On uniformly smoothing stochastic operators

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Abstract. We show that a stochastic operator acting on the Banach lattice \( L^1(m) \) of all \( m \)-integrable functions on \((X, A)\) is quasi-compact if and only if it is uniformly smoothing (see the definition below).

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Let \((X, A, m)\) be a \( \sigma \)-finite measure space. By \( \mathcal{D} \) we denote the set of all densities from \( L^1(m) \), i.e. \( m \) integrable positive functions \( f \) such that \( \int_X f \, dm = 1 \).

A linear operator \( P: L^1(m) \to L^1(m) \) is said to be stochastic if \( P(\mathcal{D}) \subseteq \mathcal{D} \).

Stochastic operators have broad applications. The reader may find appropriate references in [LM]. Among other properties, usually the asymptotic behaviour of the iterates \( P^n \) is studied. In the middle of the eighties Komornik and Lasota introduced to the theory of stochastic operators the concept of smoothness. Namely, \( P \) is said to be smoothing if

there exist a set \( F \in \mathcal{A} \) of finite measure and

\[
\lim_{n \to \infty} \int_{E \cup F^c} P^n f \, dm \leq \eta,
\]

where \( F^c \) stands here and in the sequel for the complementation \( X \setminus F \).

Smoothing stochastic operators have nice asymptotic properties. It is proved in [KL] that any smoothing stochastic operator \( P \) is asymptotically periodic i.e. there exist pairwise orthogonal densities \( g_1, \ldots, g_r \), positive functionals \( \Lambda_1, \ldots, \Lambda_r \) and a permutation \( \alpha \) of the set \( \{1, \ldots, r\} \) such that \( \lim_{n \to \infty} \| P^n f - \sum_{i=1}^r \Lambda_i(f) g_{\alpha^n(i)} \| = 0 \) and \( P g_i = g_{\alpha(i)} \) \( \quad i = 1, 2, \ldots, r \). In particular, for some constant \( d \) the sequence \( P^{nd} \) converges in the strong operator topology to \( \sum_{i=1}^r \Lambda_i \otimes g_i \). The most general result in this direction was finally obtained by Komornik. Namely, it was

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proved in [K] that any power bounded positive, and linear operator on $L^1(m)$ is asymptotically periodic.

In this note we discuss the uniform version of (S). Following [B3] (see Problem 3, page 57) we adapt here:

**Definition.** Let $0 < \eta < 1$. A stochastic operator $P$ is said to be uniformly $\eta$-smoothing if there are $F \in \mathcal{A}$ with $m(F) < \infty$, and a constant $0 < \delta$ such that for some natural $n_0$

$$(US-\eta) \quad \sup_{f \in \mathcal{D}} \int_{E \cup F^c} P^{n_0} f \, dm \leq \eta$$

for all $E \in \mathcal{A}$ satisfying $m(E) \leq \delta$.

We will show that operators satisfying $(US-\eta)$, are quasi-compact. Let us recall that an operator $P$ is quasi-compact if $\|P^n - K\| < 1$ for some compact operator $K$ and natural $n$. It is known (see for instance [B2]) that quasi-compact stochastic operators $P$ are exactly those which satisfy $\|P^{nd} - \sum_{i=1}^{r} \Lambda_i \otimes g_i\| \xrightarrow{n \to \infty} 0$, for suitable $d$, $r$, $\Lambda_i$, and $g_i$. We will exploit here the characterization of quasi-compact operators obtained in [B1]. In particular we shall apply some of the results from the mentioned paper to Markov operators acting on the Banach lattice $C(\Delta)$ of all continuous functions on $\Delta$, where $\Delta$ stands for the set of all linear and multiplicative functionals on $L^\infty(m)$ equipped with the $\ast$-weak topology, so Hausdorff and compact. We recall that a linear operator $T: C(\Delta) \to C(\Delta)$ is Markov if $T1 = 1$ and $Tf \geq 0$ for $f \geq 0$. The dual space to $C(\Delta)$ is identified with Radon, finite (signed) measures on $\Delta$. The $\ast$-weak compact (nonempty) set of all probability measures $\mu$ on $\Delta$ such that $T^* \mu = \mu$ is denoted by $P_T(\Delta)$. Clearly the adjoint to $P$ operator $T = P^*$ is markovian.

A linear operator $R$ acting on a Banach space $\mathcal{X}$ is said to be strongly ergodic if for all $x \in X$ the Cesaro means $n^{-1}(I + R + \cdots + R^{n-1})x$ are convergent in the norm of $\mathcal{X}$. Sine’s mean ergodic theorem (see [S]) provides necessary and sufficient conditions for strong ergodicity. Namely, it holds if and only if $R$-invariant vectors separate $R^*$-invariant ones. It is easy to verify that $R^*$-invariant vectors always separate $R$-invariant ones. In [B1] it is proved that a Markov operator $T$ on $C(\Delta)$ is quasi-compact if $T^*$ is strongly ergodic and the topological support $S(\mu)$ of any $\mu$ from $P_T(\Delta)$ is non-meager. Finally we notice that the quasi-compactness of $P$ is equivalent to the quasi-compactness of its adjoint $P^*$.

**Theorem.** Let $P$ be a stochastic operator on $L^1(m)$. Then $P$ is quasi-compact if and only if $P$ is $\eta$-uniformly smoothing for some (for all) $0 < \eta < 1$.

**Proof:** Assume that $P$ is $\eta$-uniformly smoothing with $F$, $n_0$, $\eta$, $\delta$ as in $(US-\eta)$, and let $X = \bigcup_{j=1}^{\infty} X_j$ where $X_j$ are pairwise disjoint with positive finite measure.
We assume that \( X_1 = F \). Now let us define a probability measure
\[
m_0 = \sum_{j=1}^{\infty} t_j m|_{X_j} \quad \text{where} \quad \sum_{j=1}^{\infty} t_j m(X_j) = 1, \quad \text{and} \quad t_j > 0.
\]

Clearly \( m_0 \) and \( m \) are equivalent, so \( L^\infty(m_0) = L^\infty(m) \). The measure \( m_0 \) may be transported on \( \Delta \) by the Gelfand transform \( \hat{\cdot} \). Then, for any \( f \in L^\infty \) we have
\[
\int_X f \, dm_0 = \int_{\Delta} \hat{f} \, d\hat{m}_0
\]
where \( \hat{f} \in C(\Delta) \) is the image of \( f \) by \( \hat{\cdot} \). By \( \sim \) let us denote the inverse operation to \( \hat{\cdot} \).

First we show that measures from \( P_T(\Delta) \) are absolutely continuous with respect to \( \hat{m}_0 \). Since \( T^* L^1(\hat{m}_0) \subseteq L^1(\hat{m}_0) \), it is sufficient to show that any \( \hat{\nu} \in P_T(\Delta) \) has a nonzero absolutely continuous with respect to \( \hat{m}_0 \) component. If not, let us suppose that for some \( \hat{\nu} \in P_T(\Delta) \) one has \( \hat{\nu} \perp \hat{m}_0 \). Then there exists a clopen set \( \hat{U} \subseteq \Delta \) so that
\[
(\star) \quad \hat{m}_0(\hat{U}) < t_1 \delta \quad \text{with} \quad \hat{\nu}(\hat{U}) = 1.
\]

Let \( \hat{f} \in C(\Delta) \) be such that \( \int \hat{f} d\hat{m}_0 = 1 \) and \( T^{*n_0}(\hat{\nu})(\hat{U}) > \frac{1}{2} + \frac{\eta}{2} \). We get
\[
\int_{\hat{U}} P_{n_0} \frac{d(\hat{f} \, d\hat{m}_0)}{dm} > \frac{1}{2} + \frac{\eta}{2} > \eta. \quad \text{This implies} \quad m(U \cap F) > \delta, \quad \text{so} \quad m_0(U \cap F) > t_1 \delta,
\]
and finally contradicting (\*\*) we get \( \hat{m}_0(\hat{U}) \geq \hat{m}_0(\hat{U} \cap \hat{F}) > t_1 \delta \). Therefore \( P_T(\Delta) \subseteq L^1(\hat{m}_0) \), which easily implies that the topological support of \( \nu \in P_T(\Delta) \) is non-meager.

Applying Sine’s mean ergodic from [S] we notice that the operator \( T \) is strongly ergodic. In particular, \( A^*_n \nu = n^{-1}(I^* + T^* + \cdots + T^{*(n-1)}) \nu \) is \( * \)-weak convergent. Since \( \Delta \) has the Grothendieck property (\( * \)-weak convergent sequences from \( C(\Delta)^* \) are weakly convergent) thus \( A^*_n \nu \) is weakly convergent. But weakly convergent Cesaro means are norm convergent. Therefore, \( T^* \) is strongly ergodic. Using results of [B1] we easily obtain quasi-compactness of \( T = P^* \). By Theorem 2 from [B2], there is a natural \( d \) such that \( P^* n d \) is convergent in the operator norm to a finite dimensional projection. This is equivalent to the norm convergence of \( P n d \), and \( P \) is quasi-compact.

To prove the opposite let us assume that a stochastic operator \( P \) is quasi-compact. For some \( d \) we have \( \lim_{n \to \infty} P n d = \sum r_{\Lambda_i} \otimes g_i \), where \( g_i \in D \) are pairwise orthogonal (i.e. \( g_i \cdot g_j = 0 \quad m \text{ a.e. for } i \neq j \)) and \( \Lambda_i(f) = \int f \, h_i \, dm \) where \( \| h_i \|_* \leq 1 \). For a given \( 0 < \eta < 1 \) we choose a set \( F \in \mathcal{A} \) of finite measure...
and positive $\delta$ that if $m(E) < \delta$ then $\int_{E \cup F^c} \sum_{j=1}^{r} g_j dm < \frac{\eta}{2}$. If $n$ is such that

$$\|P^{nd} - \sum_{j=1}^{r} \Lambda_j \otimes g_j\| < \frac{\eta}{2},$$

then we have

$$\int_{E \cup F^c} P^{nd} f \, dm = \int_{E \cup F^c} \left( P^n f - \sum_{j=1}^{r} \lambda_j(f) g_j \right) \, dm + \sum_{j=1}^{r} \lambda_j(f) \int_{E \cup F^c} g_j \, dm$$

$$\leq \frac{\eta}{2} + \int_{E \cup F^c} \sum_{j=1}^{r} g_j \, dm \leq \eta$$

where $f$ is an arbitrary density. $\square$

**References**


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