On complemented copies of $c_0$ in spaces of operators, II

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Abstract. We show that as soon as $c_0$ embeds complementably into the space of all weakly compact operators from $X$ to $Y$, then it must live either in $X^*$ or in $Y$.

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Let $X$ and $Y$ be two infinite dimensional Banach spaces. It is well known (see for instance [E1], [E3], [E4], [EJ], [F], [H], [K]) that $c_0$ can embed into $K(X,Y)$, the space of all compact operators from $X$ to $Y$ equipped with the operator norm, even if it does not embed into $X^*$ and $Y$; furthermore, such a copy of $c_0$ can be complemented in $K(X,Y)$ (see [E4], [E6]).

Recently ([E2], [E5]), we obtained some results proving that if $c_0$ embeds into either $X^*$ or $Y$ then it embeds complementably into some spaces of operators larger than $K(X,Y)$, for instance $W(X,Y)$, the space of all weakly compact operators from $X$ to $Y$. The technique we used in order to construct the complemented copy of $c_0$ requires the presence of a copy of $c_0$ in either $X^*$ or $Y$, because otherwise it does not work.

All the above facts lead us to the following natural question: Is it possible to have a complemented copy of $c_0$ inside $W(X,Y)$ even when it does not embed into $X^*$ and $Y$?

In this short note (in which we continue the research started in [E5]) we want to show that the answer to this question is negative; indeed, we prove that as soon as $c_0$ embeds complementably into $W(X,Y)$, then it must live inside either $X^*$ or $Y$. Actually, we shall prove a slightly more general result about the space $L_{w^*}(X^*, Y)$, i.e. the space of all weak*-weak continuous operators from $X^*$ to $Y$ equipped with the operator norm.

The announced result is the following

Theorem 1. Let $H$ be a complemented copy of $c_0$ in $L_{w^*}(X^*, Y)$. If $T_n$ is a basis for $H$, then there is either a $x^*_0 \in B_{X^*}$ or a $y^*_0 \in B_{Y^*}$ and a subsequence $(T_{n_k})$ of $(T_n)$ such that either the sequence $(T_{n_k}(x^*_0))$ spans a copy of $c_0$ in $Y$ or the sequence $(T_{n_k}^*(y^*_0))$ spans a copy of $c_0$ in $X$.

Proof: It is clear that for each $x^* \in B_{X^*}$ (resp. $y^* \in B_{Y^*}$) the series $\sum T_{n_k}(x^*)$ (resp. $\sum T_{n_k}^*(y^*)$) is weakly unconditionally converging in $Y$ (resp. in $X$). It will

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be enough to show that there is either a $x^*_0 \in B_{X^*}$ or a $y^*_0 \in B_{Y^*}$ and a subsequence $(T_{n_h})$ of $(T_n)$ such that either the series $\sum T_{n_h}(x^*_0)$ is not unconditionally converging in $Y$ or the series $\sum T_{n_h}^*(y^*_0)$ is not unconditionally converging in $X$, because we can thus use a well known result due to Bessaga and Pelczynski ([BP]) to conclude our proof. By contradiction we assume that for each $x^* \in B_{X^*}$ and $y^* \in B_{Y^*}$ the series $\sum T_n(x^*)$ and $\sum T_n^*(y^*)$ are unconditionally converging in $Y$ and $X$, respectively. So for any $\xi = (\xi_n) \in l_\infty$ and $x^* \in B_{X^*}$ the series $\sum \xi_n T_n(x^*)$ is unconditionally converging in $Y$. Define $T_\xi(x^*) = \sum \xi_n T_n(x^*)$ for all $x^* \in B_{X^*}$. We now show that $T_\xi$ belongs to $L_{w^*}^*(X^*, Y)$. To this aim it will be enough to consider a $w^*$-null net $(x^*_\alpha)$ in $B_{X^*}$ and a $y^*$ in $B_{Y^*}$ and to prove that

$$\lim_{\alpha} |T_\xi(x^*_\alpha)(y^*)| = 0.$$  

Since $\sum \xi_n T^*_n(y^*)$ is unconditionally converging in $X$ by our assumption, we have

$$\lim_{p} \sup_{x^* \in B_{X^*}} \big| \sum_{n=p+1}^\infty \xi_n T_n^*(y^*)(x^*) \big| = 0.$$  

Thanks to (2), given $\gamma > 0$ we can find a $\overline{p} \in N$ such that

$$\sup_{\alpha} \big| \sum_{n=\overline{p}+1}^\infty \xi_n T_n^*(y^*)(x^*_\alpha) \big| < \frac{\gamma}{2}.$$  

On the other hand,

$$\lim_{\alpha} \sum_{n=1}^{\overline{p}} \xi_n T_n(x^*_\alpha)(y^*) = 0$$  

since $T_n \in L_{w^*}(X^*, Y)$, for all $n \in N$. (3) and (4) together give (1).

Furthermore, using the Closed Graph Theorem we can prove easily that the linear map $\Psi : l_\infty \to L_{w^*}(X^*, Y)$ defined by $\Psi(\xi) = T_\xi$ is bounded. It is clear that $K = \Psi(l_\infty)$ contains $H$. If $P : L_{w^*}(X^*, Y) \to H$ is the existing projection, the operator $P|_K : l_\infty \to H$ is a quotient map of $l_\infty$ onto $c_0$. This is a well known contradiction ([D]) that concludes our proof. \qed

**Corollary 2.** Let $c_0$ embed complementably into $W(X, Y)$. Then $c_0$ embeds into either $X^*$ or $Y$.

**Proof:** It is enough to observe that $W(X, Y)$ is isomorphic with $L_{w^*}(X^{**}, Y)$.

With a proof similar to that of Theorem 1 we can prove the same result for the space $L(X, Y)$ of all bounded operators from $X$ to $Y$. One could also consider the space $UC(X, Y)$ of all unconditionally converging operators from $X$ to $Y$; in such
a case we have been able to get just a slightly less precise result than Theorem 1; indeed, the same technique used for proving Theorem 1 shows that as soon as \( c_0 \) embeds complementably in \( UC(X,Y) \), then either \( Y \) contains a copy of \( c_0 \) or there are a \( y_0^* \in B_{Y^*} \) and a subsequence \( (T_{n_k}) \) of \( (T_n) \) so that the sequence \( (T_{n_k}^*(y_0^*)) \) spans a copy of \( c_0 \) in \( X^* \), but in such a case we do not know how the copy of \( c_0 \) contained in \( Y \) is spanned.

At the end, we observe that in the paper [E5] we also considered other spaces of operators, such as spaces of Dunford-Pettis operators; we do not know if Theorem 1 can be extended to cover this case.

REFERENCES


