Remarks on special ideals in lattices

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Abstract. The author studies some characteristic properties of semiprime ideals. The semiprimeness is also used to characterize distributive and modular lattices. Prime ideals are described as the meet-irreducible semiprime ideals. In relatively complemented lattices they are characterized as the maximal semiprime ideals. $D$-radicals of ideals are introduced and investigated. In particular, the prime radicals are determined by means of $\hat{C}$-radicals. In addition, a necessary and sufficient condition for the equality of prime radicals is obtained.

Keywords: semiprime ideal, prime ideal, congruence of a lattice, allele, lattice polynomial, meet-irreducible element, kernel, forbidden exterior quotients, $D$-radical, prime radical

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1. Introduction

The notion of a semiprime ideal was introduced by Rav in [8] in the following way: An ideal $I$ of a lattice $L$ is said to be semiprime if the implication

$$(a \land b \in I \land a \land c \in I) \Rightarrow a \land (b \lor c) \in I$$

is true for every $a, b, c \in L$.

In a recent paper, a new method was used to characterize the semiprime ideals by means of lattice quotients. For a detailed description of the method see [3], whereas for a comparative study of this technique against a classical background see [1]. The semiprime ideals in lattices have been studied in [6], [2] and [4].

For completeness we include some definitions here.

Let $a, b$ be elements of a lattice $L$. If $a \leq b$, we say that these elements form a quotient $b/a$ of $L$. We write $b/a \sim_w d/c$ if either

$$b = a \lor d \quad \& \quad a \land d \geq c$$

or

$$a = b \land c \quad \& \quad b \lor c \leq d.$$ 

If there exist quotients $y_i/x_i$ such that

$$b/a = y_0/x_0 \sim_w y_1/x_1 \sim_w \cdots \sim_w y_n/x_n = d/c,$$
we write \( b/a \approx_w d/c \).

A quotient \( b/a \) is called an allele if there exists a quotient \( d/c \) satisfying \( b/a \approx_w d/c \) and such that either \( b \leq c \) or \( d \leq a \). The set of all the alleles of \( L \) will be denoted by \( A(L) \).

Let \( \hat{C}(L) \) denote the smallest congruence \( \theta \) of \( L \) for which the quotient lattice \( L/\theta \) is distributive. It can be shown \([1]\) that \((a,b) \in \hat{C}(L)\) if and only if there exist \( a_i \in L \) satisfying
\[
(1) \quad a_0 = a \land b \leq a_1 \leq a_2 \cdots \leq a_m = a \lor b
\]
and such that \( a_{i+1}/a_i \in A(L) \) for every \( i = 0, 1, \ldots, m - 1 \).

**Proposition 1.** Let \( I \) be an ideal of a lattice \( L \). Then the following conditions are equivalent:

(i) the ideal \( I \) is semiprime;

(ii) for any \( a, \tilde{a}, b \) of \( L \),
\[
(b \land a \in I \land b \land \tilde{a} \in I \land a \lor \tilde{a} \geq b) \Rightarrow b \in I;
\]

(iii) there is no allele \( b/a \) of \( L \) with \( a \in I \) and \( b \not\in I \);

(iv) for any \( x, y \) of \( L \),
\[
(x \in I \land x \leq y \land (x,y) \in \hat{C}(L)) \Rightarrow y \in I;
\]

(v) for any \( x, y \) of \( L \),
\[
(x \in I \land (x,y) \in \hat{C}(L)) \Rightarrow y \in I;
\]

(vi) the ideal \( (I)_{Id(L)} \) generated by \( I \) in the ideal lattice \( Id(L) \) is semiprime.

**Proof:** (i) \(\Leftrightarrow\) (ii). Clearly, any semiprime ideal satisfies (ii).

Suppose now that \( x \land y \in I \) and \( x \land z \in I \). Put \( a = y, \tilde{a} = z \) and \( b = x \land (y \lor z) \). From (iii) it follows that \( x \land (y \lor z) \in I \).

(i) \(\Leftrightarrow\) (iii). This is Main Theorem of \([3]\).

(iii) \(\Leftrightarrow\) (iv) and (iv) \(\Leftrightarrow\) (v). Immediate.

(i) \(\Leftrightarrow\) (vi). This has been proved by Rav \([8]\). \(\square\)

**Corollary 2.** (i) Let \( x \in L \). Then the principal ideal \( (x) \) is semiprime if and only if there is no allele \( y/x \) with \( y > x \).

(ii) An ideal \( X \) of \( L \) is semiprime if and only if there is no ideal \( Y \) satisfying \( X \nsubseteq Y \) and \( Y/X \in A(Id(L)) \).

**Proof:** (i) Suppose that \( (x) \) satisfies the condition and let \( q/i \) be an allele with \( i \in (x) \). Since \((i,q) \in \hat{C}(L), (x,x \lor q) \in \hat{C}(L) \). By the assumption and (1), \( x \lor q \in (x) \) and so \( q \in (x) \). Thus \( (x) \) is semiprime.

The remainder follows from Proposition 1 (i).

(ii) Use (i) and Proposition 1 (v). \(\square\)
2. Properties characterizing semiprime ideals

First we need some notation.

Let $I$ be an ideal of $L$ and let $M \subseteq L$. By $M^*_I$ we mean the set of all $a \in L$ such that $a \wedge m \in I$ for every $m \in M$. We write $m^*_I$ (or simply $m^*$) instead of \{m\}^*_I.

Note that the ideal $I$ is semiprime if and only if $m^*_I$ is an ideal of $L$ for every $m \in L$.

Given an ideal $I$ of $L$, let $\psi$ and $\theta$ be relations defined on $L$ in the following way:

$$(a, b) \in \psi \iff a^*_I = b^*_I; \quad (a, b) \in \theta \iff (a \wedge b)^*_I = (a \lor b)^*_I.$$  

The relation $\psi$ was used by Rav in the proof of his Main Theorem in [8]. Note that $\theta \subset \psi$. However, the converse inclusion need not be true.

**Theorem 3.** The following conditions are equivalent for any ideal $I$ of a lattice $L$:

(i) The ideal $I$ is semiprime.

(ii) The relation $\psi$ satisfies $\psi \supseteq \hat{C}(L)$.

(iii) The relation $\theta$ satisfies $\theta \supseteq \hat{C}(L)$.

(iv) The relations $\theta$ and $\psi$ satisfy $\theta = \psi \supseteq \hat{C}(L)$.

**Proof:** (i) $\Rightarrow$ (iv). Let $a^* = b^*$ and let $z \in (a \land b)^*$. Then $z \land a \land b \in I$, which gives $z \land a \land b = a^*$. Hence $z \land a \in I$ and, similarly, $z \land b \in I$. Since $I$ is semiprime, it follows that $z \land (a \lor b) \in I$. Consequently, $z \in (a \lor b)^*$ and this implies $(a \lor b)^* = (a \land b)^*$. Thus $\theta = \psi$. By [8, p. 109], $L/\theta$ is distributive and so $\theta \supseteq \hat{C}(L)$.

(iv) $\Rightarrow$ (iii). Trivial.

(iii) $\Rightarrow$ (ii). Use $\theta \subseteq \psi$.

(ii) $\Rightarrow$ (i). Let $q/i \in A(L)$ be such that $i \in I$. Then $(i, q) \in \hat{C}(L) \subseteq \psi$, and, therefore, $q^* = i^* = L$. This yields $q \in I$.

**Theorem 4.** An ideal $I$ of a lattice $L$ is semiprime if and only if

$$(a \lor b) \land c \in [a \lor (b \land c)]^*_I$$

for every $a, b, c \in I$.

**Proof:** Suppose $I$ is semiprime and let $x \in [a \lor (b \land c)]^*$. Then $x \land [a \lor (b \land c)] \subseteq I$, and, a fortiori,

$x \land c \land a \in I \ \& \ x \land c \land b \in I$.

Since $I$ is semiprime, $x \land c \land (a \lor b) \in I$. Therefore, $x \in [(a \lor b) \land c]^*$. Suppose that (2) is valid and let $a \land c \in I$ and $b \land c \in I$. Replace $a$ in (2) by $a \land c$. Then

$$(a \land c) \lor (b \land c) \in I$$

Similarly, $[(a \land c) \lor (b \land c)]^* \subseteq [(a \land c) \lor (b \land c)]^*$. Since $(a \land c) \lor (b \land c) \in I$, it is readily seen that $[(a \land c) \lor (b \land c)]^* = L$. Accordingly, $[(a \land c) \lor (b \land c) \land c \in I$, and, by (2), $c \in [b \lor (a \land c)]^* \subseteq [(b \lor a) \land c]^*$. Hence $(a \lor b) \land c \in I$. 

$\square$
Theorem 5. An ideal $I$ of a lattice $L$ is semiprime if and only if the following implication holds for every $a,b,c \in L$:

$$[(c \land a) \uparrow_I \supset (c \land b) \uparrow_I \quad \& \quad (c \lor a) \uparrow_I \supset (c \lor b) \uparrow_I] \Rightarrow a \uparrow_I \supset b \uparrow_I.$$  

Proof: First we shall suppose that $I$ is semiprime. Then we can consider the quotient lattice $L/\psi$ where $\psi$ was defined above. If $x/\psi, y/\psi \in L/\psi$, then $x/\psi \leq y/\psi$ if and only if $x* \supset y*$. Hence the antecedent of (4) can be rewritten as

$$c/\psi \land a/\psi \leq c/\psi \land b/\psi \quad \& \quad c/\psi \lor a/\psi \leq c/\psi \lor b/\psi.$$  

This, together with a result of M. Molinaro [7, p. 75], implies that $a/\psi \leq b/\psi$. Thus $a* \supset b*$.  

Finally, let (4) be valid and let $x,y$ and $z$ be arbitrary elements of $L$. Let $a = (x \lor y) \land z, b = x \lor (y \land z)$ and $c = y$. Then

$$c \land a = y \land z \leq c \land b = y \land [x \lor (y \land z)]$$

and

$$c \lor a = y \lor [(x \lor y) \land z] \leq c \lor b = x \lor y.$$  

Consequently we have

$$(c \land a)^* \supset (c \land b)^* \quad \& \quad (c \lor a)^* \supset (c \lor b)^*.$$  

By assumption, $a^* \supset b^*$. From Theorem 4 we see that $I$ is semiprime.  

Theorem 6. An ideal $I$ of a lattice $L$ is semiprime if and only if for any lattice polynomial $p(x_1,x_2,\ldots,x_n)$ and any choice of elements $a_1,a_2,\ldots,a_n \in L$ the relations

$$p(a_1,a_2,\ldots,a_n) \in I \quad \& \quad a_1 \hat{C}(L)a_2 \hat{C}(L)\ldots \hat{C}(L)a_n$$

imply $a_1,a_2,\ldots,a_n \in I$.  

Proof: Let $I$ be semiprime and let $p(a_1,a_2,\ldots,a_n) \in I$. Then

$$I = p(a_1,a_2,\ldots,a_n)/\psi = p(a_1/\psi,a_2/\psi,\ldots,a_n/\psi)$$

$$= p(a_1/\psi,a_1/\psi,\ldots,a_1/\psi) = a_1/\psi.$$  

Thus $a_1 \in I$ and the same is true for the other $a_i$.  

Now suppose that the stated implication is true and let $p(x_1,x_2) = x_1 \land x_2$. If $a \leq b$ are such that $a \in I$ and $(a,b) \in \hat{C}(L)$, then $p(a,b) = a \in I$. We therefore have from Proposition 1 (iv) that $I$ is semiprime.  

\qed
3. Semiprimeness as a descriptive tool

**Theorem 7.** A lattice $L$ is distributive if and only if every principal ideal $(a) (a \in L)$ is semiprime.

**Proof:** Let $I = ((a \land b) \lor (a \land c))$ be semiprime. Since $a \land b$ and $a \land c$ belong to $I$, we get $a \land (b \lor c) \in I$. Thus $a \land (b \lor c) \leq (a \land b) \lor (a \land c)$ and we conclude that $a \land (b \lor c) = (a \land b) \lor (a \land c)$.

Evidently, every ideal of a distributive lattice is semiprime. \hfill $\Box$

**Theorem 8.** A lattice $L$ is modular if and only if for any $a, b, c \in L$, the ideal $(a \lor [b \land (a \lor c)]$ is a semiprime ideal of the sublattice generated by $a, b, c$ in $L$.

**Proof:** Suppose that $L$ is modular and let $M$ denote the sublattice generated by $a, b, c$. Then, by modularity, $I = (a \lor [b \land (a \lor c)]) = ((a \lor b) \land (a \lor c))$. Now $M$ is isomorphic to a quotient lattice of the free modular lattice $M_{28}$ (see [5, p. 64]) with three generators $x, y, z$. However, a closer inspection of the quotient lattices of $M_{28}$ shows that in any of these quotient lattices the ideal corresponding to $((x \lor y) \land (x \lor z))$ is semiprime. Hence also our ideal $I$ is semiprime.

Conversely, suppose the ideal $I = (a \lor [b \land (a \lor c)])$ is semiprime. Note that $a \land (a \lor c) \in I$ and $b \land (a \lor c) \in I$. Consequently, $(a \lor b) \land (a \lor c) \in I$. Thus $(a \lor b) \land (a \lor c) = a \lor [b \land (a \lor c)]$ and $L$ is modular. \hfill $\Box$

**Theorem 9.** Let $I$ be a semiprime ideal of a lattice $L$. Then $I$ is prime if and only if $I$ is a meet-irreducible element of the ideal lattice $Id(L)$.

**Proof:** One easily shows that each prime ideal is a meet-irreducible element in $Id(L)$.

It remains to show that every semiprime ideal $I$ which is meet-irreducible in $Id(L)$ is also prime. To do this, consider $b, c \in L$ satisfying $b \land c \in I$.

We first note that the inclusion in $I \subset (I \lor (b)) \cap (I \lor (c))$ can be replaced by the equality sign. Indeed, let $x \in (I \lor (b)) \cap (I \lor (c))$. Then there exist $i, j \in I$ and $b_1 \leq b, c_1 \leq c$ such that $x \leq (i \lor b_1) \land (j \lor c_1)$. Hence $x \leq (h \lor b_1) \land (h \lor c_1)$ where $h = i \lor j \in I$. But $b_1 \land c_1 \leq b \land c \in I$. Therefore, $h \lor (b_1 \land c_1) \in I$.

Now $L/\hat{C}(L)$ is distributive, and so $(h \lor (b_1 \land c_1), (h \lor b_1) \land (h \lor c_1)) \in \hat{C}(L)$. Since $I$ is semiprime, we have, by Proposition 1(iv), $(h \lor b_1) \land (h \lor c_1) \in I$. Consequently, $x \in I$. Combining this with the meet-irreducibility of $I$ we can derive easily that either $b \in I \lor (b) = I$ or $c \in I \lor (c) = I$. \hfill $\Box$

**Corollary 10.** Let $(a)$ be a semiprime ideal of a lattice $L$. Then $(a)$ is prime if and only if $(a)$ is a meet-irreducible element of the lattice $L$.

**Proof:** Use the fact that $(a)$ is a meet-irreducible element of $L$ if and only if $(a)$ is a meet-irreducible element of $Id(L)$. \hfill $\Box$

By [8, p. 108], any semiprime ideal of $L$ is the kernel of a congruence of $L$. Hence the following lemma can be applied to semiprime ideals.
Lemma 11. Let $I$ be an ideal of a lattice $L$ which is the kernel of a congruence $\theta$ of $L$. Then

$$(I \land J \supset K \land J \land I \lor J \supset K \lor J) \Rightarrow I \supset K$$

for any ideals $J, K$ of $L$.

Proof: Let $k \in K$. Since $K \subset I \lor J$, there exist $i \in I$ and $j \in J$ such that $k \leq i \lor j$. At the same time, $(j, j \land k) \in J \land K \subset I$. Hence $(i, j \land k) \in \theta$ and, consequently, $(j, i \lor j) \in \theta$. From $j \leq j \lor k \leq i \lor j$ it follows that $(j, j \lor k) \in \theta$. But then $(j \land k, k) \in \theta$. Since $I$ is the kernel of $\theta$ and $j \land k \in I$, we get $k \in I$. □

Lemma 12. Let $I$ be a semiprime ideal of a lattice $L$ and let $a, b \in L$ be such that $a \land b \in I$.

Then either $(a) \lor I \neq L$ or

$$(a) \lor I = L \land b \in I.$$ 

Proof: Suppose that $(a) \lor I = L$. Put $J = (a)$, $K = (b)$ and use Lemma 11. It follows that $b \in K \subset I$. □

The following theorem generalizes a result of Chevalier [6, p. 383] stated for orthomodular lattices.

Theorem 13. Let $L$ be a relatively complemented lattice. Then a proper ideal $I$ of $L$ is prime if and only if it is a maximal semiprime ideal of $L$.

Proof: It is well-known that in a relatively complemented lattice every proper prime ideal is maximal.

What remains to be shown is that any maximal semiprime ideal $I \neq L$ is prime. Let $I$ be an ideal having these properties and let $a \land b \in I$ for some $a, b \in I$.

Suppose first that

$$(a) \lor I \neq L \land a \notin I.$$ 

Then $(a) \lor I$ is not semiprime and, by Proposition 1 (iv), there exist $p \in (a) \lor I$ and $q \notin (a) \lor I$ such that $(p, q) \in \hat{C}(L)$ with $p \leq q$. But $p \in (a) \lor I$ means that $p \leq a \lor i$ for a suitable $i \in I$. Now

$$p \leq q \land (a \lor i) \leq q \land (p, q) \in \hat{C}(L).$$

Hence $(q \land (a \lor i), q) \in \hat{C}(L)$ and, therefore,

$$(a \lor i, q \lor a \lor i) \in \hat{C}(L).$$

Let $r^+$ be a relative complement of $a \lor i$ in the interval $[i, a \lor i \lor q]$. From (6) we can see that $(i, r^+) \in \hat{C}(L)$. If $r^+$ belonged to $I$, then $r^+ \lor a \lor i$ would belong to $(a) \lor I$. But then

$$q \leq a \lor i \lor q = r^+ \lor a \lor i \in (a) \lor I;$$
a contradiction.

Thus \( r^+ \notin I, i \in I \) and, moreover, \( (i, r^+) \in \hat{C}(L) \). But this contradicts Proposition 1(iv).

We may therefore assume that (5) and a similar statement for \( b \) are not true.

However, if \( (a] \lor I = L \) or \( (b] \lor I = L \), then we can use Lemma 12. Thus either \( a \in I \) or \( b \in I \) and we are done. \( \square \)

We now turn our attention to the prime radicals. Recall [8, p. 111] that the prime radical \( \text{rad}(I) \) of an ideal \( I \) in a lattice \( L \) is the intersection of all the semiprime ideals of \( L \) which contain \( I \).

There is a simple way how to generalize this notion [4]: Given any lattice \( L \), let \( D(L) \) denote a congruence of \( L \) and let \( D \) be the class of all these congruences. We shall say that an ideal \( I \) of \( L \) is an ideal with forbidden exterior quotients in \( D \), if the implication

\[
(a \leq b \ & \ (a, b) \in D(L) \ & \ a \in I) \Rightarrow b \in I
\]

is true for any choice of \( a \) and \( b \) in \( L \).

From Proposition 1(iv) we conclude that an ideal \( I \) is semiprime if and only if it is an ideal with forbidden exterior quotients in \( \hat{C} \) where \( \hat{C} \) denotes the class of all congruences \( \hat{C}(L) \).

If \( I \) is an ideal of \( L \), we put

\[
\Gamma_D(I) = \{x \in L; (\exists i) i \in I \ & \ (i, x) \in D(L)\}
\]

calling it the \( D \)-radical of \( I \).

**Proposition 14.** The \( D \)-radical of an ideal \( I \) is equal to the intersection of all the ideals with forbidden exterior quotients in \( D \) containing \( I \).

**Proof:** Straightforward. \( \square \)

**Corollary 15.** The \( \hat{C} \)-radical of any ideal \( I \) in a lattice \( L \) is equal to the prime radical of \( I \). \( \square \)

Let \( I \) and \( J \) be ideals of a lattice \( L \). If \( \Gamma_D(I) \subset \Gamma_D(J) \), then it is clear that for any \( i \in I \) there exists \( j \in J \) such that \( (i, j) \in D(L) \). From this remark we could deduce directly a simple characterization of the case where \( \Gamma_D(I) = \Gamma_D(J) \). However, there is another approach which seems to be more fruitful:

**Theorem 16.** The following two conditions on ideals \( I, J \) of a lattice \( L \) are equivalent:

(i) \( \Gamma_D(I) = \Gamma_D(J) \).

(ii) For any \( i \in I \) and any \( j \in J \) there exist \( i_1 \in I \) and \( j_1 \in J \) such that

\[
i \leq i_1 \ & \ j \leq j_1 \ & \ (i_1, j_1) \in D(L).
\]
Suppose first that $\Gamma_D(I) = \Gamma_D(J)$ and let $i \in I$, $j \in J$.

Since $i \in \Gamma_D(I) \subset \Gamma_D(J)$, there exists $j_2 \in J$ such that $(i, j_2)$ belongs to $D(L)$. Then $(i \lor j, j_2 \lor j) \in D(L)$. It follows from $j_2 \lor j \in \Gamma_D(J) \subset \Gamma_D(I)$ that there exists $i_2 \in I$ such that $(i_2, j \lor j_2) \in D(L)$. Hence

$$
(i \lor i_2, i \lor j \lor j_2) \in D(L) \quad \text{and} \quad (i \lor j \lor i_2, i \lor j \lor j_2) \in D(L).
$$

Now $i \lor i_2 \in \Gamma_D(I) \subset \Gamma_D(J)$ and so there is $j_3 \in J$ with $(i \lor i_2, j_3) \in D(L)$. Therefore,

$$
(i \lor i_2 \lor j, j_3 \lor j) \in D(L).
$$

Put $i_1 = i \lor i_2$, $j_1 = j \lor j_3$. Then using (7) and (8), we get $(i_1, j_1) \in D(L)$ and it is evident that $i \leq i_1$ and $j \leq j_1$.

Next suppose conversely that $I$ and $J$ satisfy the condition (ii). By symmetry, it is sufficient to prove that $\Gamma_D(I) \subset \Gamma_D(J)$.

Let $x \in \Gamma_D(I)$. Then there exists $i \in I$ with $(x, i) \in D(L)$. Let $j$ be an element of $J$. By the assumption, there are $i_1 \geq i$, $j_1 \geq j$ such that $(i_1, j_1) \in D(L)$. However, from $(x, i) \in D(L)$ we obtain $(x \lor i_1 \lor j_1, i_1 \lor j_1) \in D(L)$. Similarly, $(i_1, j_1) \in D(L)$ implies that $(i_1 \lor j_1, j_1) \in D(L)$. Therefore, $(x \lor i_1 \lor j_1, j_1) \in D(L)$ and, consequently, $x \lor i_1 \lor j_1 \in \Gamma_D(J)$. Since $\Gamma_D(J)$ is an ideal, we have $x \in \Gamma_D(J)$.

\[\square\]

Corollary 17. Let $a, b$ be elements of a lattice $L$.

Then

1. the $D$-radical $\Gamma_D([a])$ is equal to the $D$-radical $\Gamma_D([b])$ if and only if $(a, b) \in D(L)$;
2. the prime radical $\text{rad}([a])$ is equal to the prime radical $\text{rad}([b])$ if and only if $(a, b) \in \hat{C}(L)$.

\[\text{Proof:} \quad \text{(i) Suppose } \Gamma_D([a]) = \Gamma_D([b]). \quad \text{By Theorem 16, there are } a_1, b_1 \text{ such that}
\]

$$
a \leq a_1 \quad \text{and} \quad b \leq b_1 \quad \text{and} \quad a_1 \in (a) \quad \text{and} \quad b_1 \in (b) \quad \text{and} \quad (a_1, b_1) \in D(L).
$$

Hence $(a, b) \in D(L)$.

Conversely, suppose $(a, b) \in D(L)$. For any $i \in (a)$ and $j \in (b)$ we then can put $i_1 = a$, $j_1 = b$ and use Theorem 16.

(ii) Now immediate. \[\square\]

References

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