On $L^{2,n}_{loc}$-regularity for the gradient of a weak solution to nonlinear elliptic systems

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Abstract. Interior $L^{2,n}_{loc}$-regularity for the gradient of a weak solution to nonlinear second order elliptic systems is investigated.

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1. Introduction

In this paper we consider the problem of the regularity of the first derivatives of weak solutions to a nonlinear elliptic system

\[(1) \quad -D_{\alpha} (A_{i}^{\alpha} (Du)) = 0, \quad (i = 1, \ldots, N)\]

in a bounded open set $\Omega \subset \mathbb{R}^n$. Throughout the whole text we use the summation convention over repeated indexes.

If $n \geq 3$, it is known that $Du$ may not be continuous. Examples are provided by nonregular solutions of elliptic systems presented by Nečas in [8]. Campanato in [2] proved that $Du \in L^{2,\lambda}_{loc} (\Omega, \mathbb{R}^N)$ with $\lambda(n) < n$, and $u \in C^{0,\alpha}_{loc} (\Omega, \mathbb{R}^N)$ for some $\alpha < 1$ if $n = 3, 4$. In this paper we give sufficient condition on $L^{2,n}_{loc}$-regularity for the gradient of a weak solution to (1). Recall that if $Du \in L^{2,n}_{loc}$, then $u$ is locally Zygmund continuous.

2. Preliminaries

Let $\Omega$ be a bounded open set in $\mathbb{R}^n$ with points $x = (x_1, \ldots, x_n)$, $n \geq 3$. The notation $\Omega_0 \Subset \Omega$ means that the closure of $\Omega_0$ is contained in $\Omega$, i.e. $\overline{\Omega}_0 \subset \Omega$. For the sake of simplicity we denote by $| \cdot |$ and $(,\,)$ the norm and scalar product in $\mathbb{R}^n$, $\mathbb{R}^N$ and $\mathbb{R}^{nN}$. If $x \in \mathbb{R}^n$ and $r$ is a positive real number, we set $B (x, r) = \{ y \in \mathbb{R}^n : |y - x| < r \}$, i.e. the open ball in $\mathbb{R}^n$, $\Omega (x, r) = B (x, r) \cap \Omega$. By $\mu (\Omega (x, r))$ we denote the $n$-dimensional Lebesgue measure of $\Omega (x, r)$. A bounded domain $\Omega \subset \mathbb{R}^n$ is said to be of type $\mathcal{A}$ if there exists a constant $\mathcal{A} > 0$ such that for every $x \in \overline{\Omega}$ and all $0 < r < diam \Omega$ it holds $\mu (\Omega (x, r)) \geq \mathcal{A} r^n$.

Let $u : \Omega \rightarrow \mathbb{R}^N$, $N \geq 1$, $u (x) = (u^1 (x), \ldots, u^N (x))$ be a vector-valued function and $Du = (D_1 u, \ldots, D_n u)$, $D_\alpha = \partial / \partial x_\alpha$. 
By \( u_{x,r} = \mu^{-1}(\Omega(x,r)) \int_{\Omega(x,r)} u(y) \, dy = \int_{\Omega(x,r)} u(y) \, dy \) we denote mean value of \( u \) over the set \( \Omega(x,r) \) provided that \( u \in L^1(\Omega, R^N) \). Besides usually used spaces as \( C_0^\infty(\Omega, R^N) \), the Hölder spaces \( C^{0,\alpha}(\bar{\Omega}, R^N) \) and the Sobolev spaces \( H^{k,p}(\Omega, R^N), H_{\text{loc}}^{k,p}(\Omega, R^N), H_0^{k,p}(\Omega, R^N) \) (see e.g. [1], [6], [7] for definitions and basic properties) we use the following Campanato and Morrey spaces.

**Definition 1** (Campanato and Morrey spaces). Let \( \lambda \in [0,n] \), \( q \in [1,\infty) \). The Morrey space \( L^{q,\lambda}(\Omega, R^N) \) is the subspace of such functions \( u \in L^q(\Omega, R^N) \) for which \( ||u||_{L^{q,\lambda}(\Omega, R^N)}^q = \sup \{r^{-\lambda} \int_{\Omega(x,r)} |u(y)|^q \, dy : r > 0, x \in \Omega \} \) is finite.

Let \( \lambda \in [0,n+q] \), \( q \in [1,\infty) \). The Campanato spaces \( \mathcal{L}^{q,\lambda}(\Omega, R^N) \) and \( \mathcal{L}_1^{q,\lambda}(\Omega, R^N) \) are subspaces of such functions \( u \in L^q(\Omega, R^N) \) for which \( ||u||_{\mathcal{L}^{q,\lambda}(\Omega, R^N)}^q = \sup \{r^{-\lambda} \int_{\Omega(x,r)} |u(y) - u_{x,r}|^q \, dy : r > 0, x \in \Omega \} \) is finite and \( ||u||_{\mathcal{L}_1^{q,\lambda}(\Omega, R^N)}^q = \sup \{\inf \{r^{-\lambda} \int_{\Omega(x,r)} |u(y) - P(y)|^q \, dy : P \in \mathcal{P}_1 \} : r > 0, x \in \Omega \} \) is finite. Here \( \mathcal{P}_1 \) is the set of all polynomials in \( n \) variables and of degree \( \leq 1 \). Let us denote \( ||u||_{\mathcal{L}^{q,\lambda}}, ||u||_{\mathcal{L}_1^{q,\lambda}} = ||u||_{L^q} + [u]_{\mathcal{L}^{q,\lambda}} \) and \( ||u||_{\mathcal{L}^{q,\lambda}} = ||u||_{L^q(\Omega, R^N)} + [u]_{\mathcal{L}_1^{q,\lambda}} \).

**Remark 1.** It is worth to recall a trivial however important property saying that \( \int_{\Omega} |u - u_\Omega|^2 \, dx = \min\{\int_{\Omega} |u - c|^2 \, dx : c \in R^N \} \) for every \( u \in L^2(\Omega, R^N) \).

**Definition 2.** The Zygmund class \( \Lambda^{1}(\bar{\Omega}, R^N) \) is the subspace of such functions \( u \in C^0(\bar{\Omega}, R^N) \) for which \( [u]_{\Lambda^1(\bar{\Omega}, R^N)} = \sup\{|u(x) + u(y) - 2u((x+y)/2)| / |x-y| : x,y,(x+y)/2 \in \bar{\Omega} \} \) is finite.

For more details see [1], [4], [6], [7]. In particular, we will use the following result.

**Proposition 1.** Let \( \Omega \) be of type \( A \) and \( 1 \leq q < \infty \). Then it holds

(a) \( L^{q,\lambda}(\Omega, R^N), \mathcal{L}^{q,\lambda}(\Omega, R^N) \) and \( \mathcal{L}_1^{q,\lambda}(\Omega, R^N) \) equipped with norms \( ||u||_{L^{q,\lambda}}, ||u||_{\mathcal{L}^{q,\lambda}} \) and \( ||u||_{\mathcal{L}_1^{q,\lambda}} \) are Banach spaces.

(b) \( \mathcal{L}^{q,\lambda}(\Omega, R^N) \) is isomorphic to the \( C^{0,(\lambda-n)/q}(\bar{\Omega}, R^N) \) if \( n < \lambda \leq n+q \),

(c) \( L^{q,n}(\Omega, R^N) \) is isomorphic to the \( L^\infty(\Omega, R^N) \subset \mathcal{L}^{q,n}(\Omega, R^N) \),

(d) \( \mathcal{L}_1^{2,n+2}(\Omega, R^N) \) is isomorphic to the \( \Lambda^1(\bar{\Omega}, R^N) \),

(e) \( C^{0,1}(\bar{\Omega}, R^N) \subset \Lambda^1(\bar{\Omega}, R^N) \subset \bigcap_{0<\alpha<1} C^{0,\alpha}(\bar{\Omega}, R^N) \).

Further, we suppose

(i) there is an \( M > 0 \) such that for every \( p \in R^{nN} \)

2. \( |A_t^\alpha(p)| \leq M (1 + |p|) \),
(ii) $A_\alpha^i(p)$ are differentiable functions on $R^{nN}$ with the bounded and continuous derivatives, i.e.

$$\left| \frac{\partial A_\alpha^i}{\partial p^j_\beta} (p) \right| \leq M \quad \text{for every } p \in R^{nN},$$

(iii) the strong ellipticity condition, i.e. there exists $\nu > 0$ such that for every $p, \xi \in R^{nN}$

$$\frac{\partial A_\alpha^i}{\partial p^j_\beta} (p) \xi^i \xi^j_\beta \geq \nu |\xi|^2.$$ 

From (ii) it follows (see [3, p.169]) the existence of a real function $\omega(s)$ defined on $[0, \infty)$, which is nonnegative, bounded, nondecreasing, concave, $\omega(0) = 0$ (moreover, $\omega$ is right continuous at 0 for uniformly continuous $\partial A_\alpha^i / \partial p^j_\beta$) and such that for all $p, q \in R^{nN}$

$$\left| \frac{\partial A_\alpha^i}{\partial p^j_\beta} (p) - \frac{\partial A_\alpha^i}{\partial p^j_\beta} (q) \right| \leq \omega \left( |p - q|^2 \right).$$

By a weak solution of (1) we mean a function $u \in H^{1,2}(\Omega, R^N)$ satisfying

$$\int_\Omega A_\alpha^i (Du) D_\alpha \phi^i dx = 0$$

for every $\phi \in H_0^{1,2}(\Omega, R^N)$.

We will also consider the pair of complementary Young functions

$$\Phi(t) = t \ln_+ at \quad \text{for } t \geq 0, \quad \Psi(t) = \begin{cases} t/a & \text{for } 0 \leq t < 1, \\ e^{t-1/a} & \text{for } t \geq 1, \end{cases}$$

where $a > 0$ is a constant, $\ln_+ at = 0$ for $0 \leq t < 1/a$ and $\ln_+ at = \ln at$ for $t \geq 1/a$. Recall Young's inequality

$$ts \leq \Phi(t) + \Psi(s), \quad t, s \geq 0.$$

For our consideration we also need to introduce quasiconvex functions.

**Definition 3** ([5, p.4]). A function $G : [0, \infty) \rightarrow R$ is said to be quasiconvex (quasiconcave) on $[0, \infty)$ if there exist a convex (concave) function $g (\tilde{g})$ and a constant $c > 0$ ($\tilde{c} > 0$) such that

$$g(t) \leq G(t) \leq cg(ct), \quad (\tilde{g}(t) \leq G(t) \leq \tilde{c} \tilde{g}(\tilde{c}t)) \quad \text{for } t \geq 0.$$
Next, we will need the following properties of quasiconvex functions:

**Lemma 1** ([5, p. 4]). Let us consider three statements:

(a) $G(t)$ is quasiconvex (quasiconcave) on $[0, \infty)$;
(b) for all $t_1, t_2 \in [0, \infty)$ and all $\lambda \in (0, 1)$

$$G(\lambda t_1 + (1 - \lambda) t_2) \leq k_1 (\lambda G(k_1 t_1) + (1 - \lambda) G(k_1 t_2))$$

$$\left( \lambda G(t_1) + (1 - \lambda) G(t_2) \leq l_1 G(l_1 (\lambda t_1 + (1 - \lambda) t_2)) \right);$$

(c) there exists a constant $k_2$ ($l_2$) such that for all $u \in L^2_{loc}(\Omega, R^N)$ and all balls $B(x, r) \subset \Omega$

$$G\left( \iint_{B(x, r)} |u|^2 \, dy \right) \leq k_2 \iint_{B(x, r)} G\left( k_2 |u|^2 \right) \, dy,$$

$$\left( \iint_{B(x, r)} G\left( |u|^2 \right) \, dy \leq l_2 G\left( l_2 \iint_{B(x, r)} |u|^2 \, dy \right) \right).$$

Then (a) $\Rightarrow$ (b) $\Rightarrow$ (c).

**Proposition 2.** For all $u, v \in L^2_{loc}(\Omega, R^N)$, all balls $B(x, r) \subset \Omega$ and an arbitrary nondecreasing quasiconvex function $G$ on $[0, \infty)$ we have

(a)

$$\int_{B(x, r)} G(|u + v|^2) \, dy \leq \frac{k_1}{2} \left( \int_{B(x, r)} G(4k_1 |u|^2) \, dy + \int_{B(x, r)} G(4k_1 |v|^2) \, dy \right),$$

(b)

$$\int_{B(x, r)} G(|u - u_{x,r}|^2) \, dy \leq c_1 \int_{B(x, r)} G(c_2 |u - c|^2) \, dy,$$

where $c_1 = \max\{k_1/2, k_2\}$, $c_2 = \max\{4k_1, 4k_1 k_2\}$ and $c \in R$ is arbitrary.
On $L^2_{\text{loc}}$-regularity for the gradient of a weak solution . . .

Proof: (a) It follows from Lemma 1(b).

(b) From (a) we get

$$\int_{B_r} G\left(|u - u_{x,r}|^2\right) dy \leq \frac{k_1}{2} \left( \int_{B_r} G\left(4k_1 |u|^2\right) dy + \int_{B_r} G\left(4k_1 |c - u_{x,r}|^2\right) dy\right).$$

Now, by means of Hölder’s inequality and Lemma 1(c)

$$\int_{B_r} G\left(4k_1 |c - u_{x,r}|^2\right) dy = \mu(B_r) G\left(4k_1 |c - u_{x,r}|^2\right)$$

$$= \mu(B_r) G\left(\int_{B_r} |u(y) - c| dy\right)^2$$

$$\leq \mu(B_r) G\left(\int_{B_r} |4k_1 |u(y) - c|^2 dy\right) \leq k_2 \int_{B_r} G\left(4k_1 k_2 |u(y) - c|^2\right) dy$$

and the result follows easily. □

**Lemma 2** ([9, p.37]). Let $\varphi: [0, \infty] \to [0, \infty]$ be a monotone function which is absolutely continuous on every closed interval of finite length. If $v \geq 0$ is measurable and $E(t) = \{x \in R^n : v(x) > t\}$, then

$$\int_{R^n} \varphi \circ v dx = \int_0^\infty \mu(E(t)) \varphi'(t) dt.$$

**Proposition 3.** Let $v \in L^2_{\text{loc}}(\Omega, R^m)$, $B(x, \sigma) \subset \Omega$, $a > 0$ and $s \in [1, \infty)$ be arbitrary. If the inequality

$$\int_{B(x, \tau \sigma)} |v - v_{x, \tau \sigma}|^2 dy \leq \int_{B(x, \sigma)} |v - v_{x, \sigma}|^2 dy$$

holds for some $\tau \in (0, 1)$, then there exists a constant $b$ such that

$$\int_{B(x, \tau \sigma)} \ln^s_+ (a |v - v_{x, \tau \sigma}|^2) dy \leq b \int_{B(x, \sigma)} \ln^s_+ (a |v - v_{x, \sigma}|^2) dy.$$

For the constant $b$ we have the following estimate

$$b \leq h \left( \int_{B(x, \sigma)} |v - v_{x, \sigma}|^2 dy \right) \left( \int_{B(x, \sigma)} \ln^s_+ (a |v - v_{x, \sigma}|^2) dy \right)^{-1},$$
where \( h(t) = (s/e(s-1))^{s/(s-1)} \) at, \( t \in [0,e^{s/(s-1)}/a] \) and \( \ln^{s/(s-1)}(at), \ t \in (e^{s/(s-1)}/a, \infty) \).

**Proof:** We set \( E_{\tau \sigma}(t) = \{ y \in B(x, \tau \sigma) : |v-v_{x,\tau \sigma}|^2 > t \} \) for \( t \geq 0 \) and \( 0 < \tau \leq 1 \).

From Lemma 2 and by means of integration by parts we get

\[
\int_{B_{\tau \sigma}} \ln^s(a|v-v_{\tau \sigma}|^2) \, dy = \frac{s}{\mu(B_{\tau \sigma})} \int_{1/a}^{\infty} \mu(E_{\tau \sigma}(t)) \frac{\ln^{s-1}(at)}{t} \, dt
\]

\[
= \frac{s}{\mu(B_{\tau \sigma})} \left[ \ln^{s-1}(at) \int_{0}^{t} \mu(E_{\tau \sigma}(\lambda)) \, d\lambda \right]_{1/a}^\infty
\]

\[
+ \frac{s}{\mu(B_{\tau \sigma})} \int_{1/a}^{\infty} \left( \int_{0}^{t} \mu(E_{\tau \sigma}(\lambda)) \, d\lambda \right) \ln^{s-1}(at) - (s-1) \frac{\ln^{s-2}(at)}{t^2} \, dt.
\]

For the sake of simplicity we put \( V_r = \int_{B(x,r)} |v-v_{x,r}|^2 \, dy \). The first integral is zero and on the second integral we can use the mean value theorem for the integrals and we have for some \( 1/a < \xi_{\tau \sigma}, \xi_{\sigma} < \infty \),

\[
\int_{B_{\tau \sigma}} \ln^s(a|v-v_{\tau \sigma}|^2) \, dy = sV_{\tau \sigma} \int_{\xi_{\tau \sigma}}^{\infty} \frac{\ln^{s-1}(at) - (s-1) \ln^{s-2}(at)}{t^2} \, dt
\]

\[
= \frac{s \ln^{s-1}(a\xi_{\tau \sigma})}{\xi_{\tau \sigma}} V_{\tau \sigma} \frac{\xi_{\sigma}}{\xi_{\tau \sigma} \ln^{s-1}(a\xi_{\sigma}) V_{\tau \sigma}} \int_{B_{\sigma}} \ln^s(a|v-v_{x,\sigma}|^2) \, dy
\]

\[
= b(\tau) \int_{B_{\sigma}} \ln^s(a|v-v_{x,\sigma}|^2) \, dy.
\]

Now the result follows from Lemma 1 (c). \( \square \)

### 3. The result

For \( x \in \Omega, r > 0 \) we set \( U_r = U(x,r) = \int_{\Omega(x,r)} |Du-(Du)_{x,r}|^2 \, dy, \ d_x = \text{dist}(x, \partial \Omega) \) and \( \alpha_n = \mu(B(0,1)) \). We define \( S_0 = \{ x \in \Omega : \lim_{r \to 0+} U(x, r) > 0 \} \).

**Remark 2.** Let \( u \) be a solution of (1). It is well known (see [9, pp.75, 122]) that \( \lim_{r \to 0+} U(x, r) = 0 \) for all \( x \in \Omega \setminus E \) where \( n - 2 + \beta \) dimensional Hausdorff measure \( H^{n-2+\beta}(E) = 0 \) for every \( \beta > 0 \).

Now we can formulate the main theorem.
Theorem. Let \( u \in H^{1,2}(\Omega, R^N) \) be a weak solution to the nonlinear system (1) under the hypotheses (i), (ii), (iii). Let \( x \in S_0 \) be arbitrary and suppose that there exists \( d \in (0, d_x/2) \) such that

\[
(8) \quad \frac{Kl_2 \omega^2}{\nu^2} \left( \int_{B(x,2d)} \ln^\ast((q-1)^{\frac{4l_2 \omega^2}{CU_2d}} |Du - (Du)_{x,2d}|^2) \, dy \right)^{1-1/q} < \frac{1}{4^r n},
\]

where \( K = c(n, N, q) (M/\nu)^8, \tau = (2^{n+5} A)^{-1/2}, l_2, A \) are the constants from Lemma 1(c), Lemma 3, \( \omega = \omega(2^n l_2 U_{2d}), \omega \) is from (5), \( C = 2^{n-8} \nu^2 \tau^n / \alpha_n A \) and \( b \) is the constant from Proposition 3 for the case \( a = 1/\nu, \sigma = 2d, v = 2\sqrt{l_2} \omega Du, s = q/(q - 1) \) where \( q \in (1, n/(n - 2)) \). Then there exists a ball \( B(x, r_x) \subset \Omega \) such that \( Du \in L^2,n(B(x, r_x), R^{nN}) \) and

\[
(9) \quad [Du]^{2,n}_{L^2,n(B(x, r_x), R^{nN})} \leq \max \{2^n (4A \tau^{-n} + 1) U_{2d}, \mu^{-1}(B_{2d}) \int_\Omega |Du - (Du)_{\Omega}|^2 \, dx \}.
\]

Proposition 4. Set \( \omega_\infty = \lim_{t \to \infty} \omega(t), V_1 = c_1 (M/\nu)^{3n+8} (\omega_\infty/\nu)^2 \) and \( V_2 = c_2 (M/\nu)^{3n+6} (\omega_\infty/\nu)^2 \). If

\[
(10) \quad V_2 \leq e^q \ & \ & q^{q-1} V_1 V_2^{1-1/q} < 1 \quad \text{or} \quad V_2 > e^q \ & \ & V_1 \ln^{q-1} V_2 < 1,
\]

then condition (8) holds for every \( x \in S_0 \). Here \( q \in (1, n/(n - 2)), c_1 = c_1(n, N, q) \) and \( c_2 = c_2(n, N) \).

Proof: Let \( x \in S_0 \) and \( d \in (0, d_x/2) \) be arbitrary such that \( U(x, 2d) > 0 \). From Proposition 3 it follows that the left hand side of (8) is equal or less than \( Kl_2 \omega_\infty^2 h^{-1-1/q} \left( 4\omega_\infty^2 U_{2d} \right) / \nu^2 \). From the definition of the function \( h(t) \) and assumption (10) it follows that (8) is satisfied.

Example. We can consider the system (1) for \( n = 3, N = 2 \) where \( A_i^\alpha(p) = (a \delta_{ij} \delta_{\alpha\beta} + b \delta_{i\alpha} \delta_{j\beta} \arctan |p|^2/2\pi) p_j^\beta, a, b \) are constants, \( 0 < b/6 < a \). We have

\[
\frac{\partial A_i^\alpha}{\partial p_j^\beta}(p) \xi_i^\alpha \xi_j^\beta \geq (a - b/6) |\xi|^2, \quad \forall \xi, p \in R^6,
\]

\( \omega_\infty \leq b \) and \( \left| \frac{\partial A_i^\alpha}{\partial p_j^\beta}(p) \right| \leq M = a + b \). Setting \( P = b/a \) we get that \( V_1 < 4c_1 P^2 (1 + P)^{3n+8} / (1 - P/6)^{3n+10}, V_2 < 4c_2 P^2 (1 + P)^{3n+6} / (1 - P/6)^{3n+8} \) and it is not difficult to see that (10) is satisfied for \( P \) sufficiently small.
Corollary 1. Let \( \Omega_0 \subseteq \Omega \) be arbitrary and of type \( A \) and the assumptions of Theorem be satisfied for every \( x \in \overline{\Omega_0} \cap S_0 \). Then there are constants \( U, d_0, r_0 > 0 \) such that \( Du \in L^{2,n}(\Omega_0, R^n_N) \) and the following estimate

\[
[Du]_{L^{2,n}(\Omega_0, R^n_N)}^2 \leq \max\{2^n(4A\tau - n + 1)U, \\
\mu^{-1}(B_{2d_0}) \int_{\Omega} |Du - (Du)_\Omega|^2 \, dx, \\
(Ar_0^n)^{-1} \int_{\Omega_0} |Du - (Du)_{\Omega_0}|^2 \, dx\}
\]

holds.

Proof: From Remark 2, Theorem and the definition of the set \( S_0 \) it follows that for every \( x \in \Omega_0 \) there exists \( B(x, r_x) \subset \Omega \) such that \( Du \in L^{2,n}(B(x, r_x), R^n_N) \). As \( \Omega_0 \) is the compact set and the system balls \( \{B(x, r_x)\} \) covers of \( \Omega_0 \) we can choose a finite subcover \( \{B(x_j, r_{x_j})\}_{j=1}^m \). If we set \( \mathcal{U} = \max\{U(x_j, 2d_{x_j}) : 1 \leq j \leq m\} \), \( r_0 = \min\{r_{x_j} : 1 \leq j \leq m\} \) and \( d_0 = \min\{d_{x_j} : 1 \leq j \leq m\} \), then the estimate follows from Remark 1.

Corollary 2. Let the assumptions of Corollary 1 be satisfied. Then \( u \in A^1(\overline{\Omega_0}, R^N) \).

Proof: It follows from Proposition 1(d), Poincaré’s inequality and Corollary 1.

4. Lemmas

The statement of the following lemma is well known (see e.g. [1], [3], [7], [8]).

Lemma 3. Let \( v \in H^{1,2}(\Omega, R^N) \) be a weak solution to the system (1) satisfying (i), (ii) and (iii), where \( \partial A^a_i/\partial p^j_\beta \) are the constants. Then there exists a constant \( A = c(n, N)(M/\nu)^6 \) such that for every \( x \in \Omega \) and \( 0 < \sigma \leq R \leq \text{dist}(x, \partial \Omega) \) the following estimate holds

\[
\int_{B(x, \sigma)} |Dv(y) - (Dv)_{x, \sigma}|^2 \, dy \leq A \left( \frac{\sigma}{R} \right)^{n+2} \int_{B(x, R)} |Dv(y) - (Dv)_{x, R}|^2 \, dy.
\]

The following lemma is possible to derive by the difference quotient method (see e.g. [1], [3], [7], [8]).

Lemma 4. Let \( u \in H^{1,2}(\Omega, R^N) \) be a weak solution to the system (1) satisfying (i), (ii) and (iii). Then \( u \in H^{2,2}_{\text{loc}}(\Omega, R^N) \) and for all \( x \in \Omega \), \( 0 < \sigma < \varrho \leq \text{dist}(x, \partial \Omega) \) we have

\[
\int_{B(x, \sigma)} |D^2 u|^2 \, dy \leq \frac{6n(M/\nu)^2}{(\varrho - \sigma)^2} \int_{B(x, \varrho)} |Du - (Du)_{x, \varrho}|^2 \, dy.
\]
Lemma 5 ([6]). Let $1 \leq p, q < \infty$, $p^{-1} - q^{-1} \leq n^{-1}$, $R > 0$, $x \in \mathbb{R}^n$. Then for $u \in H^{1,p}(B(x,R), \mathbb{R}^N)$ we have
\[
\left( \int_{B(x,R)} |u(y)|^q \, dy \right)^{1/q} \leq cR^{1+n/q-n/p} \left( R^{-p} \int_{B(x,R)} |u(y)|^p \, dy + \int_{B(x,R)} |Du(y)|^p \, dy \right)^{1/p},
\]
where $c = c(n, N, p, q)$ is a constant independent of $x$, $R$ and $u$.

Lemma 6. Let $u \in H^{1,2}(\Omega, \mathbb{R}^N)$ be a weak solution to (1) satisfying (i), (ii) and (iii). Then for every ball $B(x, 2R) \subset \Omega$ and an arbitrary constant $a > 0$ we have
\[
\int_{B(x,R)} |Du - (Du)_{x,R}|^2 \ln(a |Du - (Du)_{x,R}|^2) \, dy \leq c \left( \frac{M}{\nu} \right)^2 \left( \frac{1}{\ln^q(q-1)} \left( 4a |Du - (Du)_{x,2R}|^2 \right) \, dy \right)^{1-1/q} \int_{B(x,2R)} |Du - (Du)_{x,2R}|^2 \, dy,
\]
where $1 < q \leq n/(n-2)$ and $c = c(n, N, q)$.

**Proof:** Let $x \in \Omega$ and $0 \leq R \leq \frac{1}{4} \text{dist}(x, \partial \Omega)$. We denote $B_R = B(x, R)$ for simplicity. From Lemma 4 it follows that $Du \in H^{1,2}_{loc}(\Omega, \mathbb{R}^N)$. By means of Sobolev’s imbedding theorem $H^{1,2}(B_R, \mathbb{R}^N) \hookrightarrow L^s(B_R, \mathbb{R}^N)$, where $B_R \subset \Omega$ be arbitrary and $1 \leq s \leq 2n/(n-2)$. From this we obtain by Proposition 2 (b) and Lemma 5
\[
\int_{B_R} |Du - (Du)_R|^2 \ln(a |Du - (Du)_R|^2) \, dy \leq 4 \int_{B_R} |Du - (Du)_{2R}|^2 \ln(4a |Du - (Du)_{2R}|^2) \, dy \leq 4 \left( \int_{B_R} |Du - (Du)_{2R}|^2 \, dy \right)^{1/q} \left( \int_{B_R} \ln^{q/(q-1)}(4a |Du - (Du)_{2R}|^2) \, dy \right)^{1-1/q} \leq cR^{n(1/q-1)+2} \left( R^{-2} \int_{B_R} |Du - (Du)_{2R}|^2 + \int_{B_R} |D^2u|^2 \, dy \right) \times
\]
\[
\times \left( \int_{B_R} \ln^{q/(q-1)} \left( 4a |Du - (Du)_R|^2 \right) dy \right)^{1-1/q}
\]
\[
\leq c \left( \frac{M}{\nu} \right)^2 R^{-n(1-1/q)} \int_{B_{2R}} |Du - (Du)_R|^2 dy \times
\]
\[
\times \left( \int_{B_R} \ln^{q/(q-1)} \left( 4a |Du - (Du)_R|^2 \right) dy \right)^{1-1/q}
\]

and we finally obtain the result. \[\square\]

5. Proof of Theorem

Set \( A_{ij}^{\alpha \beta} (\zeta) = \partial A_i^{\alpha} / \partial p_j^{\beta} (\zeta) \), \( A_{ij,0}^{\alpha \beta} = A_{ij}^{\alpha \beta} ((Du)_R) \),

\[
\tilde{A}_{ij}^{\alpha \beta} = \int_0^1 A_{ij}^{\alpha \beta} ((Du)_R + t (Du - (Du)_R)) dt,
\]

\( B_R = B(x, R) \) and \( U_R = U(x, R) \) for simplicity. Then the system (1) can be rewritten as

\[
-D_\alpha \left( A_{ij,0}^{\alpha \beta} D_{\beta} u^j \right) = -D_\alpha \left( \left( A_{ij,0}^{\alpha \beta} - \tilde{A}_{ij}^{\alpha \beta} \right) \left( D_{\beta} u^j - \left( D_{\beta} u^j \right)_R \right) \right).
\]

Split \( u \) as \( v + w \) where \( v \) is the solution of the Dirichlet problem

\[
\begin{cases}
-D_\alpha \left( A_{ij,0}^{\alpha \beta} D_{\beta} v^j \right) = 0 & \text{in } B_R \\
v - u \in H_0^{1,2} (B_R, R^N). 
\end{cases}
\]

For every \( 0 < \sigma \leq R \) from Lemma 3 it follows

\[
\int_{B_\sigma} |Dv - (Dv)_\sigma|^2 dy \leq A \left( \frac{\sigma}{R} \right)^{n+2} \int_{B_R} |Dv - (Dv)_R|^2 dy,
\]

hence

\[
(12) \int_{B_\sigma} |Du - (Du)_\sigma|^2 dy \leq 2A \left( \frac{\sigma}{R} \right)^{n+2} \int_{B_R} |Dv - (Dv)_R|^2 dy + 2 \int_{B_R} |Dw|^2 dy.
\]

Now \( w \in H_0^{1,2} (B_R, R^N) \) satisfies

\[
\int_{B_R} A_{ij,0}^{\alpha \beta} D_{\beta} w^j D_{\alpha} \varphi^i dy \leq \int_{B_R} \left| A_{ij,0}^{\alpha \beta} - \tilde{A}_{ij}^{\alpha \beta} \right| \left| D_{\beta} u^j - \left( D_{\beta} u^j \right)_R \right| \left| D_{\alpha} \varphi^i \right| dy
\]

\[
\leq \left( \int_{B_R} \omega^2 \left( |Du - (Du)_R|^2 \right) |Du - (Du)_R|^2 dy \right)^{1/2} \left( \int_{B_R} |D\varphi|^2 dy \right)^{1/2}
\]
for any $\varphi \in H^{1,2}_{0}(B_{R},\mathbb{R}^{N})$, where $\omega$ is from (5). Hence, choosing $\varphi = w$, we get
\[
\nu^{2} \int_{B_{R}} |Dw|^{2} \, dy \leq \int_{B_{R}} \omega^{2} \left( |Du - (Du)_{R}|^{2} \right) |Du - (Du)_{R}|^{2} \, dy.
\]

Now applying the Young inequality (with the complementary functions (7)) on the right-hand side, we obtain for every $\varepsilon > 0$
\[
(13) \quad \nu^{2} \int_{B_{R}} |Dw|^{2} \, dy \leq \varepsilon \int_{B_{R}} |Du - (Du)_{R}|^{2} \ln_{+} \left( a\varepsilon |Du - (Du)_{R}|^{2} \right) \, dy
\] + \( \frac{2}{a} \int_{B_{R}} e^{\omega_{R}^{2}/\varepsilon - 1} \, dy \),
where $\omega_{R}^{2} = \omega^{2}(|Du - (Du)_{R}|^{2})$.

From (12) and (13) it follows
\[
(14) \quad \int_{B_{\sigma}} |Du - (Du)_{\sigma}|^{2} \, dy \leq 4A \left( \frac{\sigma}{R} \right)^{n+2} \int_{B_{R}} |Du - (Du)_{R}|^{2} \, dy
\] + \( \frac{2(2A + 1)}{\nu^{2}} \left( \varepsilon \int_{B_{R}} |Du - (Du)_{R}|^{2} \ln_{+} \left( a\varepsilon |Du - (Du)_{R}|^{2} \right) \, dy \right)^{1-1/q}
\] + \( \frac{2}{a} \int_{B_{R}} e^{\omega_{R}^{2}/\varepsilon - 1} \, dy \),

We can estimate the right-hand side by means of Lemma 1 (c) (for the quasiconcave case), Lemma 6 and we get
\[
\int_{B_{\sigma}} |Du - (Du)_{\sigma}|^{2} \, dy \leq 4A \left( \frac{\sigma}{R} \right)^{n+2} \int_{B_{R}} |Du - (Du)_{R}|^{2} \, dy
\] + \( \frac{2(2A + 1)}{\nu^{2}} \left[ \varepsilon c \left( \frac{M}{\nu} \right)^{2} \left( \int_{B_{2R}} \ln_{+}^{q/(q-1)} \left( 4a\varepsilon |Du - (Du)_{2R}|^{2} \right) \, dy \right)^{1-1/q}\right.
\] \[ \times \int_{B_{2R}} |Du - (Du)_{2R}|^{2} \, dy + \frac{2\alpha_{n}R^{n}}{a} e^{l_{2}^{2}\omega^{2}(l_{2}U_{R})/\varepsilon - 1} \] \] \] \[ \times \int_{B_{2R}} |Du - (Du)_{2R}|^{2} \, dy + \frac{2\alpha_{n}R^{n}}{a} e^{l_{2}^{2}\omega^{2}(l_{2}U_{R})/\varepsilon - 1} \] \].

Setting
\[
\phi(t) = \int_{B_{t}} |Du - (Du)_{t}|^{2} \, dy,
\] \[ F_{\varepsilon}(t) = \left( \int_{B_{t}} \ln_{+}^{q/(q-1)} \left( 4a\varepsilon |Du - (Du)_{t}|^{2} \right) \, dy \right)^{1-1/q},
\]
we can rewrite the previous inequality as follows:
\begin{equation}
\phi(s) \leq 4A\left(\frac{\sigma}{R}\right)^{n+2} \phi(R) + \frac{K\varepsilon}{\nu^2} F_\varepsilon(2R)\phi(2R) + \frac{2^4\alpha_n A}{\alpha \nu^2} e^{l_2 \omega^2 (2^n l_2 U_{2R})/\varepsilon - 1} R^n,
\end{equation}

where $K = c(n, N, q) (M/\nu)^8$. From the assumptions of Theorem it follows that there exists $d \in (0, d_x/2)$ such that (8) holds. Now we are going to prove that
\begin{equation}
\phi\left(2^{\tau k} d\right) \leq \tau^{kn} \phi(2d)
\end{equation}
for every natural number $k$ and $\tau = (2^{n+5} A)^{-1/2}$. Let $k = 1$. If we put in (15) $a = 1/CU_{2d}, \varepsilon = l_2 \omega^2 (2^n l_2 U_{2d}), \sigma = 2\tau d$ and $R = d$ we get
\[
\phi(2\tau d) \leq 2^{n+4} A \tau^{n+2} \phi(d) + \frac{K l_2 \omega^2}{\nu^2} F_\varepsilon(2d) \phi(2d) + \frac{2^4\alpha_n A}{\alpha \nu^2} CU_{2d} d^n
\leq 2^{n+4} A \tau^{n+2} \phi(d) + \frac{K l_2 \omega^2}{\nu^2} b^{1-1/q} F_\varepsilon(2d) \phi(2d) + \frac{1}{4} \tau^n \phi(2d)
\leq \left(2^{n+4} A \tau^2 + \frac{1}{4} + \frac{1}{4}\right) \tau^n \phi(2d) = \tau^n \phi(2d).
\]
Thus (16) holds for $k = 1$. Consequently $U_{2\tau d} \leq U_{2d}$ and by means of Proposition 3 we have $F_\varepsilon(2\tau d) \leq b^{1-1/q} F_\varepsilon(2d)$.

Let us suppose that (16) holds for $k \geq 1$. Similarly to consideration above we have $U_{2\tau k d} \leq U_{2d}$ and $F_\varepsilon\left(2^{\tau k} d\right) \leq b^{1-1/q} F_\varepsilon(2d)$. We will show that (16) holds for $k + 1$. Setting $a = 1/CU_{2d}, \varepsilon = l_2 \omega^2 (2^n l_2 U_{2d}), \sigma = 2\tau^{k+1} d$ and $R = \tau^k d$ in (15) we obtain
\[
\phi(2^{\tau k+1} d) \leq 2^{n+4} A \tau^{n+2} \phi\left(\tau^k d\right) + \frac{K l_2 \omega^2}{\nu^2} F_\varepsilon(2\tau^k d) \phi(2\tau^k d)
\leq 2^{n+4} A \tau^{n+2} \phi\left(2^{\tau k} d\right) + \frac{K l_2 \omega^2}{\nu^2} F_\varepsilon(2\tau^k d) \phi(2\tau^k d) + \frac{1}{4} \tau^{(k+1)n} \phi(2d)
\leq 2^{n+4} A \tau^{n+2} \tau^{kn} \phi(2d) + \frac{K l_2 \omega^2}{\nu^2} b^{1-1/q} F_\varepsilon(2d) \tau^{kn} \phi(2d) + \frac{1}{4} \tau^{(k+1)n} \phi(2d)
\leq \left(2^{n+4} A \tau^2 + \frac{1}{4} + \frac{1}{4}\right) \tau^{(k+1)n} \phi(2d) = \tau^{(k+1)n} \phi(2d).
\]
Let us consider the two possibilities:
(a) if $\tau \leq t < 1$, then $t^{-n}\phi(t_d) \leq \tau^{-n}\phi(t_d) \leq \tau^{-n}\sup_{t \in [\tau,1]} \phi(t_d)$ and also

$$\phi(t_d) \leq \left(\tau^{-n}\sup_{t \in [\tau,1]} \phi(t_d)\right)t^n,$$

(b) if $0 < t < \tau$, then there exists natural $k \geq 1$ such that $\tau^{k+1} \leq t < \tau^k$. From Proposition 3, (8), (16) and (15) with $a = 1/CU_2$, $\varepsilon = l_2\omega^2(2^nU_2)$, $\sigma = t_d$ and $R = \tau^kd$ we have

$$\phi(t_d) = \phi\left(\frac{t}{\tau^k}(\tau^k d)\right)$$

$$\leq 4A \left(\frac{t}{\tau^k}\right)^{n+2} \phi(\tau^kd) + \frac{K\varepsilon}{\nu^2} F_\varepsilon(2^k \tau d)\phi\left(2^k \tau d\right)$$

$$+ \frac{2^4\alpha_n A}{\alpha^2\nu \tau^k} e^{l^2\omega^2/t_2U_2d}/\varepsilon - 1 \tau^k d^n$$

$$\leq 4A \left(\frac{t}{\tau^k}\right)^{n+2} \tau^{kn} \phi(2d) + \frac{K\omega^2}{\nu^2} b_1^{-1/q} F_\varepsilon(2d) \tau^k \phi(2d)$$

$$+ \frac{2^4\alpha_n A}{\nu^2} CU_2d \tau^k d^n$$

$$\leq \left(4A \left(\frac{t}{\tau^k}\right)^{n+2} \tau^{kn} + \tau^{(k+1)n}\right) \phi(2d)$$

$$\leq \left(4A\tau^{-n} \left(\frac{t}{\tau^k}\right)^{n+2} + 1\right) \tau^{(k+1)n} \phi(2d) < (4A\tau^{-n} + 1) t^n \phi(2d).$$

In both cases (17) and (18) we obtain

$$t^{-n} \phi(t_d) \leq c, \quad t \in (0,1],$$

where $c = \max\{\tau^{-n}\sup_{t \in [\tau,1]} \phi(t_d), (4A\tau^{-n} + 1) \phi(2d)\} = (4A\tau^{-n} + 1) \phi(2d)$. Let $0 < r < \text{dist}(B(x,r_x), \partial\Omega)$. Hence $U(y,r)$ is uniformly continuous for fixed $r$ in $B(x,r_x) \subset \Omega$. According to Proposition 3, the expression

$$\frac{Kl_2\omega^2}{\nu^2} \left(\int_{B(y,r)} \ln^{q/(q-1)} \left(\frac{4l_2\omega^2 |Du - (Du)_{y,r}|^2}{CU(y,r)}\right) dz\right)^{1-1/q}$$

is also uniformly continuous with respect to $y$ in $\overline{B(x,r_x)}$ and we arrive at the conclusion.

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References


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