Cardinal invariants and compactifications

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Abstract. We prove that every compact space $X$ is a Čech-Stone compactification of a normal subspace of cardinality at most $d(X)^{t(X)}$, and some facts about cardinal invariants of compact spaces.

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The notions of a compactness and of a compactification are very close. Really, any compact space is a compactification of any of its dense subsets. But a useful information can be obtained from this fact only in the case when a compact space is a compactification of some certain type.

The classic type is the Čech-Stone compactification. When a compact space is a Čech-Stone compactification of some of its subsets? Theorem 1.1 says that any compact space $X$ is a Čech-Stone compactification of some normal subspace $B \subseteq X$ such that $|B| \leq d(X)^{t(X)}$, where $d(X)$ is a density of $X$, $t(X)$ is its tightness. The case, when $|X| \leq d(X)^{t(X)}$ is trivial, in this case $B = X$. In the opposite case the theorem says that "extra" points from $X \setminus B$ are constructed by a standard way, as points of Čech-Stone compactification of a "not large" normal subspace. For example, by this theorem, we can say that Fedorchuk’s compact space, that is the hereditarily separable, hereditarily normal compact space of cardinality $2^c$, is a Čech-Stone compactification of a subspace of a cardinality $c$. We prove some new facts about cardinal invariants.

The definitions and notations used here are standard, one can find them in [2], for example.

We use the notation of a sequential extension of a set $A$, that is a set $B$, $A \subseteq B \subseteq [A], |B| \leq |A|^\omega$ such that if a countable set $B' \subseteq B$ has a limit point in $X$, then there is a limit point of $B'$ in $B$. One can construct a sequential extension by induction.

**Theorem 1.1.** Let $X$ be a compact space, $A$ be a dense subset of $X$. Then $X = \beta B$, where $A \subseteq B$, $|B| \leq |A|^{t(X)}$, $B$ is a normal and countably compact subspace.

**Proof:** The case $|X| = |A|^{t(X)}$ is trivial. Let $|X| > |A|^{t(X)}$. We construct by induction the family $\{B_\alpha : \alpha < \omega_{\tau^+}\}$, where $t(X) = \tau$ such that:
(1) $B_0 = A$;
(2) $B_\beta \subseteq B_\alpha$ for $\beta \leq \alpha$;
(3) a sequential extension of $B_\alpha$ is in $B_{\alpha+1}$;
(4) $|B_\alpha| \leq |A|^\tau$ for $\alpha < \omega_\tau$;
(5) $\bigcup \{B_\alpha : \alpha < \omega_\tau +\} = B$, where $B$ is a normal, countably compact subspace and $X = \beta B$.

Let $\xi(A)$ be a choice function, defined on the set of all nonempty subsets of $X$. Let $B_0 = A$. Let $\{B_{\beta} : \beta < \alpha\}$ with conditions (1)–(4) be constructed. Let $B'_\alpha$ be a sequential extension of the set $\bigcup \{B_{\beta} : \beta < \alpha\}$ and

$$B_\alpha = B'_\alpha \cup \{\xi([T]_X \cap [T']_X) : \tau', \tau' \in \exp \tau, B'_\alpha\}.$$ 

Then $|B_\alpha| \leq |A|^\tau$ and (1)–(4) hold for $\{B_{\beta} : \beta \leq \alpha\}$. Note that $B$ is a countably compact space. Let us prove that $X = \beta B$. Let $F_1, F_2$ be disjoint closed subsets of $B$. Let $([F_1]_X \cap [F_2]_X) \setminus B \neq \emptyset$ and $x \in ([F_1]_X \cap [F_2]_X) \setminus B$. Since $t(X) = \tau$, there are $F'_1 \subseteq F_1$ and $F'_2 \subseteq F_2$ such that $F'_1, F'_2 \subseteq B_\alpha$ and $|F'_1| \leq \tau$, $|F'_2| \leq \tau$, and $x \in [F'_1]_X \cap [F'_2]_X$. There is $\alpha < \omega_\tau$ such that $F'_1, F'_2 \subseteq B_\alpha$. Since $F'_1, F'_2$ are disjoint, closed subsets of $B$, then $[F'_1]_X \cap [F'_2]_X \subseteq ([F_1]_X \cap [F_2]_X) \setminus B$. Then $\xi([F'_1]_X \cap [F'_2]_X) \in X \setminus B$, a contradiction. So $X = \beta B$. It follows from the above proof that $B$ is a normal space. The theorem is proved. \qed

Recall that a space $X$ is weakly normal if in every closed, countable, discrete set $A \subseteq X$ there is a countable $A' \subseteq A$ $C^*$-embedded in $X$ [3].

We say that $X$ is an $h$-weakly normal space if $X$ is hereditarily weakly normal.

**Definition 1.2.** A space $X$ is called $d$-normal if in every closed, countable, discrete set $A \subseteq X$ there is a countable $A' \subseteq A$ with discrete family of neighborhoods. It means that for every point $x \in A'$ there is a neighborhood $Ox$ such that $\{Ox : x \in A'\}$ is a discrete family.

We say that a space $X$ is $hd$-normal if $X$ is a hereditarily $d$-normal.

It is clear that a $hd$-normal ($d$-normal) space is $h$-weakly (weakly) normal, and a hereditarily normal space is $hd$-normal as well as it is compact first countable space or a normal first countable space.

On the other hand, the space $N^* = \beta N \setminus N$, the remainder of the Čech-Stone compactification of a countable discrete space, is an $h$-weakly normal, but not an $hd$-normal space. Really, the $h$-weakly normality of $N^*$ follows from the fact that $[D]_{N^*}$ is homeomorphic to $\beta N = \beta D$ for every countable discrete set $D \subseteq N^*$. But for a discrete set $D$ the space $X = N^* \setminus ([D]_{N^*} \setminus D)$ is not $d$-normal.

**Theorem 1.3.** Let $X$ be an $hd$-normal compact space, or an $h$-weakly normal space with countable tightness, and $A \subseteq X$ be a dense subset of $X$. Then $X = \beta B$ where $A \subseteq B$, $|B| \leq |A|^t(X)$, $B$ is a normal countably compact space such that every compact space $K \subseteq X \setminus B$ is finite.

**Proof:** Again, we suppose that $|X| > |A|^t(X)$. Let the set $B$ be as in Theorem 1.1. We prove that every compact space $K \subseteq X \setminus B$ is finite. Let $K \subseteq X \setminus B$
be an infinite compact set. There is a countable discrete (as a subspace) subset \( D \subseteq K \). We consider a set \( B \cup D \). By \( h \)-weakly normality (moreover \( hd \)-normality) of \( X \), there is a countable set \( D' \subseteq D \) \( C^* \)-embedded in \( B \cup D \). Then \( [D']_X = \beta D' \), so \( [D']_X \) is a Čech-Stone compactification of the countable discrete set. But \( \beta D' = \beta N \) is not an \( hd \)-normal and the tightness of \( \beta D' \) is not countable. This contradiction proves the theorem. \( \square \)

Theorems 1.1 and 1.3 were announced by the author in [4].

**Lemma 1.4.** Let \( X \) be an \( h \)-weakly normal compact space with countable tightness, \( X = \beta B \) for some \( B \subseteq X \). The \( \Phi \setminus F \) is a discrete (as a subspace) set for closed \( F, \Phi \subseteq X \) such that \( F \subseteq \Phi \) and \( F \cap B = \Phi \cap B \).

**Proof:** Let \( x \in \Phi \setminus F \). There is a neighborhood \( Ox \) of \( x \) such that \( [Ox]_X \cap F = \emptyset \). Therefore, \( [Ox]_X \cap \Phi = [Ox]_X \cap (\Phi \setminus F) \subseteq X \setminus B \). By Theorem 1.3, the set \( [Ox]_X \cap \Phi \) is finite, and therefore \( \Phi \setminus F \) is a discrete set. The lemma is proved. \( \square \)

Recall that a point \( x \in X \) is called a \( b \)-point if \( x = F \cap \Phi \) where \( F, \Phi \) are closed in \( X \), and \( x \) is a limit point for \( F \) and \( \Phi \) [5].

\( \alpha \)-points (limits of sequences of points of \( X \)) are \( b \)-points as well as points of non-normality (points \( x \) of a normal space \( X \) such that \( X \setminus \{x\} \) is non-normal).

**Theorem 1.5.** Let \( X \) be an \( h \)-weakly normal compact space with countable tightness. Then \( |\{x : x \text{ is a } b\text{-point in } X\}| \leq d(X)^\omega \).

**Proof:** By Theorem 1.3, \( X = \beta B \) for a normal \( B \) such that \( |B| \leq d(X)^\omega \) and every compact subset of \( X \setminus B \) is finite. We prove that none of the points of \( X \setminus B \) is a \( b \)-point. Indeed, let \( x \in X \) be a \( b \)-point, that is, \( x = F \cap \Phi \) where \( F, \Phi \) are closed in \( X \), and \( x \) is a limit point for \( F \) and \( \Phi \). Let \( F' = F \cap B \), \( \Phi' = \Phi \cap B \). Then \( x \in [F']_X \cap (\Phi')_X \). Really, by Lemma 1.4, the sets \( F \setminus [F']_X \) and \( \Phi \setminus [\Phi']_X \) are discrete and therefore \( x \in [F']_X, x \in (\Phi')_X \). But \( X \) is a Čech-Stone compactification of the normal space \( B \). This contradiction proves the theorem. \( \square \)

**Corollary 1.6.** Let \( X \) be a weakly normal compact space with countable tightness and let all points of \( X \) be \( b \)-points. Then \( |X| \leq d(X)^\omega \).

The above results have some connections with the results from [6]. Recall that \( x \in X \) is an \( hb \)-point if \( x \) is a \( b \)-point in every closed subspace \( X' \subseteq X \), if \( x \) is not isolated in \( X' \) [5].

By Arhangel’skiǐ’s theorem [7] and Theorem 1.5 we have

**Theorem 1.7.** Let \( X \) be an \( h \)-weakly normal compact space with countable tightness, \( \chi(X) \leq 2^\omega \) and every non-isolated point in \( X \) is an \( hb \)-point. Then \( |X| \leq 2^\omega \).
Proposition 1.8. Let $X$ be a countably compact $hd$-normal space, $A \subseteq X$ be a dense subset. Then there is $B \subseteq X$ such that $A \subseteq B$, $|B| \leq |A|^{\omega}$, $B$ is countably compact such that every subset $F \subseteq X \setminus B$ closed in $X$ is finite.

Proof: It is clear that $B$ is a sequential extension of $A$. The only thing we have to explain is the last part. Let $F \subseteq X \setminus B$ be an infinite closed subset of $X$. There is a countable discrete (as a subspace) set $D \subseteq F$. By an $hd$-normality of $X$ there is a countable set $D' = \{x_i : i \in \omega\}$, $D' \subseteq D$ with a discrete in $B \cup D$ family of neighborhoods $\{Ox_i : i \in \omega\}$. But $Ox_i \cap B \neq \emptyset$ for every $i \in \omega$, so we have the discrete infinite family in the compact space $B$. This contradiction proves the proposition. \qed

By the same way as Lemma 1.4 we can prove

Lemma 1.9. Let $X$ be an $hd$-normal space, $B \subseteq X$ be a dense, countably compact subspace, $F, \Phi \subseteq X$ such that $F \subseteq \Phi$, $F \cap B = \Phi \cap B$. Then $\Phi \setminus F$ is a discrete (as a subspace) set, moreover, $\Phi \setminus F$ is a free sequence in $X \setminus F$.

Lemma 1.10. Let $X$ be an $hd$-normal space, $B \subseteq X$ be a countably compact subspace. Then for every closed set $F \subseteq X$ there is a family $\pi = \{OF\}$ of neighborhoods of $F$ such that $|\pi| \leq |B|$, $(\bigcap\{OF : OF \in \pi\} \cap [B]) \setminus F \cap [B]$ is discrete; moreover, if $F \cap B = \emptyset$, then $\bigcap\{OF : OF \in \pi\} \cap [B] = \emptyset$.

Proof: We consider $F \cap [B]$. It is clear that there is a family $\pi = \{OF\}$ of neighborhoods of $F$ such that $|\pi| = |B|$, $\bigcap\{OF : OF \in \pi\} \cap B = F \cap B$. Then $(\bigcap\{OF : OF \in \pi\} \cap [B]) \setminus F \cap [B]$ is discrete by Lemma 1.9. If $F \cap B = \emptyset$, then $\bigcap\{OF : OF \in \pi\} \cap [B] \subseteq [B] \setminus B$. In the same way as in the proof of Proposition 1.7 we can prove that $\bigcap\{OF : OF \in \pi\} \cap [B]$ is finite. We can add a finite number of neighborhoods of $F$ to the family $\pi$ and get what we need. The lemma is proved. \qed

Proposition 1.11. Let $X$ be an $hd$-normal, countably compact space such that every point $x \in X$ is a limit point of a closed set of cardinality at most $d(X)^{t(X)}$. Then $|X| \leq d(X)^{t(X)}$.

Proof: By Proposition 1.8 there is a dense, countably compact space $B \subseteq X$ such that every closed set $F \subseteq X \setminus B$ is finite and $|B| \leq d(X)^{t(X)}$. Let $x \in X \setminus B$. There is a closed $F_x \subseteq X$ such that $|F_x| \leq d(X)^{t(X)}$ and $x$ is a limit point of $F_x$. Then by Lemma 1.9, $x \in [F_x \cap B]$. There is $F'_x \subseteq F_x \cap B$ such that $|F'_x| \leq t(X)$ and $x \in [F'_x]$. Then $|X \setminus B| \leq |B|^{t(X)} \leq d(X)^{t(X)}$. The proposition is proved. \qed

Proposition 1.12. Let $X$ be an $hd$-normal compact space. Then $hl(X) \leq s(X)^{\omega}$.

Proof: We prove that $\chi(F, X) \leq s(X)^{\omega}$ for every closed $F \subseteq X$. Really, for a closed $F \subseteq X$ there is a family $\pi = \{OF\}$ of neighborhoods of $F$ such that $|\pi| \leq s(X)$ and $d(\bigcap\{OF : OF \in \pi\} \setminus F) \leq s(X)$ (this is well known, see for
example [5]). By Proposition 1.8 there is a set $B \subseteq \bigcap\{\{OF : OF \in \pi\} \mid F\}$ such that $|B| \leq s(X)^\omega$, $B$ is countably compact, $[B] \supseteq (\bigcap\{\{OF : OF \in \pi\}\}) \setminus F$ and every subset of $B$ closed in $X$ is finite. By Lemma 1.9 there is a family $\pi' = \{UF\}$, $|\pi'| \leq s(X)^\omega$ of neighborhoods of $F$ such that $(\bigcap\{\{UF : UF \in \pi\}\} \setminus \{B\}) \setminus F \setminus [B]$ is discrete and therefore has cardinality at most $s(X)$. Finally, $\chi(F, X) \leq s(X)^\omega \cdot s(X) = s(X)^\omega$. The proposition is proved.

Recall that a free sequence of cardinality $\tau$ is a set $\xi\{x_\alpha : \alpha < \tau\}$ such that for all $\beta < \tau$ $\{\{x_\alpha : \alpha < \beta\} \cap \{x_\alpha : \alpha \geq \beta\}\} = \emptyset$ (see [6]).

Define $A(X) = \sup\{\tau : \tau$ is cardinality of a free sequence in $X\}$, $\varrho A(x, X) = A(X \setminus \{x\})$, $\varrho A(x, X) = \sup\{\varrho A(x, X) : x \in X\}$. A. Arhangelskii proved that $t(X) = A(X)$ for compact spaces [7]; moreover,

$$t(X) = A(X) \leq \varrho A(X) \leq s(X).$$

Note that for Alexandroff’s double circle $s(X) = 2^\omega$, $\varrho A(X) = A(X) = \omega$. The same construction with Fedorchuk’s compact space gives the space with $s(X) = 2^c$ and $\varrho A(X) = \omega$.

**Theorem 1.13.** Let $X$ be an $\text{hd}$-normal compact space. Then $\chi(x, X) \leq \varrho A(x, X)^\omega$.

**Proof:** Let there be a point $x \in X$ such that $\varrho A(x, X)^\omega < \chi(x, X)$. Define $\varrho A(x, X) = \tau$. By induction we construct a set $D = \{y_\alpha : \alpha < \omega_\tau^+\}$, a family $\{B_\alpha : \alpha < \omega_\tau^+\}$, $|B_\alpha| \leq \tau^\omega$ of neighborhoods of $x$ such that $\big(\{\{y_\alpha : \alpha < \delta\} \cap \bigcap\{\{Ox : Ox \in B_\delta\}\} \setminus \{x\}\big) = \emptyset$, $\delta < \omega_\tau^+$. Let $y_0 \in X$, $B_0 = \{Ox\}$, where $\{Ox\} \notin y_0$. Let $\{y_\alpha : \alpha < \delta\}$ and $\{B_\alpha : \alpha < \delta\}$ be constructed. If $x \notin \{\{y_\alpha : \alpha, \delta\}\}$, let $B_\delta = \{\{B_\alpha : \alpha < \delta\} \cup \{Ox\}$, where $Ox \cap \{y_\alpha : \alpha < \delta\} = \emptyset$. We choose $\delta_5$ in the set $\bigcap\{\{Ox : Ox \in B_\delta\}\} \setminus \{x\}$. If $x \in \{\{y_\alpha : \alpha < \delta\}\}$, we use Proposition 1.6 and Lemma 1.9. Since $\{\{y_\alpha : \alpha < \delta\}\} \subseteq \tau$, let us consider a family $\pi = \{Ox\}$ of neighborhoods of $x$, $|\pi| \leq \tau^\omega$ such that $T = (\bigcap\{\{Ox : Ox \in \pi\}\} \setminus \{\{y_\alpha : \alpha < \delta\}\} \setminus \{x\})$ is empty or is a free sequence in $X \setminus \{x\}$, and $|T| \leq \varrho A(x, X)$. Hence, there is a family $\pi'$ of neighborhoods of $x$ of cardinality at most $\tau^\omega$ such that $\big(\bigcap\{\{Ox : Ox \in \pi\} \setminus \{\{y_\alpha : \alpha < \delta\}\}\} \setminus \{x\}\big) = \emptyset$. Let $B_\delta \cap \big\{\{B_\alpha : \alpha < \delta\} \cup \pi', \big\}$ and choose $\delta_5$ from $\bigcap\{\{Ox : Ox \in B_\delta\}\} \setminus \{x\}$. If we continue until $\omega_\tau^+$, we get a free sequence of cardinality $\tau^+$. But this contradicts $\varrho A(x, X) = \tau$. The theorem is proved.

**References**


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