On the existence of weak solutions for degenerate systems of variational inequalities with critical growth

Martin Fuchs

Abstract. We prove the existence of solutions to systems of degenerate variational inequalities.

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In this note we give a short proof of the following Theorem obtained in [1] not relying on the partial regularity theory.

Theorem. Suppose that $\Omega \subset \mathbb{R}^n$ is a bounded open set and that $p \in (1, \infty)$ is given. For a continuous function $f : \overline{\Omega} \times \mathbb{R}^N \times \mathbb{R}^{nN} \to \mathbb{R}^N$ we consider the variational inequality

\[ \begin{align*}
\text{(V)} \quad & \text{find } u \in K \text{ such that } \\
& \int_{\Omega} |\nabla u|^{p-2} \nabla u \cdot \nabla (v - u) \, dx \geq \int_{\Omega} f(\cdot, u, \nabla u) \cdot (v - u) \, dx
\end{align*} \]

where the class $K$ is defined as \{ $v \in H^{1,p}(\Omega, \mathbb{R}^N) : v = u_0$ on $\partial \Omega$, $v(x) \in K$ \}. Here $K$ denotes the closure of a convex bounded open set in $\mathbb{R}^N$ with the boundary of class $C^2$ and $u_0$ is a given function in $H^{1,p}(\Omega, \mathbb{R}^N)$ such that $u_0(\Omega) \subset K$. Then, if $f$ satisfies the growth estimate

\[ |f(x, y, Q)| \leq a \cdot |Q|^p \]

for some constant $a \geq 0$ and if in addition

\[ a < 1 / \text{diam } K \]

holds, problem (V) admits at least one solution $u \in K$.

As shown in [1] we obtain as a
**Corollary.** If \( u_0 \in H^{1,p}(\Omega, \mathbb{R}^N) \cap L^\infty \) is given and if \( f \) satisfies (1) as well as
\[
a < \frac{1}{2\|u_0\|_\infty},
\]
then the Dirichlet problem
\[
\begin{aligned}
-\partial_\alpha(|\nabla u|^{p-2}\partial_\alpha u) &= f(\cdot, u, \nabla u) \quad \text{on } \Omega, \\
u &= u_0 \quad \text{on } \partial \Omega
\end{aligned}
\]
has at least one weak solution \( u \in H^{1,p}(\Omega, \mathbb{R}^N) \cap L^\infty \).

In the quadratic case \( p = 2 \) the above Theorem is due to Hildebrandt and Widman [5] but we did not succeed to extend their method to general \( p \). Our proof (working for all \( p \)) is based on a compensated compactness type lemma demonstrated in [2] with basic ideas taken from Landes paper [6].

**Lemma.** Suppose that we have weak convergence \( u_m \rightharpoonup u \) in the space \( H^{1,p}(\Omega, \mathbb{R}^N) \). Then there is a subsequence \( \{\tilde{u}_m\} \) such that \( |\nabla \tilde{u}_m|^{p-2}\nabla \tilde{u}_m \rightharpoonup |\nabla u|^{p-2}\nabla u \) weakly in \( L^{\frac{p}{p-1}}(\Omega, \mathbb{R}^{nN}) \) and \( \nabla \tilde{u}_m \rightharpoonup \nabla u \) pointwise a.e. provided we know
\[
\int_{\Omega} |\nabla u_m|^{p-2}\nabla u_m \cdot \nabla \varphi \, dx \leq c \cdot \|\varphi\|_\infty
\]
for all \( \varphi \in C_0^1(\Omega, \mathbb{R}^N) \) with \( 0 \leq c < \infty \) independent of \( m \) and \( \varphi \). \( \square \)

We now come to the

**Proof of the Theorem:** For \( m \in \mathbb{N} \) let
\[
f_m : \overline{\Omega} \times \mathbb{R}^N \times \mathbb{R}^{nN} \to \mathbb{R}^N,
\]
\[
f_m(x, y, Q) := \begin{cases} f(x, y, Q) : & \text{if } |f(x, y, Q)| \leq m \\
\frac{m}{|f(x, y, Q)|} \cdot f(x, y, Q) & \text{else}
\end{cases}
\]
and consider the approximate problem
\[
(V)_m \quad \begin{cases} \text{find } w \in \mathcal{K} \text{ such that} \\
\int_{\Omega} |\nabla w|^{p-2}\nabla w \cdot \nabla (v - w) \, dx \geq \int_{\Omega} f_m(\cdot, w, \nabla w) \cdot (v - w) \, dx
\end{cases}
\]
holds for all \( v \in \mathcal{K} \).

As shown in [1] the existence of solutions \( u_m \) to \( (V)_m \) can be deduced from Schauder’s fixed point theorem. Recalling (1), (2) and the definition of \( f_m \) we infer
\[
(1 - a \cdot \text{diam } K) \cdot \int_{\Omega} |\nabla u_m|^p \, dx \leq \int_{\Omega} |\nabla u_m|^{p-1} \cdot |\nabla u_0| \, dx
\]
so that \( \sup_m \|u_m\|_{H^{1,p}(\Omega)} < \infty \). Thus we may assume
\[
u_m \rightharpoonup u \quad \text{in } H^{1,p}(\Omega, \mathbb{R}^N)
\]
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at least for a subsequence. In order to proceed further we linearize the variational inequality \((V)_m\) making use of the fact that \(\partial K\) is of class \(C^2\). As in [3, Theorem 2.1, 2.2] we get for all \(\psi \in C^1_0(\Omega, \mathbb{R}^N)\)

\[
\begin{aligned}
\int_{\Omega} (|\nabla u_m|^{p-2} \nabla u_m \cdot \nabla \psi - f_m(\cdot, u_m, \nabla u_m) \cdot \psi) \, dx \\
= \int_{\Omega \cap \{u_m \in \partial K\}} \psi \cdot \mathcal{N}(u_m) b_m(\cdot, u_m, \nabla u_m) \, dx
\end{aligned}
\]

where \(\mathcal{N}(y)\) is the interior normal field of \(\partial K\) and \(b_m(\cdot, u_m, \nabla u_m)\) has the properties

\[
b_m(\cdot, u_m, \nabla u_m) \geq 0 \quad \text{a.e. on } [u_m \in \partial K],
\]

\[
b_m(\cdot, u_m, \nabla u_m) \leq \tilde{a} \cdot |\nabla u_m|^p
\]

with \(\tilde{a} \geq 0\) independent of \(m\). Now we are in the position to apply the Lemma and deduce

\[
\int_{\Omega} |\nabla u_m|^{p-2} \nabla u_m \cdot \nabla \psi \, dx \to \int_{\Omega} |\nabla u|^{p-2} \nabla u \cdot \nabla \psi \, dx
\]

(after selecting a suitable subsequence). We claim

\[
\int_{\Omega} f_m(\cdot, u_m, \nabla u_m) \cdot \psi \, dx \to \int_{\Omega} f(\cdot, u, \nabla u) \cdot \psi \, dx.
\]

To prove this we observe

\[
(x, u_m(x), \nabla u_m(x)) \to (x, u(x), \nabla u(x))
\]

for almost all \(x \in \Omega\), especially

\[
f(\cdot, u_m, \nabla u_m) \to f(\cdot, u, \nabla u) \quad \text{a.e.}
\]

But for points \(x \in \Omega\) with the property that a finite limit

\[
\lim_{m \to \infty} f(x, u_m(x), \nabla u_m(x))
\]

exists, we clearly have

\[
f_m(x, u_m(x), \nabla u_m(x)) = f(x, u_m(x), \nabla u_m(x))
\]

for \(m \gg 1\), in conclusion \(f_m(\cdot, u_m, \nabla u_m) \to f(\cdot, u, \nabla u)\) a.e. On the other hand the uniform growth estimate \(|f_m(x, y, Q)| \leq a \cdot |Q|^p\) combined with the smallness condition (2) implies Caccioppli’s inequality

\[
\int_{B_{R/2}} |\nabla u_m|^p \, dx \leq \mu \cdot R^{-p} \int_{B_R} |u_m - (u_m)_R|^p \, dx
\]
for any ball \( B_R \subset \Omega \) with \( \mu \) independent of \( m \). From this we easily get
\[
\sup_m \| \nabla u_m \|_{L^q(\Omega')} < \infty
\]
for any subregion \( \Omega' \subset\subset \Omega \) and with \( q \) slightly larger as \( p \). After passing to a subsequence we may therefore assume
\[
f_m(\cdot, u_m, \nabla u_m) \rightharpoonup g
\]
weakly in the space \( L^{q/p}_{\text{loc}}(\Omega, \mathbb{R}^N) \) for some function \( g \). Using Egoroff’s Theorem we find \( g = f(\cdot, u, \nabla u) \) which proves (5).

Next we look at the remaining integral
\[
\int_{[u_m \in \partial K]} \psi \cdot \mathcal{N}(u_m) \cdot b_m(\cdot, u_m, \nabla u_m) \, dx := I_m
\]
and specialize \( \psi = v - u \) where \( v \in \mathbb{K} \) is arbitrary but with the property \( \text{spt} (v - u) \subset\subset \Omega \). (Note that (4), (5) remain valid). We have
\[
I_m = \int_{[u_m \in \partial K] \cap \text{spt} (v-u)} (v - u_m) \cdot \mathcal{N}(u_m) \cdot b_m(\cdot, u_m, \nabla u_m) \, dx
+ \int_{[u_m \in \partial K] \cap \text{spt} (v-u)} (u_m - u) \cdot \mathcal{N}(u_m) \cdot b_m(\cdot, u_m, \nabla u_m) \, dx
=: I^1_m + I^2_m,
\]
\( I^1_m \geq 0 \) an account of \( (v - u_m) \cdot \mathcal{N}(u_m) \geq 0 \) a.e. on \([u_m \in \partial K] \cap \text{spt} (v-u)\) (due to the convexity of \( K \)) and
\[
|I^2_m| \leq \int_{\text{spt} (v-u)} \tilde{a} \cdot |\nabla u_m|^p |u_m - u| \, dx
\leq \tilde{a} \cdot \left( \int_{\text{spt} (v-u)} |\nabla u_m|^q \, dx \right)^{p/q} \cdot \left( \int_{\text{spt} (v-u)} |u_m - u|^q \, dx \right)^{1-p/q}
\xrightarrow{m \to \infty} 0,
\]
since \( \|\nabla u_m\|_{L^q(\text{spt} (v-u))} \) is uniformly bounded and
\[
\int_{\text{spt} (v-u)} |u_m - u|^q \, dx \leq \text{const}(q, p, \text{diam } K) \cdot \int_{\text{spt} (v-u)} |u_m - u|^p \, dx \to 0.
\]

Putting together our results we arrive at
\[
\int_\Omega |\nabla u|^{p-2} \nabla u \cdot \nabla (v-u) \, dx \geq \int_\Omega f(\cdot, u, \nabla u) \cdot (v-u) \, dx
\]
for all \( v \in \mathbb{K} \) such that \( \text{spt} (u - v) \subset \subset \Omega \). We have to remove the support condition on \( v \in \mathbb{K} \). Then \( v - u \in H^{1,p}(\Omega, \mathbb{R}^N) \) so that there is a sequence \( w_m \in C^\infty_0(\Omega, \mathbb{R}^N) \) such that \( w_n \rightarrow v - u \) in the strong topology of the space \( H^{1,p}(\Omega, \mathbb{R}^N) \). Let \( F : \mathbb{R}^N \rightarrow \mathbb{K} \) denote the projection onto the set \( K \). Then \( v_m := F(u + w_m) \) belongs to the class \( \mathbb{K} \), moreover (6) is valid for \( v_m \). It is easy to check that

\[
v_m \rightarrow F(v) = v
\]

weakly in \( H^{1,p}(\Omega, \mathbb{R}^N) \), hence

\[
\int_{\Omega} |\nabla u|^{p-2} \nabla u \cdot \nabla (v_m - u) \, dx \rightarrow \int_{\Omega} |\nabla u|^{p-2} \nabla u \cdot \nabla (v - u) \, dx.
\]

After passing to a subsequence we may assume \( v_m \rightarrow v \) a.e. on \( \Omega \) and since

\[
|f(\cdot, u, \nabla u)| \cdot |v_m - u| \leq a \cdot \text{diam } K \cdot |\nabla u|^p \in L^1(\Omega)
\]

we deduce from dominated convergence that

\[
\int_{\Omega} f(\cdot, u, \nabla u) \cdot (v_m - u) \, dx \rightarrow \int_{\Omega} f(\cdot, u, \nabla u) \cdot (v - u) \, dx
\]

so that \( u \) is a solution of the variational inequality (V).

From [4] we get in addition

**Corollary.** Let \( u \) denote the solution of (V) obtained in the Theorem. Then there is a relatively closed set \( \Sigma \subset \Omega \) such that \( u \in C^{\alpha}(\Omega - \Sigma) \) for some \( 0 < \alpha < 1 \) and \( H^{n-p}(\Sigma) = 0 \).

**References**


Universität des Saarlandes, Fachbereich Mathematik, D-6600 Saarbrücken, Germany

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