On one class of solvable boundary value problems for ordinary differential equation of $n$-th order

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Abstract. New sufficient conditions of the existence and uniqueness of the solution of a boundary problem for an ordinary differential equation of $n$-th order with certain functional boundary conditions are constructed by the method of a priori estimates.

Keywords: boundary problem with functional conditions, differential equations of $n$-th order, method of a priori estimates, differential inequalities

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Introduction

In the paper we give new sufficient conditions for existence and uniqueness of the solution to the problem

\begin{align*}
(1) & \quad u^{(n)} = f(t, u, \ldots, u^{(n-1)}) \\
(2_1) & \quad \ell_i(u, u^{(1)}, \ldots, u^{(k_0-1)}) = 0, \quad i = 1, \ldots, k_0 \\
(2_2) & \quad \Phi_{0i}(u^{(i-1)}) = \Phi_i(u^{(k_0)}, u^{(k_0+1)}, \ldots, u^{(n-1)}), \quad i = k_0 + 1, \ldots, n
\end{align*}

where $f : \langle a, b \rangle \times \mathbb{R}^n \to \mathbb{R}$ satisfies the local Carathéodory condition and for each $i \in \{1, \ldots, k_0\}$, $\ell_i : [C(\langle a, b \rangle)]^{k_0} \to \mathbb{R}$ is a linear continuous functional and for each $i \in \{k_0 + 1 \ldots n\}$, $\Phi_{0i}$ — the linear nondecreasing continuous functional on $C(\langle a, b \rangle)$ is concentrated on $\langle a_i, b_i \rangle \subseteq \langle a, b \rangle$, $(i = k_0 + 1, \ldots, n)$ (i.e. the value of $\Phi_{0i}$ depends only on functions restricted to $\langle a_i, b_i \rangle$, and the segment can be degenerated to a point). $\Phi_i$ $(i = k_0 + 1, \ldots, n)$ are continuous functionals on $[C(\langle a, b \rangle)]^{n-k_0}$. In general $\Phi_{0i}(1) = c_i$ $(i = k_0 + 1, \ldots, n)$, without loss of generality we can suppose $\Phi_{0i}(1) = 1$ $(i = k_0 + 1, \ldots, n)$.

Throughout the paper assume:

\begin{align*}
(3) & \quad \text{Boundary value problem } u^{(k_0)} = 0 \text{ possesses only the trivial solution with condition } (2_1).
\end{align*}
Problem for differential equation (1) together with boundary condition

\[
\sum_{j=1}^{k_0} a_{ij} \cdot u^{(j-1)}(a) + b_{ij} \cdot u^{(j-1)}(b) = 0 \quad (i = 1, \ldots, k_0)
\]

\[
u(i-1)(t_i) = c_i \quad (i = k_0 + 1, \ldots, n)
\]
is not the special case of problems in [1] and [4]. On the other hand, the boundary value problem with the same two groups of condition but in opposite order for \(c_j = 0\) is the special case of problems, which were studied in [1].

**Main result**

We adopt the following notation:

\(\langle a, b \rangle\) — a segment, \(-\infty < a \leq a_i \leq b_i \leq b < +\infty \ (i = k_0 + 1, \ldots, n)\), \(R^n\) — \(n\)-dimensional real space with points \(x = (x_i)_{i=1}^n\) normed by \(\|x\| = \sum_{i=1}^n |x_i|\),

\[R^n_+ = \{x \in R^n : x_i \geq 0 \ i = 1, \ldots, n\},\]

\(C^{n-1}(\langle a, b \rangle)\) — the space of functions continuous together with their derivatives up to the order \(n - 1\) on \(\langle a, b \rangle\) with the norm

\[\|u\|_{C^{n-1}(\langle a, b \rangle)} = \max \left\{ \sum_{i=1}^n |u^{(i-1)}(t)| : a \leq t \leq b \right\},\]

\(AC^{n-1}(\langle a, b \rangle)\) — a set of all functions absolutely continuous together with their derivatives to the \((n - 1)\)-order on \(\langle a, b \rangle\), the space \(L^p(\langle a, b \rangle)\) is the space of functions integrable on \(\langle a, b \rangle\) in \(p\)-th power with a norm

\[\|u\|_{L^p} = \left\{ \begin{array}{ll}
\int_a^b |u(t)|^p dt / p & \text{for } 1 \leq p < \infty \\
\text{vrai max}\{ |x(t)| : a \leq t \leq b \} & \text{for } p = \infty,
\end{array} \right.\]

\(L^p(\langle a, b \rangle, R^n_+) = \{u \in L^p(\langle a, b \rangle) : u(t) \geq 0, t \in \langle a, b \rangle\}\). If \(x = (x_i(t))_{i=1}^n \in [C(\langle a, b \rangle)]^n\) and \(y = (y_i(t))_{i=1}^n \in [C(\langle a, b \rangle)]^n\), then \(x \leq y\) if and only if \(x_i(t) \leq y_i(t)\) for all \(t \in \langle a, b \rangle\) and \(i = 1, \ldots, n\). A functional \(\Phi : [C(\langle a, b \rangle)]^n \to R_+\) is said to be homogeneous iff: \(\Phi(\lambda x) = \lambda \Phi(x)\) for all \(\lambda \in R_+\) \(x \in [C(\langle a, b \rangle)]^n\) and nondecreasing if \(\Phi(x) \leq \Phi(y)\) for all \(x, y \in [C(\langle a, b \rangle)]^n\), \(x \leq y\). Let us consider the problem (1), (2). Under the solution we understand the function with absolutely continuous derivatives up to the order \((n - 1)\) on \(\langle a, b \rangle\), which satisfies the equation (1) for almost all \(t \in \langle a, b \rangle\) and fulfills the boundary condition (2).

To solve (1), (2) we specify a class of auxiliary functions

\[g, \ell_1, \ell_2 \ldots \ell_{k_0}, \ h_{k_0+1} \ldots h_n, \ \Psi_{k_0+1} \ldots \Psi_n.\]
Theorem 1. Let \( \ell_i : [C(\langle a,b \rangle)]^{k_0} \to R \) (\( i = 1, \ldots, k_0 \)) be the linear continuous functionals, \( \Psi_i : [C(\langle a,b \rangle)]^{n-k_0} \to R_+ \) (\( i = k_0 + 1, \ldots, n \)) the homogeneous continuous nondecreasing functionals and \( g,h_i \in L^1(\langle a,b \rangle,R_+) \) (\( i = k_0 + 1, \ldots, n \)). If the system of differential inequalities

\[
|\varrho_i(t)| \leq |\varrho_{i+1}(t)| \quad t \in \langle a,b \rangle \quad (i = 1, \ldots, n - 1)
\]

\[
|\varrho_i(t) - g(t) \cdot \varrho_{i}(t)| \leq \sum_{j=k_0+1}^{n} h_j(t)|\varrho_j(t)|, \quad t \in \langle a,b \rangle
\]

with boundary conditions

\[
\ell_i(\varrho_1, \ldots, \varrho_{k_0}) = 0 \quad (i = 1, \ldots, k_0)
\]

\[
\min\{|\varrho_i(t)| : a_i \leq t \leq b_i\} \leq \Psi_i(|\varrho_{k_0+1}|, \ldots, |\varrho_{n}|) \quad (i = k_0 + 1, \ldots, n)
\]

has only the trivial solution, we say that \((g,\ell_1,\ell_2, \ldots, \ell_{k_0},h_{k_0+1}, \ldots, h_n,\Psi_{k_0+1}, \ldots, \Psi_n) \in \) \(LN(\langle a,b \rangle, a_{k_0+1}, \ldots, a_n, b_{k_0+1}, \ldots, b_n)\).

Remark. If \( k_0 = 0 \) we have

\(LN(\langle a,b \rangle, a_1, a_2, \ldots, a_n, b_1, \ldots, b_n) = Nic(\langle a,b \rangle, a_1, \ldots, a_n, b_1, \ldots, b_n)\)

from paper [4].

Theorem 1. Let the condition (6) be satisfied and let the data \( f, \Phi_{k_0+1}, \ldots, \Phi_n \) of (1), (2) satisfy the inequalities

\[
[f(t,x_1,x_2, \ldots, x_n) - g(t) \cdot x_n] \text{ sign } x_n \leq \sum_{j=k_0+1}^{n} h_j(t) \cdot |x_j| + \omega(t, \sum_{j=1}^{n} |x_j|)
\]

for \( t \in \langle a,b \rangle, \quad x \in R^n \)

\[
[f(t,x_1,x_2, \ldots, x_n) - g(t) \cdot x_n] \text{ sign } x_n \geq - \sum_{j=k_0+1}^{n} h_j(t)|x_j| - \omega(t, \sum_{j=1}^{n} |x_j|)
\]

for \( t \in \langle a,b \rangle, \quad x \in R^n \)

\[
|\Phi_i(u^{(k_0)}, \ldots, u^{(n-1)})| \leq \Psi_i(|u^{(k_0)}|, \ldots, |u^{(n-1)}|) + r
\]

for \( (i = k_0 + 1, \ldots, n) \),

where \( r \in R_+, \quad \omega : \langle a,b \rangle \times R_+ \to R_+ \) and \( \omega(\cdot, \varrho) \in L(\langle a,b \rangle, R_+) \) \forall \varrho \in R_+, \omega(t, \cdot) \) is nondecreasing for all \( t \in \langle a,b \rangle \) and

\[
\lim_{\varrho \to +\infty} \frac{1}{\varrho} \int_{a}^{b} \omega(t, \varrho) \, dt = 0.
\]

Then the problem (1), (2) has at least one solution.

To prove Theorem 1 the following lemma is suitable.
Lemma 1. Let the condition (6) be satisfied. Then there exists a nonnegative constant \( \varrho > 0 \) such that the estimate

\[
\|u\|_{C^{n-1}(\langle a,b \rangle)} \leq \varrho (r + \|h_0\|_{L^1(\langle a,b \rangle)})
\]

holds for each constant \( r \geq 0 \), \( h_0 \in L^1(\langle a,b \rangle, R^+) \) and for each solution \( u \in AC^{n-1}(\langle a,b \rangle) \) of the differential inequalities

\[
\begin{align*}
[u^{(n)}(t) - g(t) \cdot u^{(n-1)}(t)] \cdot \text{sign } u^{(n-1)}(t) &\leq \sum_{j=k_0+1}^{n} h_j(t) |u^{(j-1)}(t)| + h_0(t) \quad \text{for } a_n \leq t \leq b \\
[u^{(n)}(t) - g(t) \cdot u^{(n-1)}(t)] \cdot \text{sign } u^{(n-1)}(t) &\geq \sum_{j=k_0+1}^{n} h_j(t) |u^{(j-1)}(t)| - h_0(t) \quad \text{for } a \leq t \leq b_n
\end{align*}
\]

with boundary condition (2) and

\[
\min \{ |u^{(i-1)}(t)| : a_i \leq t \leq b_i \} \leq \Psi_i(\|u^{(k_0)}\|, \ldots, \|u^{(n-1)}\|) + r \\
(i = k_0 + 1, \ldots, n).
\]

Proof: Let us denote by \( M \) the set of all 3-tuples \((u, h_0, r)\) such that \( u \in AC^{n-1}(\langle a,b \rangle) \), \( h_0 \in L^1(\langle a,b \rangle) \), \( r \geq 0 \) and the relations (2), (111), (112) and (12) are satisfied. It is easy to verify that \((u, h_0, r) \in M\) if and only if the 3-tuple \((u^{(k_0)}, h_0, r)\) fulfills the assumptions of Lemma 1 in [4] (with \( n - k_0 \) in the place of \( n \)). Hence there exists \( \varrho_1 > 0 \) such that

\[
\|u^{(k_0)}\|_{C^{n-k_0}(\langle a,b \rangle)} \leq \varrho_1 (r + \|h_0\|_{L^1(\langle a,b \rangle)})
\]

holds for all \((u, h_0, r) \in M\). Furthermore, by the assumption (3) there exists the Green function \( G(t,s) \) of the boundary value problem \( u^{(k_0)} = 0 \), (2). Consequently, for any \((u, h_0, r) \in M\), the relations

\[
u^{(i-1)}(t) = \int_a^b \frac{\partial^{(i-1)}G(t,s)}{\partial t^{(i-1)}} u^{(k_0)}(s) \, ds, \quad t \in \langle a,b \rangle, \quad i = 1, 2, \ldots, k_0
\]

are true. Putting

\[
\varrho_2 = \max_{a \leq t \leq b} \sum_{i=1}^{k_0} \int_a^b \left| \frac{\partial^{(i-1)}G(t,s)}{\partial t^{(i-1)}} \right| \, ds,
\]
we obtain the relation
\[(15) \quad \|u\|_{C^k_0(\langle a,b\rangle)} \leq \varrho_1 \varrho_2 (r + \|h\|_{L^1(\langle a,b\rangle)})\]
holds for all \((u, h_0, r) \in M\). We put \(\varrho = \varrho_1 + \varrho_1 \cdot \varrho_2\), then (10) follows from (13) by (15).

**Proof of Theorem 1:** Let \(\varrho > 0\) be the constant from Lemma 1. According to (9) there exists constant \(\varrho_0 > 0\) such that
\[(16) \quad \varrho (r + \int_a^b \omega(t, \varrho_0) \, dt) \leq \varrho_0.\]
Putting
\[(17) \quad \chi(s) = \begin{cases} 
1 & \text{for } |s| \leq \varrho_0 \\
2 - \frac{|s|}{\varrho_0} & \text{for } \varrho_0 \leq |s| \leq 2\varrho_0, \\
0 & \text{for } |s| > 2\varrho_0 
\end{cases}\]
we can write
\[(18) \quad \tilde{f}(t, x_1, x_2, \ldots, x_n) = \chi(\|x\|)\int f(t, x_1, x_2, \ldots, x_n) - g(t) \cdot x_n,\]
\[(19) \quad \tilde{\Phi}_i(u^{(k_0)}, \ldots, u^{(n-1)}) = \chi(\|u\|_{C^{n-1}(\langle a,b\rangle)})\Phi_i(u^{(k_0)}, \ldots, u^{(n-1)}) \\
\quad (i = k_0 + 1, \ldots, n).\]

We consider the problem
\[(20) \quad u^{(n)}(t) = g(t)u^{(n-1)}(t) + \tilde{f}(t, u(t), \ldots, u^{(n-1)}(t))\]
with condition (21) and
\[(21) \quad \Phi_{0i}(u^{(i-1)}) = \tilde{\Phi}_i(u^{(k_0)}, \ldots, u^{(n-1)}) \quad (i = k_0 + 1, \ldots, n).\]
The relations (18), (19) immediately imply that \(\tilde{f} : \langle a, b \rangle \times \mathbb{R}^n \to \mathbb{R}\) satisfies the local Carathéodory conditions, \(\tilde{\Phi}_i : [C^{(\langle a,b\rangle)}]^{(n-k_0)} \to \mathbb{R} \quad (i = k_0 + 1, \ldots, n)\) are continuous functionals,
\[(22) \quad f_0(t) = \sup\{|\tilde{f}(t, x_1, \ldots, x_n)| : (x_i)_{i=1}^n \in \mathbb{R}^n \} \in L^1(\langle a,b\rangle)\]
and
\[(23) \quad r_i = \sup\{|\tilde{\Phi}_i(u^{(k_0)}, \ldots, u^{(n-1)})| : u \in C^{n-1}(\langle a,b\rangle)\} < +\infty.\]
Now we want to show that the homogeneous problem
\[(20_0) \quad u^{(n)} = g(t) \cdot u^{(n-1)}(t)\]
with conditions (21) and
\[\Phi_0(u^{(i-1)}) = 0 \quad (i = k_0 + 1, \ldots, n)\]
having only trivial solution. Let \(u\) be an arbitrary solution of this problem. Then
\[u^{(n-1)}(t) = c \cdot w(t)\]
where \(c = \text{const}\) and \(w(t) = \exp[\int_a^t g(s) \, ds]\).

According to (210) and the character of functional \(\Phi_0\) we get
\[\Phi_0(u^{(n-1)}) = 0 = c \cdot \Phi_0(w).\]

From \(\Phi_0(w) \geq \exp(-\int_a^b |g(t)| \, dt) \cdot \Phi_0(1) > 0\) it follows that \(c = 0\) and \(u^{(n-1)} = 0\). Similarly we have \(u^{(n-2)} = 0, \ldots, u^{(k_0)} = 0\), therefore \(u\) is a solution of the differential equation \(u^{(k_0)} = 0\) with condition (21). By hypothesis (3) we have \(u \equiv 0\). Using 2.1 from [3], we obtain that the condition (22), (23) and the unicity of trivial solution of each problem (20), (210), (21) guarantees the existence of solutions of the problem (20), (21), (21). Let \(u\) be the solution of problem (20), (21), (21). We want to show that
\[\|u\|_{C^{n-1}(a,b)} \leq \varrho_0.\]

From (18) and (7) we have
\[\|u^{(n)}(t) - g(t)u^{(n-1)}(t)\| \cdot \text{sign} \, u^{(n-1)}(t) = \]
\[= \int_a^b |f(t, u, \ldots, u^{(n-1)}(t))| \cdot \text{sign} \, u^{(n-1)}(t) = \]
\[= \chi(\sum_{i=1}^n |u^{(i-1)}(t)|)[f(t, u, \ldots, u^{(n-1)}(t)) - g(t)u^{(n-1)}(t)] \cdot \text{sign} \, u^{(n-1)}(t) \leq \]
\[\leq \chi(\sum_{j=1}^n |u^{(j-1)}(t)|)[\sum_{j=k_0+1}^n h_j(t)|u^{(j-1)}(t)| + \omega(t, \sum_{j=1}^n u^{(j-1)}(t))]| \leq \]
\[\leq \sum_{j=k_0+1}^n h_j(t)|u^{(j-1)}(t)| + \omega(t, 2\varrho_0) \quad \text{for} \quad t \in (a, b).\]

Similarly
\[\|u^{(n)}(t) - g(t)u^{(n-1)}(t)\| \cdot \text{sign} \, u^{(n-1)}(t) \geq \]
\[\geq - \sum_{j=k_0+1}^n h_j(t)|u^{(j-1)}(t)| - \omega(t, 2\varrho_0) \quad \text{for} \quad t \in (a, b).\]

From (8) and the character of functionals \(\Phi_0\) \((i = k_0 + 1, \ldots, n)\) imply that
\[\min\{|u^{(i-1)}(t)\} : a_i \leq t \leq b_i\} \leq |\Phi_0(u^{(i-1)})| \leq \Psi_i(u^{(k_0)}, \ldots, u^{(n-1)}) + r.\]

Therefore by Lemma 1 and by (16), (24) holds. Then \(\chi(\sum_{j=1}^n |u^{(j-1)}(t)|) = 1\) and hence by (18), (19) \(u\) is a solution of problem (1), (2).
Theorem 2. Let the condition (6) be satisfied and let the data \( f, \Phi_{k_0+1}, \ldots, \Phi_n \) of (1), (2) satisfy the inequalities

\[
\{ [f(t, x_{11}, \ldots, x_{1n}) - f(t, x_{21}, \ldots, x_{2n})] - g(t)[x_{1n} - x_{2n}] \} \times
\]

\[
\times \text{sign}[x_{1n} - x_{2n}] \leq \sum_{j=k_0+1}^n h_j(t)|x_{1j} - x_{2j}|
\]

(251) \quad \text{for } t \in (a, b), x_1, x_2 \in \mathbb{R}^n,

\[
\{ [f(t, x_{11}, \ldots, x_{1n}) - f(t, x_{21}, \ldots, x_{2n})] - g(t)[x_{1n} - x_{2n}] \} \times
\]

\[
\times \text{sign}[x_{1n} - x_{2n}] \geq - \sum_{j=k_0+1}^n h_j(t)|x_{1j} - x_{2j}|
\]

(252) \quad \text{for } t \in (a, b), x_1, x_2 \in \mathbb{R}^n,

\[
[\Phi_i(u^{(k_0)}, \ldots, u^{(n-1)}) - \Phi_i(v^{(k_0)}, \ldots, v^{(n-1)})] \leq
\]

\[
\leq \Psi_i(|u^{(k_0)}| - v^{(k_0)}|, \ldots, |u^{(n-1)}| - v^{(n-1)}|)
\]

(26) \quad \text{for } u, v \in C^{n-1}(a, b) (i = k_0 + 1, \ldots, n).

Then the problem (1), (2) has unique solution.

PROOF: Let us put \( \omega(t, g) = |f(t, 0 \ldots 0)|, r = \max_{i=k_0+1, \ldots, n} |\Phi_i(0, \ldots, 0)|. \) From (25), (26) and Theorem 1 follows that problem (1), (2) has a solution. We shall prove its uniqueness.

Let \( u \) and \( v \) be arbitrary solutions of the problem (1), (2). Put

\[
\varrho_i(t) = u^{(i-1)}(t) - v^{(i-1)}(t) (i = 1, \ldots, n).
\]

From (25) follows that

\[
|\varrho_i'(t) - g(t)| \leq \sum_{j=k_0+1}^n h_j|\varrho_j|.
\]

(27)

From (26) and the character of \( \ell_i \) \((i = k_0 + 1, \ldots, n)\) and \( \Phi_{0i} \) \((i = k_0 + 1, \ldots, n)\) we have

\[
\min\{|\varrho_i(t)| : a_i \leq t \leq b_i\} = \Phi_{0i}(\min\{|\varrho_i(t)| : a_i \leq t \leq b_i\}) \leq
\]

\[
\leq |\Phi_{0i}(\varrho_i)| \leq \Psi_i(|\varrho_{k_0+1}|, \ldots, |\varrho_n|) (i = k_0 + 1, \ldots, n)
\]

\[
\ell_i(\varrho_1, \ldots, \varrho_{k_0}) = 0 \quad \text{for } i = 1, \ldots, k_0.
\]

(28)

Therefore by (6) we have \( \varrho_i(t) \equiv 0 \) \((i = 1, \ldots, n)\), i.e. \( u(t) \equiv v(t) \). \qed
Effective criteria

**Theorem 3.** Let the inequalities

\[
\begin{align*}
(29) & & f(t, x_1, \ldots, x_n) \cdot \text{sign } x_n & \leq \sum_{j=k_0+1}^{n} h_j(t) |x_j| + \omega(t, \sum_{j=1}^{n} |x_j|) \\
& & \text{for } t \in (a_n, b), \ x \in \mathbb{R}^n, \\
(29) & & f(t, x_1, \ldots, x_n) \cdot \text{sign } x_n & \geq -\sum_{j=k_0+1}^{n} h_j(t) |x_j| - \omega(t, \sum_{j=1}^{n} |x_j|) \\
& & \text{for } t \in (a, b_n), \ x \in \mathbb{R}^n,
\end{align*}
\]

\[
\Phi_i(u^{(k_0)}, \ldots, u^{(n-1)}) \leq \sum_{j=k_0+1}^{n} r_{ij} \|u^{(j-1)}\|_{L^q(a,b)} + r \\
\text{for } u \in C^{n-1}(\langle a,b \rangle) \ (i = k_0 + 1, \ldots, n)
\]

hold, where \(r, r_{ij} \in \mathbb{R}_+ \ (i, j = k_0 + 1, \ldots, n)\), \(\omega : \langle a, b \rangle \times \mathbb{R}_+ \to \mathbb{R}_+\) is a measurable function nondecreasing in the second variable satisfying (9), \(h_i \in L^p(\langle a, b \rangle, \mathbb{R}_+),\ p \geq 1; 1/p + 2/q = 1\).

\[
s_i = \sum_{m=k_0+1}^{n} \{(b-a)^{1/q} \times \sum_{j=i}^{n} \left[\frac{2(b-a)}{\pi}\right]^{2(j-i)} \left(\prod_{k=i}^{j-1} \triangle_k\right) r_{jm} + \\
+\left[\frac{2(b-a)}{\pi}\right]^{2(n+1-i)} \left(\prod_{k=i}^{n} \triangle_k\right) h_{om} \} < 1 \ (i = k_0 + 1, \ldots, n),
\]

where

\[
\triangle_k = \max\{(b - a_k)^{1-2/q}, (b_k - a)^{1-2/q}\} \ (k = k_0 + 1, \ldots, n), \\
h_{0m} = \max\{\|h_m\|_{L^p(\langle a, b_m \rangle)}, \|h_m\|_{L^p(\langle a_m, b \rangle)}\} \ (m = k_0 + 1, \ldots, n).
\]

Then the problem (1), (2) has a solution.

**Theorem 4.** Let the inequalities

\[
[f(t, x_1, \ldots, x_{1n}) - f(t, x_{21}, \ldots, x_{2n})] \cdot \text{sign } [x_{1n} - x_{2n}] \leq \\
\leq \sum_{j=k_0+1}^{n} h_j(t) |x_{1j} - x_{2j}| \\
\text{for } t \in \langle a_n, b \rangle, \ x_1, x_2 \in \mathbb{R}^n,
\]

\[(32)\]
\[ [f(t, x_{11}, \ldots, x_{1n}) - f(t, x_{21}, \ldots, x_{2n})] \text{sign} [x_{1n} - x_{2n}] \geq \]

\[ \geq - \sum_{j=k_0+1}^{n} h_j(t)|x_{1j} - x_{2j}| \]

for \( t \in \langle a, b \rangle, \ x_1, x_2 \in R^n, \)

\[ |\Phi_i(u^{(k_0)}, \ldots, u^{(n-1)}) - \Phi_i(v^{(k_0)}, \ldots, v^{(n-1)})| \leq \]

\[ \leq \sum_{j=k_0+1}^{n} r_{ij} \|u^{(j-1)} - v^{(j-1)}\|_{L^q(\langle a, b \rangle)} \]

for \( u, v \in C^{n-1}(\langle a, b \rangle) (i = k_0 + 1, \ldots, n) \)

hold, where the functions \( h_i \) and constants \( r_{ij} \) and \( s_i \) satisfy the assumptions of Theorem 3. Then the problem (1), (2) has unique solution.

We consider the differential equation

\[ u'' = f(t, u, u') \]

with boundary condition

\[ \ell(u) = \int_{a}^{b} p(t) \cdot u(t) \, dt + \xi u(t_0) = 0 \]

\[ \Phi_{02}(u') = \Phi_{2}(u') \]

where \( f : \langle a, b \rangle \times R^2 \to R \) satisfies the local Carathéodory condition and \( p(t) \in C(\langle a, b \rangle), \(xi \in R, t_0 \in \langle a, b \rangle, \Phi_{02} \) — the linear non-decreasing continuous functional on \( C(\langle a, b \rangle) \) is concentrated on \( \langle a_2, b_2 \rangle \subset \langle a, b \rangle \) (e.g.

\[ \Phi_{02}(u') = \int_{a_2}^{b_2} q(t) \cdot u'(t) \, dt, \]

\( q(t) \in C(\langle a, b \rangle, R_+) \). \)

\( \Phi_2 : C(\langle a, b \rangle) \to R \) is a continuous functional.

**Theorem 5.** Let the inequalities

\[ f(t, x_1, x_2) \cdot \text{sign } x_2 \leq h(t) \cdot |x_2| + \omega(t, \sum_{i=1}^{2} |x_i|) \]

for \( a_2 \leq t \leq b, \ (x_1, x_2) \in R^2, \)

\[ f(t, x_1, x_2) \cdot \text{sign } x_2 \geq -h(t) \cdot |x_2| - \omega(t, \sum_{i=1}^{2} |x_i|) \]
for \( a \leq t \leq b, (x_1, x_2) \in \mathbb{R}^2 \).

(37) \[ |\Phi_2(u')| \leq m. \|u'\|_{L^2(\langle a, b \rangle)} + r \]

hold, where \( m, r \in \mathbb{R}_+ \), \( h(t) \in L^2(\langle a, b \rangle, \mathbb{R}_+) \),

\[
\sqrt{b - a(m + \|h\|_{L^2(\langle a, b \rangle)})} < 1, \int_a^b p(t) \, dt + \xi \neq 0,
\]

\( \omega : \langle a, b \rangle \times \mathbb{R}_+ \to \mathbb{R}_+ \) is a measurable function nondecreasing in the second

variable satisfying (9).

Then the problem (34), (35) has at least one solution.

**Proof:** We put

\[ g(t) \equiv 0; \psi_2(|x_2|) = m \cdot \|x_2\|_{L^2(\langle a, b \rangle)} \]

for \( x_2 \in C(\langle a, b \rangle) \).

By Theorem 1 we must prove that the data \((g, h, \ell, \psi_2)\) are of the class \(LN(\langle a, b \rangle, a_2, b_2)\).

Let the vector \((\varphi_1(t), \varphi_2(t))\) be the solution of the problem (38),

(381) \[ |\varphi'_1(t)| \leq |\varphi_2(t)| \quad a \leq t \leq b \]

(382) \[ |\varphi'_2(t)| \leq h(t)|\varphi_2(t)| \quad a \leq t \leq b \]

with boundary condition

(391) \[ \ell(\varphi_1) = \int_a^b p(t) \cdot \varphi_1(t) \, dt + \xi \cdot \varphi_1(t_0) = 0 \]

(392) \[ \min\{|\varphi_2(t)| : a_2 \leq t \leq b_2\} \leq m \\|\varphi_2\|_{L^2(\langle a, b \rangle)}. \]

We shall prove that this solution is zero. Let us choose \( \tau_0 \in \langle a_2, b_2 \rangle \) so that

\[ |\varphi_2(\tau_0)| = \min\{|\varphi_2(t)| : a_2 \leq t \leq b_2\}. \]

Then integrating relation (382) and using Hölder inequality we obtain

\[
|\varphi_2(t)| \leq |\varphi_2(\tau_0)| + \int_{\tau_0}^{t} h(s)|\varphi_2(s)| \, ds \leq m \\|\varphi_2\|_{L^2(\langle a, b \rangle)} + \int_{\tau_0}^{b} h(s)|\varphi_2(s)| \, ds
\]

and

\[
\|\varphi_2\|_{L^2(\langle a, b \rangle)} \leq \sqrt{b - a(m + \|h\|_{L^2(\langle a, b \rangle)})} \times \|\varphi_2\|_{L^2(\langle a, b \rangle)}.
\]

Since \( \sqrt{b - a} \cdot (m + \|h\|_{L^2(\langle a, b \rangle)}) < 1 \), it follows that \( \varphi_2(t) \equiv 0. \)

From (381) we have

\[ \varphi_1(t) \equiv C = \text{const.} \]

The relation (391) implies that \( \varphi_1(t) \equiv 0 \), because \( \int_a^b p(t) \, dt + \xi \neq 0. \)

\( \square \)
Theorem 6. Let the inequalities

\[\begin{align*}
[f(t,x_{11},x_{12}) - f(t,x_{21},x_{22})] \cdot \text{sign}[x_{12} - x_{22}] & \leq \leq h(t)|x_{12} - x_{22}| \\
\end{align*}\]

for \(a_2 \leq t \leq b; (x_{11},x_{12}), (x_{21},x_{22}) \in \mathbb{R}^2,\)

\[\begin{align*}
[f(t,x_{11},x_{12}) - f(t,x_{21},x_{22})] \cdot \text{sign}[x_{12} - x_{22}] & \geq \geq -h(t)|x_{12} - x_{22}| \\
\end{align*}\]

for \(a \leq t \leq b, (x_{11},x_{12}), (x_{21},x_{22}) \in \mathbb{R}^2,\)

\[|\Phi_2(u') - \Phi_2(v')| \leq m\|u' - v'\|_{L^2([a,b])}\]

for \(u,v \in C^1((a,b))\) hold, where the functionals \(h\) and \(m\) satisfy the assumptions of Theorem 5. Then the problem (34), (35) has unique solution.

References


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