A note on existence and uniqueness of solutions of neutral functional-differential equations with state-dependent delays

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Abstract. Existence and uniqueness theorem for state-dependent delay-differential equations of neutral type is given. This theorem generalizes previous results by Grimm and the author.

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Consider the scalar initial-value problem for state-dependent delay-differential equations of neutral type

\[ y'(t) = f(t, y(t), y(\alpha(t, y(t))), y'(\beta(t, y(t)))), \quad t \in [a, b],\]
\[ y(t) = g(t), \quad t \in [\gamma, \alpha],\]

where \( \gamma \leq a < b \), \( \gamma \leq \alpha(t, y) \leq t \), \( \gamma \leq \beta(t, y) \leq t \), and \( g \) is a given initial function.

We assume the following:

(i) \( g \) and \( g' \) are Lipschitz-continuous with constants \( L_g \) and \( L_{g'} \) respectively;
(ii) \( f(a, g(a), g(\alpha(a, g(a)))), g'(\beta(a, g(a)))) = g'(a) \), where \( g'(a) \) denotes the left hand side derivative.

Moreover, suppose that in their respective domains \( f, \alpha \) and \( \beta \) satisfy the following conditions with nonnegative Lipschitz constants:

(iii) \[ |f(t_1, y_1, u_1, z_1) - f(t_2, y_2, u_2, z_2)| \leq L_1(|t_1 - t_2| + |y_1 - y_2| + |u_1 - u_2|) + L_2|z_1 - z_2|, \quad L_2 < 1; \]
(iv) \[ |\alpha(t_1, y_1) - \alpha(t_2, y_2)| \leq A_1|t_1 - t_2| + A_2|y_1 - y_2|; \]
(v) \[ |\beta(t_1, y_1) - \beta(t_2, y_2)| \leq B_1|t_1 - t_2| + B_2|y_1 - y_2|. \]

The problem (1) with \( \gamma = a \) was studied by Grimm [1]. He proved an existence result for (1) assuming that \( f \) is bounded by some constant \( M \), \( L_2 < 1 \), and \( B_1 + B_2M \leq 1 \). He also proved a uniqueness result when \( \beta \) is independent of \( y \). In the recent paper [2] the author relaxed this very restrictive assumption at the expense of the additional condition \( L_2(1 + B_1 + B_2G) < 1 \), where \( G \) is some constant depending on \( f \) and \( g \). This condition means that the dependence of \( f \) on the last argument is not too strong. It is the purpose of this note to improve
further the results given in [1] and [2]. We prove the existence and uniqueness theorem for (1) (with \( \beta \) depending both on \( t \) and \( y \)), where the inequality \( L_2(1 + B_1 + B_2G) < 1 \) is replaced by the weaker conditions \( L_2 < 1 \) and \( B_1 + B_2G \leq 1 \).

For any continuous functions \( y \) and \( z \) on \([\gamma, b]\), put
\[
F(t, y, z) := f(t, y(t), y(\alpha(t, y(t))), z(\beta(t, y(t)))),
\]
and define
\[
M := \sup\{|F(t,0,0)| : t \in [a, b]\}; \quad C_1 := (g'_{[\gamma, a]} + M)/(1 - L_2);
\]
\[
C_2 := 2L_1/(1 - L_2); \quad Y := (g_{[\gamma, a]} + C_1/C_2)\exp((b-a)C_2);
\]
\[
Z := \max\{C_1 + C_2Y, (M + 2L_1Y)/(1 - L_2)\}; \quad G := \max\{Lg, Z\};
\]
\[
D := \max\{Lg', L_1(1 + G(1 + A_1 + A_2G))/(1 - L_2(B_1 + B_2G))\}.
\]
Here \( x_{[c,d]} := \sup\{|x(t)| : t \in [c,d]\} \) for any function \( x \). We have the following:

**Theorem.** Assume that (i)–(v) hold, \( L_2 < 1 \), and \( B_1 + B_2G \leq 1 \). Then (1) has a solution \( y \) whose derivative is Lipschitz-continuous. Moreover, this solution is unique in the space of continuously differentiable functions on \([\gamma, a]\).

**Proof:** For \( h \in J := \{h \mid h = (b - a)/n, n \geq n_0\} \), where \( n_0 \) is a positive integer, put \( t_i = a + ih, i = 0, 1, \ldots, n \), and as in [2] define the modified Euler sequences \( \{y_h\}_{h \in J} \) and \( \{z_h\}_{h \in J} \) by
\[
y_h(t_i + rh) = y_h(t_i) + rhz_h(t_i),
\]
\[
z_h(t_i + rh) = (1 - r)z_h(t_i) + rz_h(t_{i+1}), \quad z_h(t_{i+1}) = F(t_{i+1}, y_h, z_h),
\]
i = 0, 1, \ldots, n - 1, \( r \in (0, 1] \), where \( y_h(t) = g(t) \) and \( z_h(t) = g'(t) \) for \( t \in [\gamma, a] \). Note that (2) is, in general, implicit in \( z_h \), but in view of \( L_2 < 1 \) it has a unique solution \( (y_h, z_h) \) for any \( h \in J \). We will first show that \( \{y_h\}_{h \in J} \) and \( \{z_h\}_{h \in J} \) are relatively compact in the space \( C[\gamma, b] \) of continuous functions on \([\gamma, b]\). Proceeding as in [2] it follows that \( \{y_h\}_{h \in J} \) and \( \{z_h\}_{h \in J} \) are uniformly bounded by \( Y \) and \( Z \), respectively, and that \( \{y_h\}_{h \in J} \) are uniformly Lipschitz-continuous with the constant \( G \). The proof that \( \{z_h\}_{h \in J} \) are also uniformly Lipschitz-continuous is more delicate than in [2]. The proof is by induction. Assume that
\[
|z_h(t_1) - z_h(t_2)| \leq D|t_1 - t_2|, \quad t_1, t_2 \in [\gamma, t_i],
\]
and we will show that this inequality is also true for \( t_1, t_2 \in [\gamma, t_{i+1}] \) (obviously (3) holds for \( t_1, t_2 \in [\gamma, t_0] \)). Define on \([\gamma, t_{i+1}]\) the iterations \( z_h^{[\nu]}(t) = z_h(t) \) for \( t \in [\gamma, t_i], \nu = 0, 1, \ldots, \), and
\[
z_h^{[0]}(t_i + rh) = z_h(t_i),
\]
\[
z_h^{[\nu+1]}(t_{i+1}) = F(t_{i+1}, y_h, z_h^{[\nu]}),
\]
\[
z_h^{[\nu+1]}(t_i + rh) = (1 - r)z_h(t_i) + rz_h^{[\nu+1]}(t_{i+1}),
\]
It follows by the induction with respect to $\nu$ that 
\[ \{z_h^{[\nu]}\}_{\nu=0}^{\infty} \]
are uniformly bounded by $Z$ and uniformly Lipschitz-continuous on 
$[\gamma, t_{i+1}]$ with the same constant $D$. Indeed, this is true for $\nu = 0$ and, assuming 
that it is true for $\nu$, routine manipulations yield
\[
\left| z_h^{[\nu+1]}(t_{i+1}) \right| \leq M + 2L_1 Y + L_2 Z \leq Z,
\]
and
\[
\left| z_h^{[\nu+1]}(t_{i+1}) - z_h^{[\nu+1]}(t_i) \right| 
\leq L_1 (1 + G(1 + A_1 + A_2 G)) h + L_2 D(B_1 + B_2 G) h \leq Dh.
\]
The last inequality follows from the definition of $D$. In view of the Ascoli-Arzela 
theorem the sequence $\{z_h^{[\nu]}\}_{\nu=0}^{\infty}$ is relatively compact in $C[\gamma, t_{i+1}]$ and since the 
solution $(y_h, z_h)$ of (2) is unique, we have $z_h^{[\nu]} - z_h^{[\nu, t_{i+1}]} \to 0$ as $\nu \to \infty$. Therefore, 
$\{z_h\}_{h \in J}$ are uniformly Lipschitz-continuous on $[\gamma, t_{i+1}]$ with the same constant $D$. 
By induction with respect to $i$, this is also true on $[\gamma, b]$. Consequently, $\{y_h\}_{h \in J}$ and $\{z_h\}_{h \in J}$ are relatively compact in $C[\gamma, b]$ and from this point the proof is 
extactly the same as the proof of Theorem 2 in [2]. We prove the existence by 
showing that there is a subsequence of $\{y_h\}_{h \in J}$ convergent to the solution $y$ of 
(1) and we prove the uniqueness by contradiction. \hfill $\Box$

References

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