Commutative neutrix convolution products of functions

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Abstract. The commutative neutrix convolution product of the functions $x^r e^{\lambda x}$ and $x^s e^{\mu x}$ is evaluated for $r, s = 0, 1, 2, \ldots$ and all $\lambda, \mu$. Further commutative neutrix convolution products are then deduced.

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In the following we let $\mathcal{D}$ be the space of infinitely differentiable functions with compact support and let $\mathcal{D}'$ be the space of distributions defined on $\mathcal{D}$. The convolution product $f * g$ of two distributions $f$ and $g$ in $\mathcal{D}'$ is then usually defined by the equation

$$\langle (f * g)(x), \phi \rangle = \langle f(y), \langle g(x), \phi(x + y) \rangle \rangle$$

for arbitrary $\phi$ in $\mathcal{D}$, provided $f$ and $g$ satisfy either of the conditions

(a) either $f$ or $g$ has bounded support,
(b) the supports of $f$ and $g$ are bounded on the same side,

see Gel’fand and Shilov [7].

Note that if $f$ and $g$ are locally summable functions satisfying either of the above conditions then

\begin{equation}
(f * g)(x) = \int_{-\infty}^{\infty} f(t) g(x - t) \, dt = \int_{-\infty}^{\infty} f(x - t) g(t) \, dt.
\end{equation}

It follows that if the convolution product $f * g$ exists by this definition then

\begin{align}
(1) & \quad f * g = g * f, \\
(2) & \quad (f * g)' = f' * g,
\end{align}

This definition of the convolution product is rather restrictive and so the non-commutative neutrix convolution product was introduced in [2]. A commutative neutrix convolution product was given more recently in [4]. In order to define the neutrix convolution product we first of all let $\tau$ be a function in $\mathcal{D}$ satisfying the following properties:

(i) $\tau(x) = \tau(-x),$
(ii) $0 \leq \tau(x) \leq 1,$
(iii) $\tau(x) = 1$ for $|x| \leq \frac{1}{2},$
(iv) $\tau(x) = 0$ for $|x| \geq 1.$
The function $\tau_n$ is now defined by

$$\tau_n(x) = \begin{cases} 1, & |x| \leq n, \\ \tau(n^n x - n^{n+1}), & x > n, \\ \tau(n^n x + n^{n+1}), & x < -n, \end{cases}$$

for $n = 1, 2, \ldots$.

**Definition 1.** Let $f$ and $g$ be distributions in $D'$ and let $f_n = f \tau_n$ and $g_n = g \tau_n$ for $n = 1, 2, \ldots$. Then the commutative neutrix convolution product $f \boxast g$ is defined as the neutrix limit of the sequence $\{f_n * g_n\}$, provided that the limit $h$ exists in the sense that

$$\text{N} - \lim_{n \to \infty} \langle f_n * g_n, \phi \rangle = \langle h, \phi \rangle,$$

for all $\phi$ in $D$, where $N$ is the neutrix, see van der Corput [1], having domain $N' = \{1, 2, \ldots, n, \ldots\}$ and range $N''$ the real numbers, with negligible functions finite linear sums of the functions

$$n^\lambda \ln^{r-1} n, \ln^r n \quad (\lambda > 0, \ r = 1, 2, \ldots)$$

and all functions which converge to zero in the usual sense as $n$ tends to infinity.

Note that in this definition the convolution product $f_n * g_n$ is defined in Gel'fand and Shilov's sense, the distributions $f_n$ and $g_n$ both having bounded support. Note also that the non-commutative neutrix convolution, denoted by $f \circledast g$, was defined as the limit of the sequence $\{f_n * g_n\}$.

The following theorem was proved in [4], showing that the neutrix convolution product is a generalization of the convolution product.

**Theorem 1.** Let $f$ and $g$ be distributions in $D'$ satisfying either condition (a) or condition (b) of Gel'fand and Shilov’s definition. Then the neutrix convolution product $f \boxast g$ exists and

$$f \boxast g = f * g.$$

A number of neutrix convolution products have been evaluated. For example, $x^\lambda \boxast x^\mu_+$ see [4], $x^\lambda \boxast x^r_-$ see [5] and $\ln x^- \boxast x^r_+$ see [6].

In order to define further neutrix convolution products, we increase our set of negligible functions given in Definition 1 to also include finite linear sums of the functions

$$n^\lambda e^\mu n \quad (\mu > 0).$$

We now define the locally summable functions $e^{\lambda x}_+$ and $e^{\lambda x}_-$ by

$$e^{\lambda x}_+ = \begin{cases} e^{\lambda x}, & x > 0, \\ 0, & x < 0, \end{cases} \quad e^{\lambda x}_- = \begin{cases} 0, & x > 0, \\ e^{\lambda x}, & x < 0. \end{cases}$$

It follows that

$$e^{\lambda x}_- + e^{\lambda x}_+ = e^{\lambda x}, \quad x^r e^{\lambda x}_+ = x^r e^{\lambda x}_+, \quad x^r e^{\lambda x}_- = (-1)^r x^r e^{\lambda x}_-, \quad r = 0, 1, 2, \ldots.$$

We now prove
Theorem 2. The neutrix convolution product \( (x^r e^{-\lambda x}) \square (x^s e^{\mu x}) \) exists and

\[
(\lambda x) \square (\mu x) = \frac{e^{\mu x} + e^{\lambda x}}{\lambda - \mu},
\]

\[
(x^r e^{-\lambda x}) \square (x^s e^{\mu x}) = D_\lambda^r D_\mu^s \frac{e^{\mu x} + e^{\lambda x}}{\lambda - \mu} + \sum_{i=0}^{s} \left( \frac{s}{i} \right) (r + s - i)! x^i e^{\mu x} \frac{1}{(\lambda - \mu)^{r+s-i+1}} + \sum_{i=0}^{r} \left( \frac{r}{i} \right) (-1)^{r-i} (r + s - i)! x^i e^{-\lambda x} \frac{1}{(\lambda - \mu)^{r+s-i+1}},
\]

where \( D_\lambda = \partial/\partial \lambda \) and \( D_\mu = \partial/\partial \mu \), for \( \lambda \neq \mu \) and \( r, s = 0, 1, 2, \ldots \); these neutrix convolution products existing as convolution products if \( \lambda > \mu \) and

\[
(x^r e^{-\lambda x}) \square (x^s e^{\mu x}) = B(r + 1, s + 1) \text{sgn} x^r x^{r+s+1} e^{-\lambda x},
\]

where \( B \) denotes the Beta function, for all \( \lambda \) and \( r, s = 0, 1, 2, \ldots \).

Proof: We put \( (e^{-\lambda x})_n = e^{-\lambda x} \tau_n(x) \) for \( n = 1, 2, \ldots \) and suppose first of all that \( \lambda \neq \mu \). Since \( (e^{-\lambda x})_n \) and \( (e^{\mu x})_n \) are summable functions with compact support, the convolution product \((e^{-\lambda x})_n \ast (e^{\mu x})_n\) is defined by equation (1) and so

\[
(e^{-\lambda x})_n \ast (e^{\mu x})_n = \int_{-\infty}^{0} (e^{-\lambda x})_n(e^{\mu x}(x-t))_n dt = \int_{n-n-n}^{0} e^{\lambda x} \tau_n(t)e^{\mu(x-t)} \tau_n(x-t) dt.
\]

Thus if \(-n \leq x \leq 0\),

\[
(e^{-\lambda x})_n \ast (e^{\mu x})_n = \int_{n-n-n}^{x} e^{\lambda x} \tau_n(t)e^{\mu(x-t)} \tau_n(x-t) dt + \int_{n-n-n}^{-n} e^{\lambda(x-t)} \tau_n(t)e^{\mu(x-t)} \tau_n(x-t) dt
\]

\[
= e^{\lambda x} - e^{\mu x}(\lambda - \mu)n + O(n^{-n}(\lambda - \mu)n).
\]

When \( n \geq x \geq 0 \),

\[
(e^{-\lambda x})_n \ast (e^{\mu x})_n = \int_{x-n}^{0} e^{\lambda x} \tau_n(t)e^{\mu(x-t)} \tau_n(x-t) dt + \int_{x-n-n-n}^{x-n} e^{\lambda(x-t)} \tau_n(t)e^{\mu(x-t)} \tau_n(x-t) dt
\]

\[
= e^{\mu x} - e^{\lambda x}(\lambda - \mu)n + O(n^{-n}(\lambda - \mu)n).
\]

It now follows from equations (7) and (8) that for arbitrary \( \phi \) in \( \mathcal{D} \)

\[
\langle (e^{-\lambda x})_n \ast (e^{\mu x})_n, \phi(x) \rangle = (\lambda - \mu)^{-1} \langle e^{\mu x} + e^{-\lambda x}, \phi(x) \rangle + \langle e^{-\lambda x} + e^{\mu x}, \phi(x) \rangle + O(n^{-n}(\lambda - \mu)n)
\]
and so
\[
N \lim_{n \to \infty} ((e^{-\lambda x^r}_-)^n (e^{\mu x^r}_+)^n, \phi(x)) = (\lambda - \mu)^{-1} (e^{\mu x}_+^r + e^{-\lambda x}_-, \phi(x)),
\]
the usual limit existing if \( \lambda > \mu \). Equation (4) follows.

We now put \((x^r e^{-\lambda x}_-)^n = x^r e^{-\lambda x} \tau_n(x)\) and \((x^s e^{\mu x}_+)^n = x^s e^{\mu x} \tau_n(x)\). Then as above, we have
\[
(x^r e^{-\lambda x}_-)^n (x^s e^{\mu x}_+)^n = \int_{-n-n-n}^{x} t^r e^{\lambda t} \tau_n(t)(x-t)^s e^{\mu(x-t)} \tau_n(x-t) dt.
\]
Thus if \(-n \leq x \leq 0\),
\[
(x^r e^{-\lambda x}_-)^n (x^s e^{\mu x}_+)^n = \int_{-n}^{x} t^r e^{\lambda t} (x-t)^s e^{\mu(x-t)} dt + \int_{-n-n-n}^{-n} t^r e^{\lambda t} \tau_n(t)(x-t)^s e^{\mu(x-t)} \tau_n(x-t) dt
\]
\[
= D^r \lambda D^s \mu e^{\lambda x} e^{\mu x} + e^{\mu x} P(n) \cdot e^{-(\lambda - \mu)n} + O(n^{-n+r+s} e^{-(\lambda - \mu)n}),
\]
on using equation (7), where \( P \) denotes a polynomial.

When \( n \geq x \geq 0 \),
\[
(x^r e^{-\lambda x}_-)^n (x^s e^{\mu x}_+)^n = \int_{x-n}^{0} t^r e^{\lambda t} (x-t)^s e^{\mu(x-t)} dt + \int_{x-n-n-n}^{x} t^r e^{\lambda t} \tau_n(t)(x-t)^s e^{\mu(x-t)} \tau_n(x-t) dt
\]
\[
= D^r \lambda D^s \mu e^{\lambda x} e^{\mu x} \int_{x-n}^{0} e^{(\lambda - \mu)t} dt + O(n^{-n+r+s} e^{-(\lambda - \mu)n})
\]
\[
= D^r \lambda D^s \mu e^{\lambda x} e^{\mu x} \int_{x-n}^{0} e^{(\lambda - \mu)t} dt + O(n^{-n+r+s} e^{-(\lambda - \mu)n})
\]
on using equation (8).

It now follows as above from equations (9) and (10) that for arbitrary \( \phi \) in \( D \)
\[
N \lim_{n \to \infty} ((x^r e^{-\lambda x}_-)^n (x^s e^{\mu x}_+)^n, \phi(x)) = D^r \lambda D^s \mu (\lambda - \mu)^{-1} (e^{\mu x}_+^r + e^{-\lambda x}_-, \phi(x)),
\]
the usual limit existing if $\lambda > \mu$. Thus
\[
(x^r e_{-\lambda}) \ast (x^s e_{\mu+}) = D^r_{\lambda} D^s_{\mu} \left( \frac{e^{\mu x} + e^{\lambda x}}{\lambda - \mu} \right)
\]
and equation (5) follows.

Now suppose that $\lambda = \mu$. Then as above, we have
\[
(x^r e_{-\lambda}) n \ast (x^s e_{\lambda+}) n = \int_{-n-n}^{0} t^r \tau(x-t)^s \tau(t)(x-t)^s \tau_n(x-t) dt.
\]
Thus if $-n \leq x \leq 0$,
\[
(x^r e_{-\lambda}) n \ast (x^s e_{\lambda+}) n =
\]
\[
e^{\lambda x} \sum_{i=0}^{s} \binom{s}{i} (-1)^i \int_{-n}^{x} x^{s-i} t^{r+i} dt + O(n^{-n+r+s})
\]
\[
e^{\lambda x} \sum_{i=0}^{s} \binom{s}{i} (-1)^i x^{r+s+1} - \binom{s}{i} (-1)^i \frac{x^{r+s+1} - (-n)^{r+i+1} x^{s-i}}{r + i + 1}
\]
\[
+ O(n^{-n+r+s})
\]
\[
= B(r + 1, s + 1) x^{r+s+1} e^{\lambda x} + e^{\lambda x} \sum_{i=0}^{s} \binom{s}{i} \frac{(-1)^i x^{s-i} n^{r+i+1}}{r + i + 1} +
\]
\[
+ O(n^{-n+r+s}),
\]
where $B$ denotes the Beta function.

When $x \geq 0$,
\[
(x^r e_{-\lambda}) n \ast (x^s e_{\lambda+}) n =
\]
\[
e^{\lambda x} \sum_{i=0}^{s} \binom{s}{i} \frac{(-1)^{i+1} x^{s-i} (x-n)^{r+i+1}}{r + i + 1} + O(n^{-n+r+s})
\]
and it follows that
\[
N \lim_{n \to \infty} (x^r e_{-\lambda}) n \ast (x^s e_{\lambda+}) n = x^{r+s+1} e^{\lambda x} \sum_{i=0}^{s} \binom{s}{i} \frac{(-1)^{i+1}}{r + i + 1}
\]
\[
= -B(r + 1, s + 1) x^{r+s+1} e^{\lambda x},
\]
when $x \geq 0$.

It now follows as above from equations (11) and (12) that for arbitrary $\phi$ in $D$

\begin{equation}
N \lim_{n \to \infty} (x^r e_{\pm}^{\lambda x})_n * (x^s e_{\pm}^{\mu x}, \phi(x)) = B(r + 1, s + 1)(x^{r+s+1} e_{\pm}^{\lambda x} - x^{r+s+1} e_{\mp}^{\lambda x}, \phi(x))
\end{equation}

and equation (6) follows.

**Corollary.** The neutrix convolution products $(x^r e_{\pm}^{\lambda x}) \boxplus (x^s e_{\pm}^{\mu x})$ and $(x^r e_{\mp}^{\lambda x}) \boxplus (x^s e^{\mu x})$ exist and

\begin{align}
(x^r e_{\pm}^{\lambda x}) \boxplus (x^s e_{\pm}^{\mu x}) &= \pm D_\lambda^r D_\mu^s e_{\pm}^{\lambda x}, \\
(x^r e_{\mp}^{\lambda x}) \boxplus (x^s e_{\mp}^{\mu x}) &= 0,
\end{align}

for $\lambda \neq \mu$ and $r, s = 0, 1, 2, \ldots$ and

\begin{align}
(x^r e_{+}^{\lambda x}) \boxplus (x^s e_{+}^{\mu x}) &= \pm B(r + 1, s + 1)x^{r+s+1} e_{+}^{\lambda x}, \\
(x^r e_{-}^{\lambda x}) \boxplus (x^s e_{-}^{\mu x}) &= -B(r + 1, s + 1) \text{sgn} x.x^{r+s+1} e_{-}^{\lambda x},
\end{align}

for all $\lambda$ and $r, s = 0, 1, 2, \ldots$.\)

**Proof:** We will suppose first of all that $\lambda \neq \mu$. It was proved in [3] that

\begin{align}
(x^r e_{+}^{\lambda x})^\ast (x^s e_{+}^{\mu x}) &= D_\lambda^r D_\mu^s e_{+}^{\lambda x} - e_{+}^{\mu x}, \\
(x^r e_{-}^{\lambda x})^\ast (x^s e_{-}^{\mu x}) &= D_\lambda^r D_\mu^s e_{-}^{\lambda x} - e_{-}^{\mu x}.
\end{align}

It follows that

\begin{equation}
(x^r e^{\lambda x}) \boxplus (x^s e^{\mu x}) = (x^r e_{+}^{\lambda x} + x^r e_{-}^{\lambda x}) \boxplus (x^s e_{+}^{\mu x}) = D_\lambda^r D_\mu^s e_{+}^{\lambda x},
\end{equation}

on using equations (5) and (17) and noting that the neutrix convolution product is distributive with respect to addition.

Similarly,

\begin{equation}
(x^r e^{\lambda x}) \boxplus (x^s e^{\mu x}) = (x^r e_{+}^{\lambda x} + x^r e_{-}^{\lambda x}) \boxplus (x^s e_{-}^{\mu x}) = -D_\lambda^r D_\mu^s e_{-}^{\lambda x},
\end{equation}

on using equations (5) and (18). Equations (13) are proved.

We now have

\begin{equation}
(x^r e^{\lambda x}) \boxplus (x^s e^{\mu x}) = (x^r e_{+}^{\lambda x}) \boxplus (x^s e_{+}^{\mu x} + x^s e_{-}^{\mu x}) = 0,
\end{equation}
on using equations (13), proving equation (14).

Now suppose that $\lambda = \mu$. It was proved in [3] that in this case

\begin{equation}
(x^r e^{\lambda x}_+) \ast (x^s e^{\lambda x}_-) = B(r + 1, s + 1)x^{r+s+1} e^{\lambda x}_+.
\end{equation}

\begin{equation}
(x^r e^{\lambda x}_-) \ast (x^s e^{\lambda x}_+) = -B(r + 1, s + 1)x^{r+s+1} e^{\lambda x}_-.
\end{equation}

It follows that

\begin{equation}
(x^r e^{\lambda x}) \boxtimes (x^s e^{\lambda x}) = (x^r e^{\lambda x}_+ + x^r e^{\lambda x}_-) \boxtimes (x^s e^{\lambda x}_+) = B(r + 1, s + 1)x^{r+s+1} e^{\lambda x}_+
\end{equation}
on using equations (5) and (19).

Similarly,

\begin{equation}
(x^r e^{\lambda x}) \boxtimes (x^s e^{\lambda x}) = (x^r e^{\lambda x}_+ + x^r e^{\lambda x}_-) \ast (x^s e^{\lambda x}_-) = -B(r + 1, s + 1)x^{r+s+1} e^{\lambda x}_+,
\end{equation}
on using equations (5) and (20) and then

\begin{equation}
(x^r e^{\lambda x}) \boxtimes (x^s e^{\lambda x}) = (x^r e^{\lambda x}_+ + x^r e^{\lambda x}_-) \ast (x^s e^{\lambda x}_-) = -B(r + 1, s + 1) \text{sgn} x^r x^{r+s+1} e^{\lambda x}_+,
\end{equation}
on using equations (21) and (22). Equations (15) and (16) are now proved.

The non-commutative neutrix convolution product $(x^r e^{\lambda x} - ) \ast (x^s e^{\mu x} )$ was evaluated in [3]. Note that

\begin{equation}
(x^r e^{\lambda x}_-) \boxtimes (x^s e^{\mu x}_+) = (x^r e^{\lambda x}_- \ast (x^s e^{\mu x}_+) = (x^r e^{\lambda x}_- \ast (x^s e^{\mu x}_+),
\end{equation}

for $\lambda \neq \mu$, but

\begin{equation}
(x^r e^{\lambda x}_-) \boxtimes (x^s e^{\lambda x}_- \neq (x^r e^{\lambda x}_- \ast (x^s e^{\lambda x}_+).
\end{equation}

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