Note on Petrie and Hamiltonian cycles in cubic polyhedral graphs

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Abstract. In this note we show that deciding the existence of a Hamiltonian cycle in a cubic plane graph is equivalent to the problem of the existence of an associated cubic plane multi-3-gonal graph with a Hamiltonian cycle which takes alternately left and right edges at each successive vertex, i.e. it is also a Petrie cycle. The Petrie Hamiltonian cycle in an \( n \)-vertex plane cubic graph can be recognized by an \( O(n) \)-algorithm.

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Throughout this note we shall consider cubic polyhedral graphs, i.e. 3-valent plane 3-connected graphs (see Grünbaum [4], Malkevitch [6]).

Many papers are devoted to the study of the existence of Hamiltonian cycles in cubic plane graphs, see e.g. Holton and McKay [5] or Malkevitch [6] for recent surveys. In Fleischner [2, Chapter VI] there is proved that the problem of finding a Hamiltonian cycle in a cubic plane graph is equivalent to the problem of finding an \( A \)-trail, that is an Eulerian trail whose consecutive edges (including the last and the first) lie on a common face, in an associated Eulerian plane graph.

In this note we show that the cubic hamiltonian problem is equivalent to the problem of finding a cubic multi-3-gonal plane graph \( M \) (i.e. having sizes of all faces \( \equiv 0 \) (mod 3)) with a Petrie cycle which passes through all vertices of \( M \). A cycle \( C \) in a cubic graph is said to be a Petrie cycle if every two, but no three, consecutive edges of \( C \) (including the last and the first) lie on a common face. A path with this property is known to be a Petrie path (a Petrie arc), cf. Coxeter [1], Grünbaum [4, p. 258].

Petrie cycles do not always exist in cubic plane graphs. For example, a graph of a \( k \)-side prisma, \( k \geq 3 \), has a Petrie cycle if and only if \( k \equiv 0 \) (mod 4). Because every Petrie cycle is uniquely determined by arbitrary two of its consecutive edges, the existence of an \( O(n) \)-algorithm which decides if there is a Petrie cycle crossing all vertices of an \( n \)-vertex cubic plane graph is easily seen. Such cycle is called a Petrie Hamiltonian cycle (a \( PH \)-cycle in the sequel).

Let \( G \) be a cubic plane graph and \( A \) be its vertex adjacent to the vertices \( B_1, B_2, B_3 \) and incident with faces \( \alpha_1, \alpha_2, \alpha_3 \). By a cutting off the vertex \( A \) of \( G \) we mean the placing new vertices \( A_1 \) and \( A_2 \) on the edges \( AB_1 \) and \( AB_2 \) of \( G \), respectively, and joining them by a new edge \( A_1 A_2 \) (i.e. a replacing of the vertex
A by a triangle $AA_1A_2$). This changes the graph $G$ into a cubic plane graph $G^*$ with a new triangle $AA_1A_2$ and new faces $\alpha'_1, \alpha'_2, \alpha'_3$ instead of the faces $\alpha_1, \alpha_2, \alpha_3$ of $G$. If the face $\alpha_i, i = 1, 2, 3$ is an $r_i$-gon, the face $\alpha'_i$ is an $(r_i + 1)$-gon. The change $G$ into $G^*$ is denoted by $G^* = G\nabla A$. Let $S = \{A_i|1 \leq i \leq t\}$ be a set of vertices of $G$. Let $G_0 = G, G_i = G_{i-1}\nabla A_i$ for all $i = 1, 2, \ldots, t$. We put

$$G\nabla S := G_t.$$ 

**Lemma 1.** Let $C$ be a cycle of the length $2k$ in a cubic plane graph $G$. Then there is a set $S$ of, say $t$, vertices of $C$ such that $G^* = G\nabla S$ has a Petrie cycle $C^*$ of the length $2(k + t)$.

**Proof:** Denote the vertices of cycle $C$ successively $A_0, A_1, \ldots, A_{2k-1}$. Let $h$ be an edge incident with the vertex $A_0$ lying outside of $C$. We will construct $G^*$ together with its Petrie cycle $C^*$. Let $G_0 = G$. Suppose we have a graph $G_i, i = 0, 1, \ldots, k - 2$ with a Petrie path $P_i$ starting in $A_0$ and ending in $A_{2i}$ and such that for continuation of $P_i$ the right edge in the vertex $A_{2i}$ must be chosen. In the graph $G_i$ one of the four situations (a), (b), (c), (d) depicted in Fig. 1 appears.

![Diagram](image-url)
In the situation (a) of Fig. 1 we put $G_{i+1} := G_i$ and $P_{i+1} := P_i \cup A_{2i+1}A_{2i+2}$. In the situation (b) of Fig. 1 we cut off the vertex $A_{2i+1}$ as it is shown in Fig. 1 (e) and put $G_{i+1} = G_i \nabla A_{2i+1}$ and $P_{i+1} := P_i \cup D_{2i}E_{2i}A_{2i+1}A_{2i+2}$. Changes for the situation (c) and (d) are depicted in Fig. 1 (f) and (g), respectively.

In the graph $G_{k-1}$ we have the Petrie path from $A_0$ to $A_{2k-2}$ and, because of $h$, only the situation of Figure (a) or (b) appears. In the first case we put $G^* := G_{k-1}$ and $C^* := P_{k-1} \cup A_{2k-1}A_0$. In the second case $G^* := G_{k-1} \nabla A_{2k-1}$ and $C^* := P_{k-1} \cup D_{2k-2}E_{2k-2}A_{2k-1}A_0$.

The proposition concerning the length of $C^*$ is clear from the above.

**Corollary 2.** If $C$ is a Hamiltonian cycle in a cubic plane graph, then there is a set $S$ of vertices of $G$ such that $G\nabla S$ has a Hamiltonian cycle $C^*$ which is also a Petrie cycle.

**Theorem 3.** A cubic plane graph $G$ is Hamiltonian if and only if there exists a set $S$ of vertices of $G$ such that the graph $G\nabla S$ has a $PH$-cycle.

**Proof:** Since $G$ is cubic it has even number of vertices and the necessity follows from Lemma 1 and Corollary 2.

Sufficiency. Let $H^*$ be a $PH$-cycle in $G\nabla S$. It is easy to see that each triangle of $G\nabla S$ has two of its edges on $H^*$. Let $\tau_1, \tau_2, \ldots, \tau_s$, $s \geq 1$, be triangles obtained by cutting off the vertices from $S$ in $G$. If we delete from $G\nabla S$ the edge of $\tau_j$, for any $1 \leq j \leq s$, not lying on $H^*$ and then forget the vertices of degree two, we get a Hamiltonian cycle $H$ in $G$.

The problem of deciding the existence of Hamiltonian cycles in cubic, planar, 3-connected graphs, is an $NP$-complete problem, see Garey et al [3]. Therefore one could think, to find a Hamiltonian cycle by using Theorem 3, it is necessary to consider as set $S$ all of $2^n$ subsets of the vertex set of an $n$-vertex cubic plane graph. But the following theorem provides some restrictions on $S$.

**Theorem 4.** If a cubic polyhedral $n$-vertex graph $G$ has a $PH$-cycle then

(i) all faces of $G$ are multi-3-gonal,

(ii) $4 \leq n \equiv 0 \pmod{4}$, $n \neq 8$.

**Proof:** Suppose $C$ is a $PH$-cycle in $G$. Then it is easy to see that each third edge of any face in $G$ is a chord of $C$. Further there is the same number, say
of chords in the interior and in the exterior of $C$. Every chord makes two non-adjacent vertices of $C$ trivalent. Hence $C$ must have $4t$ vertices.

Let $G$ be a cubic polyhedral graph on 8 vertices and with a $PH$-cycle $C$. Let the vertices of $C$ be successively $A_1, A_2, \ldots, A_8$. Without loss of generality we can assume that the edges $A_1A_3$ and $A_5A_7$ lie inside of $C$. Because of planarity of $G$, the edges $A_2A_6$ and $A_4A_8$ cannot exist in $G$. The existence of an edge $A_2A_4$ or $A_2A_8$ leads to the contradiction with the 3-connectivity of $G$. □

Note that for any $n$, $4 \leq n \equiv 0 \pmod{4}$, $n \neq 8$, there exists an $n$-vertex cubic polyhedral graph with $PH$-cycle. The proof of this statement is left to the reader.

As the cutting off a vertex $A$ of a graph $G$ leads to the increasing of the number in $G \nabla A$ by two, Theorem 4 yields

**Corollary 5.** Let $G$ be an $n$-vertex plane cubic graph having $PH$-cycle, then

$$|S| \equiv \frac{n}{2} \pmod{2}.$$  

□

Here and in the sequel, $S$ is as in Theorem 3.

Many other restrictions on $S$ are given by (i) of Theorem 4. To obtain a multi-3-gonal face from an $m$-gonal face $\alpha$, $m \equiv j \pmod{3}$, $j = 1, 2, 3$, $3t - j$ vertices must be cut off for some $t = 1, 2, \ldots, \lfloor \frac{m}{3} \rfloor$. By this we have

**Corollary 6.** Let $p_k(G)$ denote the number of $k$-gonal faces of an $n$-vertex cubic plane graph $G$ having $PH$-cycle and $K = 2 \sum_{k \geq 1} p_{3k+1}(G) + \sum_{k \geq 1} p_{3k+2}(G)$. Then

$$\frac{K}{3} \leq |S| \leq n - \frac{K}{3}.$$  

□

**References**

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