Whitney blocks in the hyperspace of a finite graph

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Abstract. Let $X$ be a finite graph. Let $C(X)$ be the hyperspace of all nonempty sub-continua of $X$ and let $\mu : C(X) \to \mathbb{R}$ be a Whitney map. We prove that there exist numbers $0 < T_0 < T_1 < T_2 < \cdots < T_M = \mu(X)$ such that if $T \in (T_{i-1}, T_i)$, then the Whitney block $\mu^{-1}(T_{i-1}, T_i)$ is homeomorphic to the product $\mu^{-1}(T) \times (T_{i-1}, T_i)$. We also show that there exists only a finite number of topologically different Whitney levels for $C(X)$.

Keywords: hyperspaces, Whitney levels, Whitney blocks, finite graphs

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Introduction

Throughout this paper $X$ denotes a finite graph, i.e. a compact connected metric space which is the union of finitely many segments joined by their end points. A segment of $X$ is one of those segments. A subgraph of $X$ is a graph contained in $X$ formed by some of those segments. Let $SG(X) = \{ A \subset X : A$ is a subgraph of $X \}$.

The hyperspace of subcontinua of $X$ is $C(X) = \{ A \subset X : A$ is a nonempty, closed, connected subset of $X \}$ metrized with the Hausdorff metric. Let $F_1(X) = \{ \{x\} \in C(X) : x \in X \}$. A map is a continuous function. A Whitney map for $C(X)$ (see [8, 0.50]) is a map $\mu : C(X) \to \mathbb{R}$ such that $\mu(\{x\}) = 0$ for every $x \in X$, $\mu(A) < \mu(B)$ if $A \subset B \neq A$ and $\mu(X) = 1$. A Whitney level is a set of the form $\mu^{-1}(t)$, where $t \in [0, 1]$. A Whitney block is a set of the form $\mu^{-1}(t, s)$, where $0 \leq t < s \leq 1$. From now on, $\mu$ will denote a Whitney map for $C(X)$.

In [1], R. Duda made a detailed study of the polyhedral structure of $C(X)$ by giving a good decomposition of $C(X)$ into balls. In [2], he gave a characterization of those polyhedra which are hyperspaces of acyclic finite graphs.

Whitney levels of finite graph have been studied by H. Kato. In [4] he showed that they are always polyhedra and that if $t_0 = \min\{ \mu(A) : A$ is a simple closed curve contained in $X \}$ and $0 \leq t < t_0$, then $\mu^{-1}(t)$ is homotopically equivalent to $X$. In [4] and [6] he gave bounds for the fundamental dimension of Whitney levels of finite graphs and, in [5] he proved that Whitney levels of finite graphs admit all homotopy types of compact connected ANRs.

This paper was motivated by the following result of I. Puga ([10, Theorem 2.5]): “There exists $t \in [0, 1)$ and there exists a homeomorphism $\varphi : (\text{Cone over} \ \mu^{-1}(t))$
\[ \rightarrow \mu^{-1}(t, 1) \text{ such that } \varphi(A, 0) = A, \varphi(A, 1) = X \text{ and } s < t \text{ implies that } \varphi(A, s) \subset \varphi(A, t) \text{ for each } A \in \mu^{-1}(t). \]

She expressed this property by saying that the hyperspace of subcontinua of a finite graph is conical pointed.

In this paper, we prove:

**Theorem 1.** Suppose that \( \mu(SG(X)) \cup \{0\} = \{T_0, T_1, \ldots, T_M\} \), where \( 0 = T_0 < T_1 < \cdots < T_M = 1 \). If \( 1 \leq i \leq M \) and \( T \in (T_{i-1}, T_i) \), then there exists a homeomorphism \( \varphi : \mu^{-1}(T) \times (T_{i-1}, T_i) \rightarrow \mu^{-1}(T_{i-1}, T_i) \) such that \( \varphi(A, T) = A \) and \( \varphi(A, s) \subset \varphi(A, t) \) if \( s < t \) for every \( A \in \mu^{-1}(T) \) and, for each \( t \in (T_{i-1}, T_i) \), \( \varphi|\mu^{-1}(T) \times \{t\} \) is a homeomorphism from \( \mu^{-1}(T) \times \{t\} \) onto \( \mu^{-1}(t) \).

**Theorem 2.** There is only a finite number of topologically different Whitney levels for \( C(X) \).

1. **Preliminaries**

The vertices of \( X \) are the end points of the segments of \( X \). Notice that the set \( SG(X) \) of subgraphs of \( X \) depends on the choice of the segments. We are interested in having as less subgraphs as possible, so we will suppose that \( X \) is not a simple closed curve and each vertex of \( X \) is either an end point of \( X \) or a ramification point of \( X \). With this restriction two extremes of a segment of \( X \) may coincide and then such a “segment” would be a simple closed curve. The set of segments of \( X \) is denoted by \( S \). For each \( J \in S \), we fix an orientation and then we identify \( J \) with a closed interval \( [(−1)J, (1)J] \). Notice that it is possible that \( (−1)J = (1)J \). We write \(-1\) (resp. \(1\)) instead of \((−1)J\) (resp. \( (1)J\)) if no confusion arrives.

In order to define the map \( \varphi \) in Theorem 1, we will describe its action in each \( J \in S \). For each \( A \in \mu^{-1}(T) \), we consider \( A \cap J \) and we enlarge or shrink this set. To illustrate how this shrinking of \( A \cap J \) has to be done, let us consider the following diagram:
Here, $L$ and $M$ are segments of $X$ and $J$ is a segment in $X$ such that the end points of $J$ coincide (that is, $J$ is a simple closed curve). The subcontinua $A_1$, $A_2$ and $A_3$ have been outlined in thicker lines. The subcontinuum $A_2$ contains $J$ and $M$ and one half of $L$, $A_1 \cap L$ and $A_3 \cap L$ are a little bit larger that $A_2 \cap L$ while $A_1 \cap J$ and $A_3 \cap J$ are a little bit smaller than $A_2 \cap J$. In this example, $T_{i-1} = \mu(J \cup M)$.

If we shrink $A_2 \cap J$, then we have to cut it at some place of the circle $J$. Since $A_1$ is very close to $A_2$, the continuity of the shrinking implies that we have to cut $A_1 \cap J$ at a similar position as $A_2 \cap J$. Then, the connectedness of the shrinking of $A_1 \cap J$ implies that $A_2 \cap J$ has to be cut only on the upper part of $J$. But, since $A_3$ is very close to $A_2$, in the same way as above, $A_2 \cap J$ has to be cut only on the lower part of $J$. This contradiction shows that it is not possible to shrink $A_2 \cap J$.

However, we have to shrink the continuum $A_2$ and the shrinkings have to take all the sizes in the interval $(T_{i-1}, \mu(A_2)]$. Then, the shrinking of $A_2$ will be carried out by making the arc $A_2 \cap L$ shorter and shorter. Since $A_1$ and $A_3$ are very close to $A_2$, then the shrinking of $A_1 \cap J$ and $A_3 \cap J$ have to be almost imperceptible compared with the shrinking of $A_1 \cap L$ and $A_3 \cap L$, respectively.

The map $\varphi$ in Theorem 1 will be an appropriate reparametrization and restriction of the following map $F$, so the behaviour of $F$ will be similar to the behaviour of $\varphi$ and the discussion concerning the shrinking of the subcontinua of $X$ is applicable to $F$. 
Observing that to get the effect of shrinking some intervals very slowly compared with others, we strongly use the asymptoteness of the graph of the map $g$ to the lines $y = \pm 1$ in the Euclidean plane.

2. Auxiliary maps

Consider the map $f : (-1, 1) \to \mathbb{R}$ given by $f(t) = \tan(t\pi/2)$ and let $g : \mathbb{R} \to (-1, 1)$ be the inverse map of $f$. Then $f(t) = -f(t)$ for every $t \in (-1, 1)$, $g(-s) = -g(s)$ for every $s \in \mathbb{R}$ and $-g$ is the inverse map of $-f$. Define $C^\vee(X) = C(X) - (SG(X) \cup F_1(X))$.

Define $F : C^\vee(X) \times \mathbb{R} \to C^\vee(X)$ by $F(A, t) = \bigcup \{F_j(A, t) : J \in S\}$, where $F_j : C^\vee(X) \times \mathbb{R} \to \{E : E$ is a closed subset of $J\}$ is defined as follows:

$$F_j(A, t) = \begin{cases} 
(a) & A \cap J \quad \text{if} \quad A \cap J = \emptyset, \{-1\}, \{1\}, \{-1, 1\} \quad \text{or} \quad J, \\
(b) & [-1, g(f(b) + t)) \quad \text{if} \quad A \cap J = [-1, b] \quad \text{and} \quad -1 < b < 1, \\
(c) & [g(f(a) - t), 1) \quad \text{if} \quad A \cap J = [a, 1] \quad \text{and} \quad -1 < a < 1, \\
(d) & [a + e(m - a), b + e(m - b)] \quad \text{where} \quad m = \frac{a + b}{2 + a - b} \quad \text{and} \quad e = 1 + \frac{1 + g(f(b - a - 1 + t))}{a - b}, \quad \text{if} \quad A \cap J = [a, b] \quad \text{and} \quad -1 < a < b < 1 \quad \text{and}, \\
(e) & [-1, a + e(m - a)] \cup [b + e(m - b), 1], 
\quad \text{where} \quad m = \frac{a + b}{2 + a - b} \quad \text{and} \quad e = 1 + \frac{1 + g(f(b - a - 1 - t))}{a - b}, \quad \text{if} \quad A \cap J = [-1, a] \cup [b, 1], \\
& \quad -1 < a < b < 1 \quad \text{and} \quad -1 < a \quad \text{or} \quad b < 1. 
\end{cases}$$

In case (e), $a(1 + a) \leq b(1 + a)$ and $a(1 - b) \leq b(1 - b)$, then $2a + a^2 - ab \leq a + b \leq 2b + ab - b^2$, so $a \leq m \leq b$, where $a < m$ or $b < m$. Notice that $e$ is a strictly increasing function of $t$. If $t \to \infty$, $e \to 1$, $a + e(m - a) \to m$ and $b + e(m - b) \to m$. If $t \to -\infty$, $e \to 1 + \frac{2}{a - b} a + e(m - a) \to -1$ and $b + e(m - b) \to 1$. Thus $F_j(A, t)$ is a proper subset of $J$, $\{-1, 1\} \subset F_j(A, t) \neq \{-1, 1\}$; if $t < s$, then $F_j(A, t) \subset F_j(A, s) \neq F_j(A, t)$, $F_j(A, t) \to J$ as $t \to \infty$ and $F_j(A, t) \to \{-1, 1\}$ as $t \to -\infty$.

Similarly, in case (d), $F_j(A, t)$ is a proper subset of $J$, $-1, 1 \not\in F_j(A, t)$, $m \in F_j(A, t)$; if $t < s$, then $F_j(A, t) \subset F_j(A, s) \neq F_j(A, t)$, $F_j(A, t) \to J$ as $t \to \infty$ and $F_j(A, t) \to \{m\}$ as $t \to -\infty$.

In all the cases, if $A \cap J$ is a nonempty proper subset of $J$, then $F_j(A, t)$ is a nonempty proper subset of $J$. Moreover, $-1$ (resp. 1) belongs to $A$ if and only if $-1$ (resp. 1) belongs to $F_j(A, t)$. It follows that, for each $t$, a vertex $p$ of $X$ belongs to $A$ if and only if $p$ belongs to $F(A, t)$ and $F(A, t) \in C^\vee(X)$. Therefore $F$ is well defined.

We will need the following properties of function $F$:
II. For a fixed $t < s$, then $F(A, t) \subset F(A, s) \neq F(A, t)$.
It follows from the fact that in cases (b), (c), (d) and (e), if $t < s$, then $F_J(A, t) \subset F_J(A, s) \neq F_J(A, t)$.

II. For a fixed $A \in C^\gamma(X)$, if $t \to -\infty$, $F(A, t)$ tends to a one-point set or to a subgraph of $X$ which is contained in $A$ and, if $t \to \infty$, then $F(A, t)$ tends to a subgraph of $X$ which contains $A$.

III. $F$ is continuous.

Let $((A_n, t_n))n$ be a sequence in $C^\gamma(X) \times \mathbb{R}$ which converges to an element $(A, t)$ in $C^\gamma(X) \times \mathbb{R}$. We may suppose that if $J \in \mathcal{S}$ and $A \cap J = \emptyset$, then $A_n \cap J = \emptyset$ for every $n$. Let $S^* = \{J \in \mathcal{S} : A \cap J \neq \emptyset\}$. Since $F(A, t)$ has no isolated points, if we can find a finite set $E$ such that $F(A_n, t_n) \cup E \to F(A, t)$, then we will have that $F(A_n, t_n) \to F(A, t)$. In order to find such a set $E$, it is enough to show that, for each $J \in S^*$, there exists a finite set $E_J$ such that $F_J(A_n, t_n) \cup E_J \to F_J(A, t)$. Then take $J \in S^*$. Here it is necessary to consider the following cases:

1. $A \cap J = J$,
2. $A \cap J = [-1, b]$ with $-1 < b < 1$,
3. $A \cap J = [a, 1]$ with $-1 < a < 1$,
4. $A \cap J = [a, b]$ with $-1 < a < b < 1$,
5. $A \cap J = [-1, a] \cup [b, 1]$ with $-1 < a < b < 1$,
6. $A \cap J = [1, a] \cup [b, 1]$ with $1 < a < 1$,
7. $A \cap J = \{-1\}$ or $A \cap J = \{1\}$.
8. $A \cap J = \{-1, 1\}$.

We only check cases 1 and 6; the others are similar. For case 1, the sequence $(A_n)n$ can be partitioned into subsequences $(B_k)k$ where each $B_k$ lies in one of the following subcases:

(a) $B_k \cap J = J$. Then $F_J(B_k, t_{n_k}) = J \to F_J(A, t)$.

(b) $B_k \cap J = [-1, b_k]$ with $-1 < b_k < 1$. Since $B_k \to A$, $b_k \to 1$, then $F_J(B_k, t_{n_k}) = [-1, g(f(b_k) + t_{n_k})] \to [-1, 1] = F_J(A, t)$.

(c) $B_k \cap J = [a_k, 1]$ with $-1 < a_k < 1$. It is similar to (b).

(d) $B_k \cap J = [a_k, b_k]$ with $-1 < a_k < b_k < 1$. Then $a_k \to -1$ and $b_k \to 1$, so $e_k = 1 + [1 + g(f(b_k - a_k - 1) + t_{n_k})]/(a_k - b_k) \to 0$. Thus $b_k + e_k(m_k - b_k) - (a_k + e_k(m_k - a_k)) = (b_k - a_k)(1 - e_k) \to 2$. Therefore

$$F_J(B_k, t_{n_k}) = [a_k + e_k(m_k - a_k), b_k + e_k(m_k - b_k)] \to [-1, 1] = F_J(A, t).$$

(e) $B_k \cap J = [-1, a_k] \cup [b_k, 1]$, with $-1 < a_k < b_k < 1$ and $-1 < a_k$ or $b_k < 1$. Then $b_k - a_k \to 0$. Thus $b_k + e_k(m_k - b_k) - (a_k + e_k(m_k - a_k)) = (b_k - a_k)(1 - e_k) = (b_k - a_k)(1 + g(f(b_k - a_k - 1) + t_{n_k})/(a_k - b_k)) \to 0$.

Thus $F_J(B_k, t_{n_k}) \to F_J(A, t)$.

Therefore $F_J(A_n, t_n) \to F_J(A, t)$.

In case 6, define $E_J = \{1\}$. Note that $F_J(A, t) = [-1, g(f(a) + t)] \cup \{1\}$. We must consider the following subcases:

(a) $B_k \cap J = [-1, b_k]$ with $-1 < b_k < 1$. Since $B_k \to A$, $b_k \to a$, then
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This contradiction shows that this case is not possible.

Since \( a \) and \( r \)
\[ m_k = (a_k + b_k)/(2 + a_k - b_k) \rightarrow -1 \] and \( e_k \rightarrow 1 + [1 + g(f(a) + t)]/(-1 - a) \). Thus \( F_J(B_k, t_{n_k}) \cup E_J = [a_k + e_k(m_k - a_k), b_k + e_k(m_k - b_k)] \cup E_J = [-1, g(f(a) + t)] \cup \{1\} = F_J(A, t) \).

(c) \( B_k \cap J = [-1, a_k] \cup [b_k, 1] \), with \(-1 \leq a_k < b_k \leq 1\) and \(-1 < a_k \) or \( b_k < 1\). Then \( a_k \rightarrow a \), \( b_k \rightarrow 1 \), \( m_k \rightarrow 1 \) and \( e_k \rightarrow (a - g(f(a) + t))/(a - 1) \). Thus, \( F_J(B_k, t_{n_k}) \cup E_J = [-1, a_k + e_k(m_k - a_k)] \cup [b_k + e_k(m_k - b_k), 1] \rightarrow [-1, g(f(a) + t)] \cup \{1\} = F_J(A, t) \).

Hence, \( F_J(A_n, t_n) \cup E_J \rightarrow F_J(A, t) \).

Therefore, \( F \) is continuous.

IV. If \((A, t), (B, s) \in C^\infty(X) \times \mathbb{R}\) are such that \( A - B \neq \emptyset \) and \( F(A, t) = F(B, s) \), then \( t < s \).

To prove this, choose a point \( p \in A - B \), let \( J \in S \) be such that \( p \in J \). If \( p \) is a vertex of \( X \), then \( p \in F(A, t) = F(B, s) \), so \( p \in B \). This contradiction proves that \( p \) is not a vertex of \( X \). Then \( J \) is the unique segment of \( X \) which contains \( p \).

We consider some cases:

(a) \( A \cap J = J \). Then \( J \subset F(B, s) \). This implies that \( B \cap J = J \) and \( p \in B \). This contradiction shows that this case is not possible.

(b) \( A \cap J = [-1, b] \) with \(-1 < b < 1\). Since \( F(A, t) = F(B, s) \), then \( B \cap J \) is of the form \( B \cap J = [-1, b_1] \) with \(-1 < b_1 < b \) and \([-1, g(f(b) + t)] = [-1, g(f(b_1) + s)] \). Then \( f(b) + t = f(b_1) + s \). Thus \( t < s \).

(c) \( A \cap J = [a, 1] \) with \(-1 < a < 1\). This case is similar to case (b).

(d) \( A \cap J = [-1, a] \cup [b, 1] \) with \(-1 \leq a < b \leq 1\) and \(-1 < a \) or \( b < 1\).

Since \( F(A, t) = F(B, s) \), then \( B \cap J \) is of the form \( B \cap J = [-1, a_1] \cup [b_1, 1] \), with \(-1 \leq a_1 < b_1 \leq 1\) and \(-1 < a_1 \) or \( b_1 < 1\). Moreover, \( a + e(m - a) = a_1 + e_1(m_1 - a_1) \) \( b + e(m - b) = b_1 + e_1(m_1 - b_1) \) \( m = (a + b)/(2 + a - b) \), \( m_1 = (a_1 + b_1)/(2 + a_1 - b_1) \), \( e - 1 = (1 + g(f(b - a - 1) - t))/(a - b) \) and \( e_1 - 1 = (1 + g(f(b_1 - a_1 - 1) - s))/(a_1 - b_1) \).

From (1) and (2), \((1 - e)a - (1 - e_1)a_1 = (1 - e)b - (1 - e_1)b_1 \), then \((1 - e)(a - b) = (1 - e_1)(a_1 - b_1) \) \( e_1 = 1 + r/(a_1 - b_1) \). So, (1) and (2) imply: \( m + r(m - a)/(a - b) = m_1 + r(m_1 - a_1)/(a_1 - b_1) \) and \( m + r(m - b)/(a - b) = m_1 + r(m_1 - b_1)/(a_1 - b_1) \).

Using definitions of \( m \) and \( m_1 \), \( m - r(1 + a)/(2 + a - b) = m_1 - r(1 + a)/(2 + a - b) \) and \( m + r(1 - b)/(2 + a - b) = m_1 + r(1 - b)/(2 + a - b) \).

Then \( m - m_1 = r[(1 + a)/(2 + a - b) - (1 + a)/(2 + a_1 - b_1)] \). Hence \( m - m_1 = r(a - a_1 + b_1 - a_1 b_1 + b_1 a)/(2 + a - b)(2 + a_1 - b_1) \). While, from definitions of \( m \) and \( m_1 \), \( m - m_1 = 2(a - a_1 + b - b_1 - a_1 b - a b_1)/(2 + a - b)(2 + a_1 - b_1) \).

Since \( r < 2 \), \((a - a_1 + b - b_1 - a_1 b - a b_1)/(2 + a - b)(2 + a_1 - b_1) = 0 \). Therefore \( m = m_1 \).
From (6) we have \((1+a)/(2+a-b) = (1+a_1)/(2+a_1-b_1)\) and \((1-b)/(2+a-b) = (1-b_1)/(2+a_1-b_1)\). Since \(p \in (A \cap J) - (B \cap J)\), then \(a_1 < a\) or \(b < b_1\). In the first case, \(1+a_1 < 1+a\), so \(2+a-b > 2+a_1-b_1\) and \(f(b-a-1) < f(b_1-a_1-1)\), then (5) implies \(t < s\). Analogously, in the second case, \(t < s\).

(e) \(A \cap J = [a, b]\) with \(-1 < a < b < 1\). This case is similar to case (d). Then \(t < s\).

This completes the proof of Property IV.

Define \(G : C^\vee(X) \times \mathbb{R} \to C^\vee(X)\) by \(G(B, t) = \bigcup \{G_J(B, t) : J \in \mathcal{S}\}\), where \(G_J : C^\vee(X) \times \mathbb{R} \to \{E : E\) is a closed subset of \(J\}\) is defined as follows:

\[
G_J(B, t) = \begin{cases} 
(a) & B \cap J = \emptyset, \{-1\}, \{1\}, \{-1, 1\} \text{ or } J, \\
(b) & [-1, g(f(b) - t)] \text{ if } B \cap J = [-1, b] \text{ and } -1 < b < 1, \\
(c) & [g(f(a) + t), 1] \text{ if } B \cap J = [a, 1] \text{ and } -1 < a < 1, \\
(d) & [(a-e'm)/(1-e''), (b-e'm)/(1-e'')] \text{ where } m = \frac{a+b}{2+a-b}, \\
& \text{and } e' = 1 - \frac{b-a}{1+g(t-f(\frac{a-b-1}{t}))} \text{ if } B \cap J = [-1, a] \cup [b, 1], -1 \leq a < b \leq 1 \text{ and } -1 < a \text{ or } b < 1.
\end{cases}
\]

In case (e), let \(a_1 = (a-e'm)/(1-e'')\) and \(b_1 = (b-e'm)/(1-e'')\), then \(a_1 < b_1\). Note that \(e'\) is an increasing continuous function of \(t\). If \(t \to -\infty, e' \to (2+a-b)/2, \) if \(t \to -\infty, e' \to -\infty. \) Then \(e' < (2+a-b)/2 \) for every \(t \in \mathbb{R}\). Thus \(e'(1+m) = e'2(1+a)/(2+a-b) \leq 1+a\) and \(e'(1-m) = e'2(1)-b/(2+a-b) \leq 1-b. \) This implies that \(-1 \leq (a-e'm)/(1-e'') = a_1\) (equality holds if and only if \(-1 = a\) and \(b_1 = (b-e'm)/(1-e'') \leq 1\) (equality holds if and only if \(b = 1\)).

If \(t \to \infty, a_1 \to -1\) and \(b_1 \to 1. \) If \(t \to -\infty, a_1 \to m\) and \(b_1 \to m. \) Since \(a+b-2e'm = m(2+a-b-2e'), \) \(m = (a-e'm+b-e'm)/(2(1-e') + a-b) = (a_1+b_1)/(2+a_1-b_1). \) Therefore \(m = \frac{a_1+b_1}{2+a_1-b_1}. \) Define \(e = 1 + \frac{1+g(f(b_1-a_1-1)+t)}{a_1+b_1}. \) Note that \(b_1 - a_1 - 1 = (b-a - (1-e'))/(1-e'') = -g(-t-f(b-a-1)).\) This implies that \(e = e'.\) Thus \(a_1 + e(m-a_1) = a\) and \(b_1 + e(m-b_1) = b. \)

Therefore, \(G_J(B, t)\) is a continuous function of \(t, G_J(B, t) \to J \to -\infty, \) \(G_J(B, t) \to \{-1, 1\} \) as \(t \to -\infty, \) \(G_J(B, 0) = B \cap J\) and supposing that \(G(B, t) \in C^\vee(X),\) we have that \(F_J(G(B, t), t) = [-1, a] \cup [b, 1] = B \cap J\) for every \(t \in \mathbb{R}.

The analysis of cases (a), (b), (c) and (d) is similar and we conclude that \(G(B, t) \in C^\vee(X)\) for each \(t \in \mathbb{R}, \) \(F_J(G(B, t), t) = B \cap J\) for every \(t \in \mathbb{R},\) then \(F(G(B, t), t) = B\) for every \(t \in \mathbb{R}, \) \(G(B, t)\) depends continuously on \(t, G(B, t)\) tends to one-point set or to a subgraph of \(X\) which is contained in \(B\) as \(t \to \infty\) and \(G(B, t)\) tends to a subgraph of \(X\) which contains \(B\) as \(t \to -\infty.\)
3. Proof of Theorem 1

Define $\mathcal{A} = \mu^{-1}(\mathbb{T}) \subset C^\prime(X)$ and $\mathcal{B} = \mu^{-1}(T_i-1, T_i)$. For each $A \in \mathcal{A}$, let $r(A) = \inf\{t \in \mathbb{R} : F(A, t) \in \mathcal{B}\}$ and $R(A) = \sup\{t \in \mathbb{R} : F(A, t) \in \mathcal{B}\}$. Since $F_J(A, 0) = A \cap J$ for every $J \in \mathcal{S}$, we have that $F(A, 0) = A \in \mathcal{B}$ for each $A \in \mathcal{A}$. Then $r(A)$ and $R(A)$ are defined and $-\infty \leq r(A) < 0 < R(A) \leq \infty$. Let $\mathcal{C} = \{(A, t) \in \mathcal{A} \times \mathbb{R} : r(A) < t < R(A)\}$. We will prove that the function $F_0 = F | \mathcal{C}$ is a homeomorphism from $\mathcal{C}$ onto $\mathcal{B}$.

Property I implies that $F_0(A, t) \in \mathcal{B}$ for every $(A, t) \in \mathcal{C}$. In order to prove that $F_0$ is injective, suppose that $F_0(A, t) = F_0(B, s)$. If $A \neq B$, since $\mu(A) = \mu(B)$, then $A - B \neq \emptyset$ and $B - A \neq \emptyset$. Property IV implies that $t < s$ and $s < t$. This contradiction implies that $A = B$. Thus, by Property I, $(A, t) = (B, s)$. Therefore $F_0$ is injective. To prove that $F_0$ is onto, let $B \in \mathcal{B} \subset C^\prime(X)$. Since $G(B, t)$ tends to one-point set or to a subgraph of $X$ which is contained in $\mathcal{B}$ as $t \to \infty$ and $G(B, t)$ tends to a subgraph of $X$ which contains $B$ as $t \to -\infty$. Then $\lim_{t \to -\infty} \mu(G(B, t)) \leq T_i-1$ and $\lim_{t \to -\infty} \mu(G(B, t)) \geq T_i$. Thus there exists $t \in \mathbb{R}$ such that $A = G(B, t) \in \mathcal{A}$. The continuity of $F$ implies that $r(A) < t < R(A)$. Then $F_0(A, t) = B$. Therefore $F_0$ is surjective.

Let $K : \mathcal{B} \to \mathcal{C}$ be the inverse function of $F_0$. We will show that $K$ is continuous. It is enough to prove that if $(B_n)n$ is a sequence in $\mathcal{B}$ which is convergent to an element $B \in \mathcal{B}$ and the sequence $(K(B_n))n$ converges to an element $(A_0, t_0) \in \mathcal{A} \times [-\infty, \infty]$, then $(A_0, t_0) = K(B)$.

Let $(A, t) = K(B)$ and, for each $n$, let $(A_n, t_n) = K(B_n)$. Then $(A_n, t_n) \to (A_0, t_0)$. If $r(A_0) < t_0 < R(A_0)$, then $F_0(A, t) = B = \lim_{n \to \infty} B_n = \lim_{n \to \infty} F_0(A_n, t_n) = F_0(A_0, t_0)$, so $(A_0, t_0) = K(B)$. If $t_0 \leq r(A_0)$, take a number $t^* > r(A_0)$. Then there exists $N$ such that $t_n < t^*$ for each $n \geq N$. Then $B_n \subset F(A_n, t_n) \subset F(A_n, t^*)$ for each $n \geq N$. Thus $B \subset F(A_0, t^*)$ for every $t^* > r(A_0)$. If $r(A_0) > -\infty$, then $B \subset F(A_0, r(A_0)) \subset F(A_0, 0) = A_0$. Thus $T_i-1 < \mu(B) \leq \mu(F(A_0, r(A_0))) \leq \mu(A_0) < T_i$. Then there exists $r < r(A_0)$ such that $T_i-1 < \mu(F(A_0, r)) < T_i$ which is a contradiction with the definition of $r(A_0)$. If $r(A_0) = -\infty$, then $B \subset \lim_{n \to \infty} F(A_0, -n)$ which is a subgraph of $X$ or a one-point set contained in $A_0$. Thus $\mu(B) \leq T_i-1$ which is a contradiction. Similar contradictions are obtained supposing that $t_0 \geq R(A_0)$. This completes the proof that $(A_0, t_0) = K(B)$. Therefore $K$ is continuous.

Hence $F$ is a homeomorphism.

In order to define $\varphi$, let $g_1 : \mathcal{A} \times \mathbb{R} \to \mathcal{A}$ and $g_2 : \mathcal{A} \times \mathbb{R} \to \mathbb{R}$ be the respective projection maps. Define $\psi : \mathcal{B} \to \mathcal{A} \times (T_i-1, T_i)$ by $\psi(B) = (g_1(K(B)), \mu(B))$. Then $\psi$ is continuous.

Let $(A, t) \in \mathcal{A} \times (T_i-1, T_i)$. Since $F(A, n)$ converges to a subgraph of $X$ which contains $A$, then $\lim_{n \to \infty} \mu(F(A, n)) \geq T_i$. Thus there exists $n_1 > 1$ such that $\mu(F(A, n_1)) > t$. Similarly, there exists $n_2 > 1$ such that $\mu(F(A, -n_2)) < t$. Hence there exists a unique $s \in \mathbb{R}$ such that $\mu(F(A, s)) = t$. Define $\varphi(A, t) = F(A, s)$.

Property I implies that if $t_1 < t_2$, then $\varphi(A, t_1) \subset \varphi(A, t_2)$. Note that
\[ \psi(\varphi(A,t)) = \psi(F(A,s)) = (A,t). \] Since \( \mu(F(\varphi_1(K(B)), \varphi_2(K(B)))) = \mu(B), \) then \( \varphi(\psi(B)) = \psi((\varphi_1(K(B)), \varphi_2(K(B)))) = F(K(B)) = B. \) Then \( \psi \) is the inverse map of \( \varphi. \) Since \( \mu(F(A,0)) = \mu(A) = T, \) then \( \varphi(A,T) = A \) for every \( A \in \mathcal{A}. \)

To prove that \( \varphi \) is continuous, it is enough to prove that if \((A_n, t_n)\) is a sequence in \( \mathcal{A} \times (T_{i-1}, T_i) \) which converges to an element \((A,t)\) in \( \mathcal{A} \times (T_{i-1}, T_i) \) and \( \varphi(A_n, t_n) \) converges to an element \( B \in C(X) \), then \( B = \varphi(A,t). \) Set \( \varphi(A_n, t_n) = F(A_n, s_n), \) where \( \mu(F(A_n, s_n)) = t_n \) and set \( \varphi(A, t) = F(A, s) \) where \( \mu(F(A, s)) = t. \) Then \( t_n = \mu(\varphi(A_n, t_n)) \rightarrow \mu(B), \) so \( \mu(B) = t \in (T_{i-1}, T_i). \) Thus \( B \in \mathcal{B}. \) Set \( K(B) = (A^*, r). \) Then \( (A^*, r) = \lim_{n \to \infty} K(\varphi(A_n, t_n)) = \lim_{n \to \infty} K(F(A_n, s_n)) = \lim_{n \to \infty} F(A_n, s_n) \) Thus \( A_n \rightarrow A^* \) and \( s_n \rightarrow r. \) Hence \( A^* = A. \) Since \( t_n = \mu(F(A_n, s_n)) \rightarrow \mu(F(A, r)), \) then \( t = \mu(F(A, r)). \) Hence \( B = \varphi(A,t). \)

This completes the proof that \( \varphi \) is a homeomorphism and the proof of Theorem 1.

\[ \square \]

Corollary ([10, Theorem 2.5]). \( C(X) \) is conical pointed. That is, for each Whitney map \( \mu : C(X) \rightarrow \mathbb{R} \) there exists \( T \in (0, 1) \) such that \( \mu^{-1}([T, 1]) \) is homeomorphic to the topological cone of \( \mu^{-1}(T). \)

4. Proof of Theorem 2

Definition. Let \( \mathcal{A} \) and \( \mathcal{B} \) be two Whitney levels for \( C(X) \) and let \( C \in C(X). \) We say that \( C \) is placed between \( \mathcal{A} \) and \( \mathcal{B} \) if there exists \( A \in \mathcal{A} \) and \( B \in \mathcal{B} \) such that \( A \subset C \subset B \neq A \) or \( B \subset C \subset A \neq B. \)

Theorem. Let \( \mathcal{A} \) and \( \mathcal{B} \) be two Whitney levels. Suppose that no element in \( SG(X) \cup F_1(X) \) is placed between \( \mathcal{A} \) and \( \mathcal{B}. \) Then \( \mathcal{A} \) and \( \mathcal{B} \) are homeomorphic.

Proof: Set \( \mathcal{A} = \mu^{-1}(t) \) and \( \mathcal{B} = \nu^{-1}(s) \) where \( \mu, \nu : C(X) \rightarrow \mathbb{R} \) are Whitney maps and \( t, s \in [0, 1]. \) Let \( A \in \mathcal{A} - \mathcal{B} \), we will prove that there exists a unique \( r \in \mathbb{R} \) such that \( \nu(F(A, r)) = s. \) If \( \nu(A) < s, \) taking an order arc from \( A \) to \( X \) (see [8, Theorem 1.8]), there exists \( B_0 \in \mathcal{B} \) such that \( A \subset B_0 \neq A, \) then \( A \notin SG(X) \cup F_1(X). \) Therefore \( A \in C^\vee(X). \) Let \( D = \lim_{n \to \infty} F(A, n). \) Then \( D \) is a subgraph of \( X \) which contains \( A. \) If \( \nu(D) \leq s, \) there exists \( B \in \mathcal{B} \) such that \( D \subset B. \) Then \( \nu(A) < \nu(B) \) and \( A \subset D \subset B \neq A \) which contradicts our assumption. Thus \( \nu(D) > s. \) Then \( \nu(F(A, 0)) = \nu(A) < s = \lim_{n \to \infty} \nu(F(A, n)). \) This proves the existence of \( r \) in this case. The case \( \nu(A) > s \) is similar. In both cases \( r \) is unique by Property I.

Analogously, for each \( B \in \mathcal{B} - \mathcal{A}, \) \( B \in C^\vee(X) \) and there exists a \( z \in \mathbb{R} \) such that \( \mu(G(B, z)) = t. \)

Define \( \gamma : \mathcal{A} \rightarrow \mathcal{B} \) by \( \gamma(A) = A \) if \( A \in \mathcal{A} \cap \mathcal{B} \) and \( \gamma(A) = F(A, r) \in \mathcal{B} \) if \( A \in \mathcal{A} - \mathcal{B}. \)

Note that \( A \subset \gamma(A) \) or \( \gamma(A) \subset A. \) To prove that \( \gamma \) is surjective, let \( B \in \mathcal{B}. \) If \( B \in \mathcal{A}, \) then \( B = \gamma(B). \) If \( B \in \mathcal{B} - \mathcal{A}, \) let \( z \in \mathbb{R} \) be such that \( \mu(G(B, z)) = t. \)
Then $F(G(B, z), z) = B$ and $G(B, z) \in A$. Thus $\gamma(G(B, z)) = B$. Hence $\gamma$ is surjective. To prove that $\gamma$ is injective, let $A_1, A_2 \in A$ with $A_1 \neq A_2$. If $A_1, A_2 \in B$, then $\gamma(A_1) = A_1 \neq A_2 = \gamma(A_2)$. If $A_1 \in B$ and $A_2 \notin B$, then $A_2 \subset A_1$ or $\gamma(A_2) \subset A_2$ $\neq \gamma(A_2)$, so $\gamma(A_2) \notin A$, and $\gamma(A_2) \neq A_1 = \gamma(A_1)$. If $A_1, A_2 \notin B$, since $A_1 - A_2 \neq \emptyset$ and $A_2 - A_1 \neq \emptyset$, Property IV implies that $F(A_1, r_1) \neq F(A_2, r_2)$ for every $r_1, r_2 \in \mathbb{R}$. Hence $\gamma(A_1) \neq \gamma(A_2)$. Therefore $\gamma$ is injective.

Finally, we will prove that $\gamma$ is continuous. It is enough to prove that if $(A_n)_n$ is a sequence in $A$ which converges to an element $A \in A$ and $\gamma(A_n) \rightharpoonup B \in B$, then $\varphi(A) = B$. We may suppose that $A_n \in B$ for each $n$ or $A_n \notin B$ for each $n$. The first case is immediate. In the second case, set $\gamma(A_n) = F(A_n, r_n)$. We consider two subcases:

(a) $A \in A - B$, set $\gamma(A) = F(A, r)$. We suppose, for example, that $r \leq r_n$ for each $n$. Then $F(A_n, r) \subset F(A_n, r_n) = \gamma(A_n)$, then $\gamma(A) = F(A, r) = \lim_{n \to \infty} F(A_n, r) \subset \lim_{n \to \infty} \gamma(A_n) = B$. Since $\gamma(A), B \in B$, we have that $\gamma(A) = B$.

(b) $A \in B$. Since $A_n \subset \gamma(A_n)$ or $\gamma(A_n) \subset A_n$ for every $n$, then $A \subset B$ or $B \subset A$ and $A, B \in B$. Thus $A = B$. This completes the proof that $\gamma$ is continuous.

Therefore $\gamma$ is a homeomorphism.

**Proof of Theorem 2:** Let $\mathcal{A} = \{ A \subset C(X) : A$ is a Whitney level for $C(X), A \neq F_1(X) \text{ and } A \neq \{ X \} \}$. Let $\mathcal{P} = \{ E : E \subset SG(X) \}$. Then $\mathcal{P}$ is finite.

Define $\sigma : \mathcal{A} \to \mathcal{P} \times \mathcal{P} \times \mathcal{P}$ by:

\[ \sigma(A) = \{ \{ E \in SG(X) : \text{there exists } A \in A \text{ such that } E \subset A \neq E \}, SG(X) \cap A, \{ E \in SG(X) : \text{there exists } A \in A \text{ such that } A \subset E \neq A \} \}. \]

In order to prove Theorem 2, it is enough to show that if $\sigma(A) = \sigma(B)$, then $A$ is homeomorphic to $B$.

Suppose then that $\sigma(A) = \sigma(B)$. By the previous theorem, it is enough to prove that no element in $SG(X)$ is placed between $A$ and $B$. Suppose, for example, that there exists $A \in A, B \in B$ and $E_0 \in SG(X)$ such that $A \subset E_0 \subset B \neq A$. If $A = E_0$, then $E_0 \in SG(X) \cap A = SG(X) \cap B \subset B$, so $E_0, B \in B$ and $E_0 \subset B \neq E_0$ which is a contradiction. If $A \neq E_0$, $F(A) = F(B)$ implies that there exists $B_1 \in B$ such that $B_1 \subset E_0 \neq B_1$. Thus $B_1 \subset B \neq B_1$ which is also a contradiction.

Therefore $A$ is homeomorphic to $B$.

**References**


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