On the Jacobson radical of strongly group graded rings

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Abstract. For any non-torsion group $G$ with identity $e$, we construct a strongly $G$-graded ring $R$ such that the Jacobson radical $J(R_e)$ is locally nilpotent, but $J(R)$ is not locally nilpotent. This answers a question posed by Puczyłowski.

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Several interesting results of ring theory establish the local nilpotency of the Jacobson radical of some ring constructions (cf. [9]). In this paper we consider an analogous question for strongly group graded rings. Let $G$ be a group. An associative ring $R = \bigoplus_{g \in G} R_g$ is said to be strongly $G$-graded if $R_g R_h = R_{gh}$ for all $g, h \in G$. Strongly group graded rings have been intensively investigated for several years (cf., for example, [12],[15],[20]). In [18] the following question was posed: is it true that for every free group $G$ of rank $\geq 2$ the Jacobson radical of each strongly $G$-graded ring is locally nilpotent? (As it is noted in [18], the question is also connected with [14], Problem 24, and with a problem on the local nilpotency of the Jacobson radical of a skew polynomial ring, cf. [19].) It follows from the results of [6] that the answer is positive in the case when $R_e$ satisfies the ascending chain condition for left annihilators, where $e$ is the identity of $G$. It is also known that the answer is positive for group rings of free groups of rank $\geq 2$ (cf. [18]). The answer to the analogous question for the rings of polynomials in at least two non-commuting variables is also positive (cf. [18]).

We shall show that in general the answer is negative. Namely, for an arbitrary group $G$, we construct a strongly $G$-graded ring $R$ such that the Jacobson radical $J(R)$ is not nil. On the other hand, we shall prove that, for the positive answer to the question above, it suffices to assume that $J(R_e)$ is left $T$-nilpotent. It will also be shown that the weaker condition that $J(R_e)$ is equal to the Baer radical $B(R_e)$, is not sufficient for the local nilpotency of $J(R)$.

Our proofs are based on the previous results of [7], [10] and [21].

Theorem 1. For each group $G$, there exists a strongly $G$-graded ring $R$ such that the Jacobson radical $J(R)$ is not nil.
Lemma 1. Let $R = \bigoplus_{g \in G} R_g$ be a $G$-graded ring, and let $h \in G$. Then there exists a $G$-graded ring $Q = \bigoplus_{g \in G} Q_g$ such that $Q \supseteq R$, $J(Q) \supseteq J(R)$, $Q_g \supseteq R_g$ and $Q_g Q_h \supseteq R_{gh}$ for each $g, h \in G$.

Proof: Let $Z$ be the ring of integers, $R^1$ the ring $R$ with identity 1 adjoined, $Z[x, y]$ the ring of polynomials with non-commuting variables $x, y$. Denote by $W$ the free product of $R$ and $Z[x, y]$. For $w \in W$, let $\langle w \rangle$ be the subring generated in $W$ by $w$. Put $M = R + Ry + xR + xRy$, $S = M + \langle xy \rangle + \langle xy \rangle x + y \langle xy \rangle + \langle yx \rangle$. To simplify the notation, we shall denote by the same letters elements and their images in the quotient rings which will be introduced. If we factor out the ideal generated in $W$ by $x^2, y^2, yR, Rx$ and all $r - yxr, r - ryx$, where $r$ runs over $R$, then the resulting quotient ring $Q$ is equal to $Z + \langle x \rangle + \langle y \rangle + S$. Clearly, $S$ and $M$ are ideals of $Q$. It is routine to verify that

$$M = \begin{bmatrix} R & Ry \\ xR & xRy \end{bmatrix}$$

and

$$S/M = \begin{bmatrix} \langle xy \rangle & \langle xy \rangle x \\ y\langle xy \rangle & \langle yx \rangle \end{bmatrix}$$

are Morita contexts (cf. [1]). Further, $R, xRy \cong R$ and $Ry, xR \cong R^0$, where $R^0$ stands for the ring with zero multiplication defined on the additive group of $R$. Since $\langle xy \rangle$ and $\langle yx \rangle$ are semiprime rings and $S/M$ satisfies the left annihilator condition in the sense of [21], then [21], Lemma 2.6, implies that $S/M$ is semiprime. Therefore $J(S) = J(M)$.

Take any $q \in J(Q)$, say $q = a + bx + cy + s$, where $a, b, c \in Z, s \in S$. If $a \neq 0$, then $qxy \notin M$ and so $0 \neq qxy \in J(S/M)$, a contradiction. If $a = 0, b \neq 0$, then $0 \neq qy \in J(S/M)$ gives a contradiction. Finally, if $a = b = 0, c \neq 0$, then $0 \neq qx \in J(S/M)$, a contradiction again. Therefore $a = b = c = 0$, that is $q \in J(M)$. Thus $J(Q) = J(M)$.

Denote by $I$ the ideal generated in $Q$ by $J(R)$. Then

$$I = \begin{bmatrix} J(R) & J(R)y \\ xJ(R) & xJ(R)y \end{bmatrix}.$$ 

Clearly, $I$ is the largest ideal of $M$ satisfying the property that $I \cap R \subseteq J(R)$ and $I \cap xRy \subseteq J(xRy) = xJ(R)y$. In view of [10], Corollary 1, and [11], Corollary 6, we conclude $I = J(M)$. Hence $I = J(Q)$. In particular, $J(Q) \supseteq J(R)$.

To make $Q$ a $G$-graded ring, we put $x \in Q_h, y \in Q_{h-1}, Z \subseteq Q_e$, and then the grading naturally comes from $R$. For example, $xR_g y \subseteq Q_{gh-1} \subseteq Q_{hg}$. Since $R_{gh} y \subseteq Q_g$ and $x \in Q_h$, we get $Q_g Q_h \supseteq (R_{gh} y) x = R_{gh}$, as required. □
Lemma 2. Let \( R = \bigoplus_{g \in G} R_g \) be a \( G \)-graded ring. Then there exists a \( G \)-graded ring \( Q = \bigoplus_{g \in G} Q_g \) such that \( Q \supseteq R, J(Q) \supseteq J(R), Q_g \supseteq R_g \) and \( Q_g Q_h \supseteq R_{gh} \) for all \( g, h \in G \).

**Proof:** Denote by \( R^{(h)} \) the ring constructed by \( R \) and \( h \) in Lemma 1. We may order the set \( G \), identify the elements of \( G \) with ordinal numbers and define an ascending chain of \( G \)-graded rings \( T_\alpha \) by putting \( T_1 = R^{(1)}, T_\alpha = ( \bigcup_{\beta < \alpha} T_\beta )^{(\alpha)} \).

The transfinite induction shows that \( J(T_\alpha) \supseteq J(R) \) in view of Lemma 1. However, \( G = \{ \alpha | \alpha \leq \tau \} \) for some \( \tau \). Hence a straightforward verification shows that \( Q = \bigcup_{\alpha \leq \tau} T_\alpha \) is the desired ring. \( \square \)

Lemma 3. Let \( R = \bigoplus_{g \in G} R_g \) be a \( G \)-graded ring. Then there exists a strongly \( G \)-graded ring \( Q = \bigoplus_{g \in G} Q_g \) such that \( Q \supseteq R, J(Q) \supseteq J(R), \) and \( Q_g \supseteq R_g \) for all \( g \in G \).

**Proof:** Denote by \( R' \) the ring constructed in Lemma 2, and put \( R_1[1] = R', R_{n+1} = (R_n)^' \). Then it is routine to verify that \( Q = \bigcup_{n=1}^{\infty} R_n \) is the required example. \( \square \)

**Proof of Theorem 1** easily follows from Lemma 3 if we take any quasi-regular but not nil ring \( R \) and make it \( G \)-graded with \( R_e = R \).

Now we shall give a new condition sufficient for the Jacobson radical of a ring strongly graded by a free group to be locally nilpotent. In fact, our condition is applicable not only to free groups, but also to all u.p.-groups. A group \( G \) is called a unique product (u.p.-)group if, for any non-empty subsets \( X, Y \) of \( S \), there exists at least one element uniquely presented in the form \( xy \), where \( x \in X, y \in Y \) (cf. [16], §13.1). The radicals of rings graded by u.p.-groups were considered, in particular, in [6] and [7]. A ring \( R \) is said to be left \( T \)-nilpotent if, for every sequence \( x_1, x_2, \ldots \in R \), there exists \( n \) such that \( x_1 \ldots x_n = 0 \). The class of left \( T \)-nilpotent rings lies strictly between the class of nilpotent rings and the Baer radical class (cf. [5]).

**Theorem 2.** Let \( G \) be a u.p.-group, \( R = \bigoplus_{g \in G} R_g \) a strongly \( G \)-graded ring. If \( J(R_e) \) is left \( T \)-nilpotent, then \( J(R) \) is locally nilpotent.

**Proof:** Given that \( G \) is a u.p.-group, it follows from [7], Theorem 2.2, that the Levitzky radical \( L(R) \) is homogeneous, i.e. \( L(R) = \bigoplus_{g \in G} L(R) \cap R_g \). Since \( R/L(R) \) is...
is strongly $G$-graded, we may assume that from the very beginning $L(R) = 0$.

Suppose to the contrary that $J(R) \neq 0$. For $r \in R$, $g \in G$, denote by $r_g$ the projection of $r$ on $R_g$, and put $\text{supp}(r) = \{ g \in G | r_g \neq 0 \}$. Let $l(r) = |\text{supp}(r)|$. Choose a non-zero element $s$ in $J(R)$ with minimal length $l(s)$, and take any $h \in \text{supp}(s)$. If $s_h R_{h-1} = 0$, then $s_h R_{h-1} R = 0$, and so $s_h \in A = \{ r \in R | r R = 0 \}$. However, $A \subseteq L(R) = 0$, because $A^2 = 0$. Thus $s_h R_{h-1} \neq 0$. Therefore the set $P = \{ r_r | r \in J(R), \ l(r) = l(s) \}$ is non-zero. Given that $G$ is a u.p.-group, Theorem 3.2 of [7] tells us that $P \subseteq J(R_e)$. Denote by $I$ the ideal generated in $R$ by $P$. We claim that $I$ is left $T$-nilpotent.

Suppose that there exists a sequence of elements $x_1, x_2, \ldots$ of $I$ such that $x_1 \ldots x_n \neq 0$ for all $n$. Each $x_i$ is a finite sum of elements of the form $ar_e b$, where $r \in J(R)$, $l(r) = l(s)$, $a$ and $b$ are homogeneous elements of $R^1$. We may assume that all $b$ belong to $R$. (Indeed, if $b \in Z$, then we can replace $x_i$ by $x_i x_{i+1}$, and consider the sequence $x_1, \ldots, x_i x_{i+1}, x_{i+2}, \ldots$) Denote by $S(x)$ the set of such summands of $x_i$. For arbitrarily large $n$ we can pick $y_1 \in S(x_1), \ldots, y_n \in S(x_n)$ such that $y_1 \ldots y_n \neq 0$. Since all the $S(x_i)$ are finite, a standard argument shows that there exists an infinite sequence $y_1, y_2, \ldots$ where $y_i \in S(x_i)$ and $y_1 \ldots y_n = 0$ for all $n$. Then $y_i = a^{(i)} r_e^{(i)} b^{(i)}$ where $r^{(i)} \in J(R)$, $l(r^{(i)}) = l(s)$, $a^{(i)}$ and $b^{(i)}$ are homogeneous elements of $R^1$, and $b^{(i)} \in R$. Given that $R$ is strongly graded, $b^{(2)} \in R$, and $G$ is a group, it follows that $b^{(2)} = c^{(2)} d^{(2)}$ for some homogeneous elements $c^{(2)}, d^{(2)}$ such that $b^{(1)} a^{(2)} r_e^{(2)} c^{(2)} \in R_e$. Similarly, for any $i \geq 3$, there exist homogenous elements $c^{(i)}, d^{(i)}$ such that $b^{(i)} = c^{(i)} d^{(i)}$ and $d^{(i-1)} a^{(i)} r_e^{(i)} c^{(i)} \in R_e$. Let $z_1 = r_e^{(1)}$, $z_2 = b^{(1)} a^{(2)} r_e^{(2)} c^{(2)}$, and $z_i = d^{(i-1)} a^{(i)} r_e^{(i)} c^{(i)}$ for $i \geq 3$. Then, $z_1, z_2, z_3, \ldots \in P$. Since $J(R_e)$ is left $T$-nilpotent and contains $P$, we get $z_1 \ldots z_n = 0$ for some $n > 1$. Hence $y_1 \ldots y_n = a^{(1)} z_1 \ldots z_n d^{(n)} = 0$.

This contradiction shows that $I$ is left $T$-nilpotent, and so $I \subseteq L(R) = 0$. Therefore $J(R) = 0$, which completes the proof. \hfill \Box

Let $\mathbb{P}$ denote the set of positive integers. The well-known Golod’s example of a nil but not locally nilpotent ring $R$ is $\mathbb{P}$-graded (cf. [16]). Therefore one cannot replace strongly graded rings by arbitrary graded rings in Theorem 2. Now we shall show that the left $T$-nilpotence cannot be weakened to Baer radicalness, either.

**Theorem 3.** Let $G$ be a non-periodic group with identity $e$. Then there exists a strongly $G$-graded ring $Q$ such that $J(Q_e) = B(Q_e)$ but $J(Q) \neq L(Q)$.

**Proof:** Let $R$ be the Golod ring. Since $R$ is $\mathbb{P}$-graded, $R$ can easily be made $G$-graded with $R_e = 0$. Take any $h \in G$ and denote by $Q, S, M$ the rings constructed by $R$ as in the proof of Lemma 1. It has been proved that $J(Q) = J(M)$. The same reasoning shows that $B(Q) = B(M)$. Further,

$$M \cong \begin{bmatrix} R & Ry \\ xR & xRy \end{bmatrix},$$
whence

\[ M_e = \begin{bmatrix} R_e & R_h y \\ x R_{h^{-1}} & x R_{e y} \end{bmatrix}. \]

Evidently \( R_e \) and \( x R_{e y} \) are isomorphic to \( R_e \) which satisfies \( J(R_e) = B(R_e) \), because it is zero. It follows from [10], Corollary 1, and [11], Corollary 6, that \( J(M_e) \) is equal to the largest ideal \( I \) of \( M_e \) with the property that \( I \cap R_e \subseteq J(R_e) \) and \( I \cap x R_{e y} \subseteq J(x R_{e y}) \). Besides, [10], Corollary 3, and [11], Corollary 6, imply that \( B(M_e) \) is the largest ideal of \( M_e \) with the property that \( I \cap R_e \subseteq J(R_e) \) and \( I \cap x R_{e y} \subseteq J(x R_{e y}) \). Therefore \( J(M_e) = B(M_e) \). Further, \( Q_e / S_e \cong \mathbb{Z} \) and \( S_e / M_e = \langle xy \rangle + \langle yx \rangle \) imply \( J(Q_e) = B(Q_e) \). It follows from [21], Lemma 2.3, that \( J(Q_e) \supseteq J(R_e) \). This and transfinite induction show that all rings \( Q \) obtained from \( R \) in Lemmas 2 and 3 satisfy \( J(Q_e) = B(Q_e) \). However \( J(Q) \) is not locally nilpotent, because \( J(Q) \supseteq J(R) \). Thus \( Q \) is the required example. \( \square \)

Note that in the opposite case, where \( G \) is locally finite, it follows from the results of [2] and [3] that \( J(R_e) = L(R_e) \) implies \( J(R) = L(R) \) (cf. [13], Lemma 1.1). Analogous results were obtained in [2] for the more general case of rings graded by locally finite semigroups. A sufficient condition for the Jacobson radical of an algebra graded by a finite group to be nilpotent follows from the main theorem of [17].

### References


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