}$, there are no special subsets whatsoever of $D^\omega_{seq}$. In other words, $\tau_0$ is an uncountable cardinal. Thus, the following is the last brick needed to conclude the proof of Lemma 23.

28. **Claim.** $\tau_0$ is a limit cardinal.

**Proof:** Suppose $\tau_0 = \lambda^+$ for some infinite cardinal $\lambda$. Let $\{A_{\alpha,\zeta} : \alpha < \lambda, \zeta < \tau_0\}$ be an Ulam matrix of subsets of $\tau_0$. That is,

(A) $A_{\alpha,\zeta} \cap A_{\beta,\zeta} = \emptyset$ for all $\alpha < \beta < \lambda$ and $\zeta < \tau_0$;
(B) $A_{\alpha,\zeta} \cap A_{\alpha,\xi} = \emptyset$ for all $\alpha < \lambda$ and $\xi < \zeta < \tau_0$;
(C) $\bigcup_{\alpha<\lambda} A_{\alpha,\zeta} = \tau_0 \setminus (\zeta + 1)$ for all $\zeta < \lambda$. 
By Claims 25 and 26, for each $\zeta < \tau$, there exists a subfamily $A_\zeta \subset \{A_\alpha, \zeta : \alpha < \lambda\} \cup \{\zeta + 1\}$ of size $k(\zeta) < \omega$ such that $\bigcup A_\zeta \notin cl_{seq}(U)$. (For this proof, we only need that $\bigcup A_\zeta \notin U$.) Let $(\zeta_n : n \in \omega)$ be an increasing sequence of ordinals less than $\tau_0$ such that $k(\zeta_n) = k$ for a fixed $k$ and all $n \in \omega$. Let $\xi < \tau_o$. By (B), $\xi$ belongs to at most $k$ among the sets $\bigcup A_{\zeta_n}$, and hence there exists a subsequence $(\zeta_{n_k})_{k \in \omega}$ such that either $\lim_{k \to \infty} \bigcup A_{\zeta_{n_k}} = \emptyset$, or $\lim_{k \to \infty} \bigcup A_{\zeta_{n_k}} = \bigcup_{k \in \omega} \zeta_{n_k}$. This is impossible, since both $\emptyset \in U$ and $\bigcup_{k \in \omega} \zeta_{n_k} \in U$. Since $U$ is open in $D_{seq}$, an element of $U$ cannot be the limit of a convergent sequence whose terms are all outside $U$. \hfill $\Box$

**Problem 2:** Determine exactly how large the cardinal $\tau_0$ has to be. Is it equal to the first cardinal $\tau$ such that there exists a sequentially continuous, discontinuous map from $D^\tau$ into some Tychonoff space?

**References**


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