A $\sigma$-porous set need not be $\sigma$-bilaterally porous

J. NÁJARES R., L. ZAJÍČEK*

Abstract. A closed subset of the real line which is right porous but is not $\sigma$-left-porous is constructed.

Keywords: sigma-porous, sigma-bilaterally-porous, right porous

Classification: Primary 26A99; Secondary 28A5

1. Introduction

Let $E \subseteq \mathbb{R}$ be a set, and let $I$ be an interval. Then we denote by $\lambda(E, I)$ the length of the largest open subinterval of $I$ which does not intersect $E$. The right porosity of $E$ at $x \in \mathbb{R}$ is defined as

$$p^+(E, x) = \lim_{h \to 0^+} \frac{\lambda(E, (x, x+h))}{h}.$$ 

The left porosity $p^-(E, x)$ is defined by the symmetrical way.

We say that:

(i) $E$ is right porous at $x$ if $p^+(E, x) > 0$,
(ii) $E$ is left porous at $x$ if $p^-(E, x) > 0$,
(iii) $E$ is bilaterally porous at $x$ if it is porous both on the right and on the left at $x$.

The set $E$ is said to be right (left, bilaterally) porous if it is right (left, bilaterally) porous at each of its points and $\sigma$-right-porous ($\sigma$-left-porous, $\sigma$-bilaterally-porous) if it is a countable union of right (left, bilaterally) porous sets. It is easy to see that a set is $\sigma$-bilaterally porous iff it is bilaterally $\sigma$-porous (i.e. it is both $\sigma$-right-porous and $\sigma$-left-porous). The main aim of the present article is to prove the following result.

Theorem. There exists a closed set $F \subset \mathbb{R}$ which is right porous but is not $\sigma$-left-porous.

We obtain the example slightly modifying the ideas of [F] and [Za 1].

We essentially use Lemma 5 which is a special case of the generalized Foran lemma [Za 3], which enables us to give a simple proof that our set $F$ is not

*Supported by Research Grant GAUK 363.
σ-left-porous. Another ingredient of our proof is Proposition, which is analogous to Proposition 4.4 from [Za 2]. We believe that it can be also of some independent importance. Note that for symmetrical porosity an analogical proposition does not hold [E-H-S].

2. Proposition and lemmas

Definition 1. If \( c > 0 \), \( M \subset \mathbb{R} \) and \( r > 0 \) are given, then we define

\[
S(c, r, M) = \bigcup \{ x \ominus (y - \sigma, y) ; y \in \mathbb{R}, \ 0 < \sigma < r, (y - \sigma, y) \cap M = \emptyset \},
\]

where \( c \ominus (y - \sigma, y) = (y - c\sigma, y) \).

We shall need the following lemmas which are obvious.

Lemma 1. If \( p^+(M, x) \geq c > 0 \), then \( x \in \bigcap \{ S(\frac{2}{c}, r, M) ; r > 0 \} \).

Lemma 2. If \( c > 1 \), \( x \in M \) and \( x \in \bigcap \{ S(c, r, M) ; r > 0 \} \), then \( p^+(M, x) \geq \frac{1}{c} \).

Proposition. Let \( A \) be a \( \sigma \)-right-porous set (\( \sigma \)-left-porous) and \( c < 1 \). Then there exists a sequence \( \{ A_n \}_{n=1}^\infty \) such that \( A = \bigcup_{n=1}^\infty A_n \) and \( p^+(A_n, x) \geq c \) (\( p^-(A_n, x) \geq c \), respectively) for any \( n \in \mathbb{N} \) and \( x \in A_n \).

Proof: It is sufficient to give the proof for right porosity only. By definition \( A = \bigcup_{n=1}^\infty B_n \) where \( B_n \) is a right porous set for any \( n \in \mathbb{N} \). Putting

\[
B_{n,k} = \{ x \in B_n ; p^+(B_n, x) \geq \frac{1}{k} \}
\]

we have that \( A = \bigcup_{n,k=1}^\infty B_{n,k} \) and

\[
p^+(B_{n,k}, x) \geq \frac{1}{k}
\]

for any \( x \in B_{n,k} \).

Thus it is sufficient to prove the following statement:

If \( M \subset \mathbb{R} \), \( a > 0 \) and, for each \( x \in M \), the inequality \( p^+(M, x) \geq a \) holds, then \( M = \bigcap_{i=1}^\infty M_i \), where \( p^+(M_i, y) \geq c \) for any \( y \in M_i \).

We can suppose \( a < c < 1 \), the case \( a \geq c \) being trivial. Choose \( n \in \mathbb{N} \) such that \( (\frac{1}{c})^n \geq \frac{2}{a} \) and define \( C_k = M \cap \bigcap_{r>0} S(c^{-k}, r, M) \). By Lemma 1 \( M = C_n \) and therefore \( M = \bigcup_{k=2}^n (C_k \setminus C_{k-1}) \cup C_1 \). By Lemma 2, we have \( p^+(C_1, x) \geq c \) for any \( x \in C_1 \).

For \( k = 2, \ldots, n \) define \( T_{k,m} = C_k \setminus S(c^{-k+1}, m^{-1}, M) \). Then

\[
\bigcup_{m=1}^\infty T_{k,m} = C_k \setminus \bigcap_{m=1}^\infty S(c^{-k+1}, m^{-1}, M) = C_k \setminus C_{k-1}.
\]

Since \( T_{k,m} \subset C_k \), for each \( z \in T_{k,m} \) and \( r > 0 \), there exist \( y \) and \( t \) such that \( 0 < t < \min(r, m^{-1}) \), \( (y-t, y) \cap M = \emptyset \) and \( z \in c^{-k} \ominus (y-t, y) \). Put \( J = c^{-k+1} \ominus (y-t, y) \). Then \( z \in c^{-1} \ominus J \) and \( J \cap T_{k,m} = \emptyset \), since \( J \subset S(c^{-k+1}, m^{-1}, M) \).
Thus, for each \( z \in T_{k,m} \), we have \( z \in \bigcap_{r>0} S(c^{-1}, c^{-k+1}r, T_{k,m}) \) and therefore \( p^+(T_{k,m}, z) \geq c \) by Lemma 2, which proves our statement. \( \square \)

For the sake of brevity, in the following we shall say that \( E \) is \( V \)-porous at \( x \) if \( p^-(E, x) > \frac{100}{101} \). The following lemma is easy to prove.

**Lemma 3.** Let \( E \subset \mathbb{R} \), \( x \in \mathbb{R} \) and a natural number \( p \) be given such that \( x - 10^{-k} \) or \( x - 10^{-(k+1)} \) belongs to \( E \) for each natural \( k > p \). Then \( E \) is not \( V \)-porous at \( x \).

The following lemma is an immediate consequence of Proposition.

**Lemma 4.** A set \( E \subset \mathbb{R} \) is \( \sigma \)-left-porous iff it is \( \sigma \)-\( V \)-porous.

**Definition 2.** Let \( \mathcal{F} \subset \exp \mathbb{R} \) be a non-\( \sigma \)-\( V \)-porosity family if the following conditions hold:

(a) \( \mathcal{F} \) is a nonempty family of nonempty closed sets,

(b) for each \( F \in \mathcal{F} \) and each open set \( G \subset \mathbb{R} \) with \( F \cap G \neq \emptyset \), there exists \( F^* \in \mathcal{F} \) such that \( \emptyset \neq F^* \cap G \subset F \cap G \) and \( F \) is \( V \)-porous at no point of \( F^* \cap G \).

We shall need the following lemma which is a special case of [Za 3, Lemma 4.3].

**Lemma 5.** Let \( \mathcal{F} \) be a non-\( \sigma \)-\( V \)-porosity family. Then no set from \( \mathcal{F} \) is \( \sigma \)-\( V \)-porous.

### 3. Proof of theorem

Our theorem stated in Introduction immediately follows from Lemma 7 and Lemma 8 below. To formulate them, we need some notions.

**Definition 3.** Let \( x \in (0, 1) \). As usual, we write \( x = 0, a_1a_2... \) if \( x = \sum_{i=1}^{\infty} a_i10^{-i} \) and \( a_i \in \{0, 1, ..., 9\} \). The uniqueness of the expansion is obtained using terminating 0’s whenever \( x \) has two expansions. Let \( a \in \{0, 1, ..., 9\} \) be a digit. The density and the upper density of \( a \) in the expansion of \( x \) are defined as

\[
\begin{align*}
d(a, x) &= \lim_{n \to \infty} \frac{\# \{k; 1 \leq k \leq n, a_k(x) = a \}}{n}, \\
\bar{d}(a, x) &= \lim_{n \to \infty} \frac{\# \{k; 1 \leq k \leq n, a_k(x) = a \}}{n}.
\end{align*}
\]

The following easy fact is well known and easy to prove.

**Lemma 6.** The function \( x \mapsto \bar{d}(a, x) \) is Borel measurable on \((0, 1)\).
Let a natural number $N$, $\varepsilon > 0$, $1 > \alpha > 0$ and digits $a_1, \ldots, a_{N^2} \in \{0, 1, \ldots, 9\}$ be given. Then we define the set $A(\alpha, a_1, \ldots, a_{N^2}, \varepsilon)$ as the set of all $x \in (0, 1)$ for which

(1) \[ a_1(x) = a_1, \ldots, a_{N^2}(x) = a_{N^2} \quad \text{and} \]

(2) \[ 1 - \frac{\varepsilon}{n^\alpha} \leq \frac{c(x, n)}{2n + 1} < 1 \quad \text{whenever} \quad n \geq N. \]

Lemma 7. Let $0 < \alpha < 1$, $\varepsilon > 0$ and digits $a_1, \ldots, a_{N^2} \in \{0, 1, \ldots, 9\}$ such that

(3) \[ N > \max \left( (1 + \varepsilon)^{1/\alpha}, \varepsilon^{1/\alpha-1} \right) \]

be given.

Then $A(\alpha, a_1, \ldots, a_{N^2}, \varepsilon)$ is a closed set which is not $\sigma$-left-porous.

Proof: Obviously (2) implies that

(4) \[ c(x, n) \neq 0 \quad \text{whenever} \quad n \geq N \quad \text{and} \quad x \in A(\alpha, a_1, \ldots, a_{N^2}, \varepsilon). \]

Now suppose that $x_n \in A(\alpha, a_1, \ldots, a_{N^2}, \varepsilon)$ and $x_n \to x$. On account of (4) we easily obtain that

\[ (a_1(x_n), a_2(x_n), \ldots) \to (a_1(x), a_2(x), \ldots) \]

in the space $\mathbb{N}^\mathbb{N}$ and consequently $x \in A(\alpha, a_1, \ldots, a_{N^2}, \varepsilon)$. Thus we have that $A(\alpha, a_1, \ldots, a_{N^2}, \varepsilon)$ is closed.

The condition (2) is equivalent to

\[ c(x, n) \in \left[ (1 - \frac{\varepsilon}{n^\alpha})(2n + 1), \ 2n + 1 \right] := I_n \quad \text{for} \quad n \geq N. \]

If $n \geq N$, we have by (3)

\[ (1 - \frac{\varepsilon}{n^\alpha})(2n + 1) > (1 - \frac{\varepsilon}{1 + \varepsilon})(2(1 + \varepsilon)^{\frac{1}{\alpha}}) > 2 \quad \text{and} \]

\[ (2n + 1) - (1 - \frac{\varepsilon}{n^\alpha})(2n + 1) = \frac{\varepsilon}{n^\alpha}(2n + 1) > 2\varepsilon n^{1-\alpha} > 2. \]

Thus we have $I_n \subset (2, 2n + 1)$ and $\text{length}(I_n) > 2$ for $n \geq N$; consequently $A(\alpha, a_1, \ldots, a_{N^2}, \varepsilon) \neq \emptyset$ and $c(x, n) > 2$ whenever $x \in A(\alpha, a_1, \ldots, a_{N^2}, \varepsilon)$ and $n \geq N$.

Now let $\mathcal{F}$ denote the family of all sets of the form $A(\alpha, a_1, \ldots, a_{N^2}, \varepsilon)$ for which (3) holds. By Lemma 4 and Lemma 5 it is sufficient to prove that $\mathcal{F}$ is a non-$\sigma$-$V$-porosity family. To this end suppose that $F = A(\alpha, a_1, \ldots, a_{N^2}, \varepsilon) \in \mathcal{F}$ and an open set $G \subset \mathbb{R}$ such that $F \cap G \neq \emptyset$ are given.
Choose an arbitrary \( y \in F \cap G \) and find a natural \( M \) so large that

\[
M > N, \quad M > \left( \frac{\varepsilon}{2} \right)^{\frac{1}{\alpha}} \quad \text{and} \quad F^* := A(\alpha, a_1, \ldots, a_{N^2}, a_{N^2+1}(y), \ldots, a_{M^2}(y), \frac{1}{2}\varepsilon) \subset G.
\]

Clearly \( F^* \subset F \). On account of (3) and (5) we have

\[ M > \max \left( (1 + \frac{\varepsilon}{2})^\frac{1}{\alpha}, (\frac{\varepsilon}{2})^\frac{1}{\alpha-1} \right) \]

and therefore \( F^* \in F \). Thus it is sufficient to prove that \( F \) is \( V \)-porous at no point \( z \in F^* \). To prove this, fix an arbitrary \( z \in F^* \) and consider an arbitrary natural \( k > (M + 1)^2 \). By Lemma 3 it is sufficient to prove that at least one of the points \( z_k^- = z - 10^{-k} \), \( z_{k+1}^- = z - 10^{-(k+1)} \) belongs to \( F \). It is easy to see that

\[
c(z, n) - 1 \leq c(z_k^-, n) \quad \text{and} \quad c(z, n) - 1 \leq c(z_{k+1}^-, n), \quad \text{for each} \quad n.
\]

Since \( z \in F^* \), we have \( c(z, M) > 0 \) (we know even \( c(z, M) > 2 \)) and therefore

\[
a_s(z) = a_s(z_k^-) = a_s(z_{k+1}^-) \quad \text{for} \quad s \leq M^2.
\]

Now suppose that \( x \in \{z_k^-, z_{k+1}^-\} \). Then (7) says that

\[ a_s(x) = a_s(z) \quad \text{for} \quad s \leq M^2. \]

For \( n \geq M \) the definition of \( F^* \), (6) and (5) yield

\[
\frac{c(x, n)}{2n+1} \geq \frac{c(z, n) - 1}{2n+1} \geq 1 - \frac{\varepsilon}{2n^\alpha} - \frac{1}{2n+1} > 1 - \frac{\varepsilon}{n^\alpha}.
\]

Thus it is sufficient to establish that, for \( x = z_k^- \) or \( x = z_{k+1}^- \),

\[
e(x, n) \neq 0, \quad \text{for each} \quad n \geq M.
\]

To this end suppose that

\[ e(z_k^-, n) = 0 \quad \text{for some} \quad n \geq M. \]

Since \( c(z, n) \neq 0 \), this condition easily implies that

\[
k = n^2 + i \quad \text{where} \quad i \in \{1, \ldots, 2n\},
\]

\[
a_{n^2+1}(z) = 0, \ldots, a_{n^2+i}(z) = 0 \quad \text{and} \quad a_{n^2+i+1}(z) = 9, \ldots, a_{(n+1)^2}(z) = 9.
\]

Consequently it is easy to see that (8) holds for \( x = z_{k+1}^- \). \( \square \)
Lemma 8. If $\frac{1}{2} < \alpha < 1$, then the set $A(\alpha, a_1, ..., a_{N2}, \varepsilon)$ from Lemma 7 is right porous.

Proof: Choose an arbitrary $x \in A(\alpha, a_1, ..., a_{N2}, \varepsilon)$.

For each natural $n$, let $m_n$ be the maximum of those natural $i$, for which there exist natural numbers $u, v$ such that

$$n^2 \leq u < v \leq (n + 1)^2, \quad a_s(x) = 9 \text{ for each } u < s \leq v$$

and $v - u = i$. It is easy to see that

$$2n + 1 - e(x, n) = c(x, n) \leq m_n(e(x, n) + 1) \text{ and consequently}$$

$$m_n \geq \frac{2n + 1 - e(x, n)}{e(x, n) + 1}. \quad (10)$$

On account of (2) we have that

$$e(x, n) \leq \frac{\varepsilon(2n + 1)}{n^\alpha} \text{ for } n \geq N$$

and therefore (10) implies that there exists $c > 0$ and a natural $n_0$ such that

$$m_n \geq cn^\alpha \text{ for all } n \geq n_0. \quad (11)$$

Now, for each $n$, choose $u_n, v_n$ such that

$$v_n - u_n = m_n \text{ and } (9) \text{ holds for } u = u_n, \ v = v_n.$$ 

Put

$$y_n = x + 10^{-v_n} \quad \text{and} \quad z_n = x + 10^{-v_n + 1}.$$ 

It is easy to see that, for each $t \in (y_n, z_n)$, we have

$$a_s(t) = 0, \text{ for each } u_n < s \leq v_n - 1$$

and therefore

$$c(t, n) \leq 2n + 1 - (m_n - 1). \quad (12)$$

If $n$ is so big that $n > n_0$, $n > N$ and $2n + 2 - cn^\alpha < (2n + 1)(1 - \frac{\varepsilon}{n^\alpha})$, we have by (12) and (11)

$$c(t, n) \leq 2n + 2 - cn^\alpha < (2n + 1)(1 - \frac{\varepsilon}{n^\alpha}).$$

Thus we obtain by (2) that $t \notin A(\alpha, a_1, ..., a_{N2}, \varepsilon)$. Consequently

$$p^+\left(A(\alpha, a_1, ..., a_{N2}, \varepsilon), x\right) \geq \lim_{n \to \infty} \frac{10^{-(v_n - 1)} - 10^{-v_n}}{10^{-v_n + 1}} = \frac{9}{10}.$$
A $\sigma$-porous set need not be $\sigma$-bilaterally porous

References


[Za 2] ———, Sets of $\sigma$-porosity and sets of $\sigma$-porosity ($\varrho$), Časopis Pěst. Mat. 101 (1976), 350–359.


Department of Mathematical Analysis, Charles University, Sokolovská 83, 186 00 Praha 8, Czech Republic

(Received January 19, 1994)