Necessary and sufficient conditions for weak convergence of random sums of independent random variables

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Abstract. Let \( \{X_n, n \geq 1\} \) be a sequence of independent random variables such that \( EX_n = a_n, E(X_n - a_n)^2 = \sigma_n^2, n \geq 1 \). Let \( \{N_n, n \geq 1\} \) be a sequence of positive integer-valued random variables. Let us put \( S_{N_n} = \sum_{k=1}^{N_n} X_k, L_n = \sum_{k=1}^{n} a_k, s_n^2 = \sum_{k=1}^{n} \sigma_k^2, n \geq 1 \). In this paper we present necessary and sufficient conditions for weak convergence of the sequence \( \{(S_{N_n} - L_n)/s_n, n \geq 1\} \), as \( n \to \infty \). The obtained theorems extend the main result of M. Finkelstein and H.G. Tucker (1989).

Keywords: random sums, weak convergence, stable law, nonrandom centering, measure of dependence between \( \sigma \)-fields

Classification: Primary 60F05; Secondary 60G50

1. Introduction.

Let \( \{X_n, n \geq 1\} \) be a sequence of independent random variables, defined on a probability space \( (\Omega, A, P) \), such that \( EX_n = a_n, E(X_n - a_n)^2 = \sigma_n^2 < \infty, n \geq 1 \). Let us put

\[
S_n = \sum_{k=1}^{n} X_k, \quad L_n = \sum_{k=1}^{n} a_k, \quad s_n^2 = \sum_{k=1}^{n} \sigma_k^2, \quad n \geq 1.
\]

Let \( \{N_n, n \geq 1\} \) be a sequence of positive integer-valued random variables, defined on the same probability space \( (\Omega, A, P) \).

Recently many authors have studied limit behaviour of the following sequences:

\[
\{(S_n - L_n)/s_n, n \geq 1\}, \quad \{(S_n - EL_n)/\sigma(S_n), n \geq 1\}, \quad \{(S_n - L_n)/s_n, n \geq 1\},
\]

under the assumption that for each \( n \geq 1 \) the random variables \( N_n, X_1, X_2, \ldots \) are independent. Also the rate of convergence to the obtained limit law has extensively been studied (cf. [3], [6], [9], [4], [5], [8] and the references given there).

The limit distribution of the sequence \( \{(S_{N_n} - L_n)/s_n, n \geq 1\} \) is presented in [3]. Namely, M. Finkelstein and H.G. Tucker [3] have obtained the following very interesting result.
Theorem A. Let \( \{X_n, n \geq 1\} \) be a sequence of independent and identically distributed random variables such that \( EX_1 = \mu \neq 0 \) and \( E(X_1 - \mu)^2 = \sigma^2 > 0 \). If \( \{N_n, n \geq 1\} \) is a sequence of positive integer-valued random variables independent of \( X_n, n \geq 1 \), then the condition

\[
(S_{N_n} - n\mu)/\sigma \sqrt{n} \xrightarrow{D} \text{(some) } Z
\]

holds if and only if the condition

\[
(N_n - n)/\sqrt{n} \xrightarrow{D} \text{(some) } U
\]

holds, in which case the distribution of \( Z \) is that of \( X + Y \), where \( X \) and \( Y \) are independent random variables, \( X \) being \( N(0, 1) \) and \( Y \) having the same distribution as \( \mu U/\sigma \).

The main aim of this paper is to extend Theorem A in the following directions:

(i) We consider the random variables \( X_n, n \geq 1 \), not necessarily identically distributed.

(ii) We omit the assumption that the random variables \( X_n, n \geq 1 \), have finite moments, and therefore we consider weak convergence to the Levy class distribution functions.

(iii) We do not assume that the random variables \( N_n, n \geq 1 \), are independent of \( X_n, n \geq 1 \). We study limit distribution of the sequence \( \{(S_{N_n} - L_n)/s_n, n \geq 1\} \), under the assumption that for some \( 1 \leq q \leq \infty \)

\[
r(n) = R_{1,q}(\sigma\{N_k, k \geq 1\}, \sigma\{X_k, k \geq n\}) \to 0
\]

or

\[
R(n) = R_{1,q}(\sigma\{N_n\}, \sigma\{X_k, k \geq 1\}) \to 0
\]

as \( n \to \infty \), where \( R_{p,q}(\mathcal{F}, \mathcal{G}) \) denotes the measure of dependence between \( \sigma \)-fields \( \mathcal{F} \) and \( \mathcal{G} \) introduced in [2] (cf. (1.1)). Namely, for \( 1 \leq p, q \leq \infty \)

\[
R_{p,q}(\mathcal{F}, \mathcal{G}) = \sup |Ef g - Ef Eg|/\|f\|_p\|g\|_q,
\]

where the sup is taken over all \( f \) and \( g \) such that \( f \) is simple, real-valued, and \( \mathcal{F} \)-measurable and \( g \) is simple, real-valued, and \( \mathcal{G} \)-measurable. (0/0 is presented to be 0.) Of course, \( R_{p,q} \) is simply a norm of the bilinear form covariance.

In Section 2 we present the results. In Section 3 some auxiliary lemmas are given. The proofs of the main results are presented in Section 4.
2. Results.

**Theorem 1.** Let \( \{X_n, n \geq 1\} \) be a sequence of independent random variables and let \( \{N_n, n \geq 1\} \) be a sequence of positive integer-valued random variables satisfying (1.3) or (1.4) for some \( 1 \leq q \leq \infty \). Let \( \{L_n, n \geq 1\} \) and \( \{s_n, n \geq 1\} \) be sequences of real numbers and positive real numbers, respectively. Denote

\[
\begin{align*}
a_n &= L_n - L_{n-1}, \\
S_n &= \sum_{k=1}^{n} X_k, \\
S_{N_n} &= \sum_{k=1}^{N_n} X_k, \\
L_{N_n} &= \sum_{k=1}^{\infty} L_k I[N_n = k], \\
s_{N_n} &= \sum_{k=1}^{\infty} s_k I[N_n = k], \quad n \geq 1.
\end{align*}
\]

Assume

\[
\max_{1 \leq k \leq n} P[|X_k - a_k| \geq \varepsilon s_n] \to 0 \quad \text{as} \quad n \to \infty,
\]

and

\[
(2.1) \quad (S_n - L_n)/s_n \xrightarrow{D} F(\cdot) \quad \text{as} \quad n \to \infty,
\]

where

\[
(2.2) \quad \int e^{itx} F(dx) = \exp\{i\gamma t + \oint (e^{itx} - 1 - itx/(1 + x^2)) (1 + x^2)/x^2 G(dx)\},
\]

\( \gamma \) is a real number, \( G(\cdot) \) is nondecreasing bounded function (\( \oint \) means that the integrand is equal to \(-t^2/2\) for \( x = 0 \)) and not identically equal to a constant, and

\[
(2.3) \quad N_n \xrightarrow{P} \infty \quad \text{as} \quad n \to \infty
\]

or for every \( k, n \in \mathbb{N} \) and some constant \( C > 0 \)

\[
(2.4) \quad |L_n - L_k| \geq C|n - k| \quad \text{and} \quad n/s_n \to \infty \quad \text{as} \quad n \to \infty,
\]

or

\[
(2.5) \quad L_n/s_n \to \infty \quad \text{or} \quad L_n/s_n \to \infty, \quad \text{as} \quad n \to \infty,
\]

If

\[
(2.6) \quad (s_{N_n}/s_n, (L_{N_n} - L_n)/s_n) \to (\text{some}) \quad A(\cdot, \cdot),
\]

where \( A \) is a two-dimensional distribution function, then

\[
(2.7) \quad (S_{N_n} - L_n)/s_n \xrightarrow{D} (\text{some}) \quad \Psi(\cdot),
\]
where

\[(2.8) \quad \int e^{itx} \Psi(dx) = \int \int \exp \{i \gamma ty + itz + \oint (e^{i(ty)x} - 1 - i(ty)x/(1 + x^2)) (1 + x^2)/x^2 G(dx)) \} A(dy, dz). \]

If (2.7) holds with some distribution function \(\Psi(\cdot)\), then the sequence \\{\((s_{N_n}/s_n, (L_{N_n} - L_n)/s_n), n \geq 1\)\} is tight.

It is known that the set of possible weak limits of sums of independent random variables (cf. for e.g. [7, IV, §3]) is the class of Levy distribution function \(F(\cdot)\) which may be characterized by (2.2) and: For every \(0 < \alpha < 1\), there exists the characteristic function \(f_\alpha(t)\) such that

\[\int e^{itx} F(dx) = \int e^{it\alpha x} F(dx) f_\alpha(t), \quad t \in \mathbb{R}.\]

Furthermore, by Lemma 11 [7, IV, §3], (2.1) implies

\[(2.9) \quad s_{n+1}/s_n \to 1, \quad \text{and} \quad s_n \to \infty \quad \text{as} \quad n \to \infty.\]

The condition that \(G(\cdot)\) is not identically equal to a constant implies

\[\oint (e^{itx} - 1 - itx/(1 + x^2)) (1 + x^2)/x^2 G(dx) \neq 0\]

so that \(F(\cdot)\) in (2.1) is not a degenerate distribution function.

We note that the condition (2.3) may be expressed as follows:

For some sequence \(\{\alpha(n), n \geq 1\}\) such that \(\alpha(n) \to \infty\) as \(n \to \infty\),

\[(2.10) \quad P(N_n < \alpha(n)) \to 0 \quad \text{as} \quad n \to \infty.\]

Let us observe that if for each \(n \geq 1\), the random variables \(N_n, X_1, X_2, \ldots\) are independent, then (1.3) and (1.4) hold with \(r(n) = \mathbb{R}(n) = 0\) for every \(q \geq 1\). The next result deals with the convergence to the stable limit law. Assume

\[(2.11) \quad P(X_n > x)/P(|X_n| > x) \to c_{1,n}/(c_{1,n} + c_{2,n}) \quad \text{as} \quad x \to \infty,\]

where \(\{c_{j,n}, n \geq 1\}, j = 1, 2, \) are some sequences of nonnegative numbers such that \(c_{1,n} + c_{2,n} > 0, n \geq 1\).
For some $0 \leq \alpha \leq 2$ we define

$$e_1 = \int_0^\infty u^{-\alpha} \sin(u) \, du, \quad e_2 = \begin{cases} -\int_0^\infty u^{-\alpha} \cos(u) \, du, & \text{if } \alpha < 1, \\ 1 & \text{if } \alpha = 1, \\ \int_0^\infty u^{-\alpha}(1 - \cos(u)) \, du, & \text{otherwise} \end{cases}$$

$$\sigma_n^\alpha = (c_{1,n} + c_{2,n})e_1, \quad s_n^\alpha = \sum_{i=1}^n \sigma_i^\alpha, \quad s_0 = 1,$$

$$a_n = \begin{cases} 0, & \text{if } \alpha < 1, \\ EX_n, & \text{if } \alpha > 1, \\ \int_0^1 d_n(x) \, dx + \int_1^\infty (d_n(x) - (c_{2,n} - c_{1,n})/x) \, dx + \\ \quad + \sum_{i=1}^{n-1} (c_{1,i} - c_{2,i})e_2 \ln(s_i/s_{i-1}) + \\ \quad + (c_{1,n} - c_{2,n})e_2 \ln(s_n) + (c_{2,n} - c_{1,n})\gamma, & \text{otherwise} \end{cases}$$

$$L_n = \sum_{i=1}^n a_i, \quad n \geq 1,$$

where $d_n(x) = P(X_n > x) - P(X_n < -x)$, $\gamma$ is the Euler’s constant and $s_n = (s_n^\alpha)^{1/\alpha}$. Furthermore, let

$$\beta_n = \sum_{i=1}^n (c_{1,i} - c_{2,i})e_2.$$

Let $G_{\alpha,\beta,\nu,\lambda}(\cdot)$ denote the stable law with parameters $\alpha, \beta, \nu, \lambda$, $\alpha \in (0, 2]$, $\beta \in [-1, 1]$, $\lambda > 0$, $\nu \in \mathbb{R}$, i.e.

$$\int e^{itx} G_{\alpha,\beta,\nu,\lambda}(dx) = \exp\{i\nu t - \lambda |t|^\alpha (1 + i \text{sgn}(t)\omega(t, \alpha, \beta))\},$$

where $\omega(t, \alpha, \beta) = \beta t g(\pi\alpha/2)$ for $\alpha \neq 1$ and $\omega(t, \alpha, \beta) = -(2/\pi) \ln |t|$ for $\alpha = 1$.

**Theorem 2.** Let $\{X_n, n \geq 1\}$ be a sequence of independent random variables satisfying (2.11). Assume, for some $\alpha \in (0, 2]$,

$$\beta_n/s_n^\alpha \to \beta \quad \text{as } n \to \infty$$

and

$$\begin{array}{l}
(2.12) \quad (S_n - L_n)/s_n \xrightarrow{D} G_{\alpha,\beta,0,1}(\cdot) \quad \text{as } n \to \infty,
\end{array}$$

hold.

Let $\{N_n, n \geq 1\}$ be a sequence of positive integer-valued random variables satisfying (1.3) or (1.4) and (2.3) or (2.4). If

$$\begin{array}{l}
(2.13) \quad (s_n^\alpha)_{L_n - L_n}/s_n \xrightarrow{D} (\text{some}) A(\cdot, \cdot, \cdot),
\end{array}$$

$$\begin{array}{l}
(2.14) \quad \frac{s_n^\alpha}{s_N^\alpha} \left(\beta_n - \beta_n \right)/s_n^\alpha \xrightarrow{D} \text{ (some) } A(\cdot, \cdot, \cdot),
\end{array}$$

$$\begin{array}{l}
(2.15) \quad \frac{L_n - L_n}{s_n} \xrightarrow{D} G_{\alpha,\beta,0,1}(\cdot) \quad \text{as } n \to \infty,
\end{array}$$

$$\begin{array}{l}
(2.16) \quad \frac{L_n - L_n}{s_n} \xrightarrow{D} G_{\alpha,\beta,0,1}(\cdot) \quad \text{as } n \to \infty,
\end{array}$$

then

$$\begin{array}{l}
(2.17) \quad (S_n - L_n)/s_n \xrightarrow{D} G_{\alpha,\beta,0,1}(\cdot) \quad \text{as } n \to \infty.
\end{array}$$
where $A$ is a three-dimensional distribution function, then

$$
(2.15) \quad (S_{N_n} - L_n)/s_n \xrightarrow{D} \Psi(\cdot) \text{ as } n \to \infty,
$$

where

$$
\int e^{itx} \Psi(dx) = \int \int \int \exp\{-|t|^{\alpha}(x + i \text{sgn}(t)\omega(\alpha, \beta, t)) - \}
$$

$$
|t|^{\alpha} i \text{sgn}(t)\omega(\alpha, 1, t)y + etz\} A(dx, dy, dz).
$$

If (2.15) holds with some distribution function $\Psi$, then the sequence $\{(s_{N_n}^{\alpha}/s_n^{\alpha}, (\beta_{N_n} - \beta_n)/s_n^{\alpha}, (L_{N_n} - L_n)/s_n), n \geq 1\}$ is tight.

Note that for $\alpha < 1$ we have $L_k = 0$ for all $k$, hence $(L_{N_n} - L_n) = 0, n \geq 1$. The given result seems to be interesting in case of i.i.d. random variables, but because in case $\alpha < 1$ the centralization is equal to 0, we formulate two corollaries for $\alpha > 1$ and $\alpha = 1$ only.

**Corollary 1.** Let $\{X_n, n \geq 1\}$ be a sequence of independent and identically distributed random variables. Assume $X_1$ belongs to the area of attraction of a stable law $G_{\alpha, \beta, 0, \lambda}(\cdot), \alpha \in (1, 2]$. Let $\{N_n, n \geq 1\}$ be a sequence of positive integer-valued random variables satisfying (1.3) or (1.4). If

$$
(2.16) \quad (N_n - n)/n^{1/\alpha} \xrightarrow{D} \text{(some) } A(\cdot), \text{ as } n \to \infty,
$$

then

$$
(2.17) \quad (S_{N_n} - nEX_1)/(n\lambda)^{1/\alpha} \xrightarrow{D} \text{(some) } \Psi(\cdot), \text{ as } n \to \infty,
$$

where $\lambda = e_1(c_{1,1} + c_{2,1})$, and

$$
\int e^{itx} \Psi(dx) = \exp\{-|t|^{\alpha}\lambda(1 + i \text{sgn}(t)\omega(\alpha, \beta, t))\}
$$

$$
\int \exp\{-|t|^{\alpha}i \text{sgn}(t)\omega(\alpha, \beta, t)
$$

$$
(c_{1,1} - c_{2,1})e_2/\lambda^{1/\alpha} + itEX_1/\lambda^{1/\alpha}\} A(dx).
$$

If (2.17) holds, then the sequence given in (2.16) is tight.

**Corollary 2.** Let $\{X_n, n \geq 1\}$ be a sequence of independent and identically distributed random variables. Assume $X_1$ belongs to the area of attraction of Cauchy law. Let $\{N_n, n \geq 1\}$ be a sequence of positive integer-valued random variables satisfying (1.3) or (1.4). If

$$
(2.18) \quad (N_n/n, (N_n \ln(N_n) - n \ln(n))/n) \xrightarrow{D} \text{(some) } A(\cdot, \cdot), \text{ as } n \to \infty,
$$

then

$$
(2.19) \quad (S_{N_n} - n\mu - r(n \ln(n))/n\lambda) \xrightarrow{D} \text{(some) } \Psi(\cdot), \text{ as } n \to \infty,
$$
where \( \lambda = e_1(c_{1,1} + c_{2,1}) \), \( r = e_1(c_{1,1} - c_{2,1}) \),

\[
\mu = \int_0^1 d_1(x) \, dx + \int_1^\infty (d_1(x) - (c_{1,1} - c_{2,1})/x) \, dx + (c_{1,1} - c_{2,1}) \left[ \gamma + e_2 \ln(\lambda) \right],
\]

and

\[
\int e^{itx} \Psi(dx) = \int \int_{\mathbb{R}^2} \exp\left\{-|t|\lambda(x+(2x-1)i \sgn(t)\omega(1, \beta, t)) + it(x+1)\mu + it\beta(y+1)\right\} A(dx, dy).
\]

If (2.19) holds, then the sequence given in (2.18) is tight.

The next result deals with the central limit theorem. Here we can formulate a stronger result than in Theorems 1 and 2 (cf. Lemma 6 in Section 3).

**Theorem 3.** Let \( \{X_n, n \geq 1\} \) be a sequence of independent random variables such that \( EX_n = a_n \) and \( E(X_n-a_n)^2 = \sigma_n^2 < \infty \), \( n \geq 1 \). Let \( \{N_n, n \geq 1\} \) be a sequence of positive integer-valued random variables satisfying (1.3) or (1.4). Let us put

\[
S_n = \sum_{k=1}^n X_k, \quad L_n = \sum_{k=1}^n a_k, \quad s_n^2 = \sum_{k=1}^n \sigma_k^2,
\]

\[
S_{N_n} = \sum_{k=1}^{N_n} X_k, \quad L_{N_n} = \sum_{k=1}^{N_n} a_k, \quad s_{N_n}^2 = \sum_{k=1}^{N_n} \sigma_k^2, \quad n \geq 1.
\]

If

\[
(2.20) \quad (S_n - L_n)/s_n \xrightarrow{D} N(0,1) \quad \text{as} \quad n \to \infty,
\]

and (2.3) or (2.4) or (2.5) hold, then the following conditions are equivalent:

\[
(2.21) \quad \left( s_{N_n}^2/s_n^2, (L_{N_n} - L_n)/s_n \right) \xrightarrow{D} \text{(some)} \quad A(\cdot, \cdot),
\]

where \( A \) is a two-dimensional distribution function,

\[
(2.22) \quad \left( S_{N_n} - L_n \right)/s_n \xrightarrow{D} \text{(some)} \quad \Psi(\cdot),
\]

where \( \Psi \) is a distribution function.

The distribution functions \( A \) and \( \Psi \) are such that

\[
(2.23) \quad \int_{-\infty}^{\infty} \exp(itx) \Psi(dx) = \int \int_{\mathbb{R}^2} \exp(-t^2x^2/2 + ity) A(dx, dy).
\]
Corollary 3. Let \( \{X_n, n \geq 1\} \) be a sequence of independent random variables such that \( EX_n = \mu \neq 0, E(X_n - \mu)^2 = \sigma^2 < \infty, n \geq 1, \) and
\[
(2.24) \quad (S_n - n\mu)/\sigma\sqrt{n} \xrightarrow{D} N(0, 1) \quad \text{as} \quad n \to \infty.
\]
Let \( \{N_n, n \geq 1\} \) be a sequence of positive integer-valued random variables satisfying (1.3) or (1.4). Then the following conditions are equivalent:
\[
(2.25) \quad (N_n - n)/\sqrt{n} \xrightarrow{D} \text{ (some) } G(\cdot), \quad \text{as} \quad n \to \infty,
\]
and
\[
(2.26) \quad (S_{N_n} - n\mu)/\sigma\sqrt{n} \xrightarrow{D} \text{ (some) } \Psi(\cdot), \quad \text{as} \quad n \to \infty.
\]
The distribution functions \( G \) and \( \Psi \) are such that
\[
\int e^{itx} \Psi(dx) = \exp(-t^2/2) \int e^{it\mu x/\sigma} G(dx).
\]

Let us observe that if, in addition, \( X_n, n \geq 1, \) are identically distributed, then (2.24) holds. Thus Corollary 3, under the assumption that the random variables \( N_n, X_1, X_2, \ldots \) are independent for each \( n \geq 1, \) gives Theorem A.

3. Auxiliary lemmas.

In the proofs of the main results we need some lemmas. Let \( \mathcal{L}(X) \) denote the distribution of the random variable \( X. \)

Lemma 1. Let \( \{X_n, n \geq 1\} \) be a sequence of independent random variables and let \( \{N_n, n \geq 1\} \) be a sequence of positive integer-valued random variables satisfying (1.3) or (1.4). Let \( \{Y_n, n \geq 1\} \) be a sequence of independent random variables and independent of \( \{X_n, n \geq 1\} \) and \( \{N_n, n \geq 1\} \) such that \( \mathcal{L}(X_n) = \mathcal{L}(Y_n), n \geq 1. \) Let \( \{s_n, n \geq 1\} \) and \( \{L_n, n \geq 1\} \) be sequences of real numbers such that \( s_n > 0, n \geq 1, \) and \( s_n \to \infty \) as \( n \to \infty. \) Let \( Z_n = Y_1 + \cdots + Y_n, n \geq 1. \) Assume (2.1) holds. Then the following conditions are equivalent:
\[
(3.1) \quad (S_{N_n} - L_n)/s_n \xrightarrow{D} \text{ (some) } \Psi(\cdot), \quad \text{as} \quad n \to \infty.
\]
and
\[
(3.2) \quad (Z_{N_n} - L_n)/s_n \xrightarrow{D} \text{ (some) } G(\cdot), \quad \text{as} \quad n \to \infty.
\]
in which case \( \Psi(\cdot) \equiv G(\cdot). \)

Proof: Let us observe that \( \mathcal{L}(S_n) = \mathcal{L}(Z_n), n \geq 1, \) but in general \( \mathcal{L}(S_{N_n}) \neq \mathcal{L}(Z_{N_n}), n \geq 1, \) since \( N_n \) is independent of \( Y_n, n \geq 1, \) but may be dependent of \( X_n, n \geq 1. \)
Assume (1.4) holds. Then

\[ I_n(t) = |E \exp\{it(S_{N_n} - L_n)/s_n\} - E \exp\{it(Z_{N_n} - L_n)/s_n\}| = \]

\[ = | \sum_{m=1}^{\infty} [EI(N_n = m) \exp\{it(S_m - L_n)/s_n\} - EI(N_n = m) E \exp\{it(Z_m - L_n)/s_n\}]| \leq \]

\[ \leq \sum_{m=1}^{\infty} \mathbb{R}(n)P(N_n = m) = \mathbb{R}(n) \rightarrow 0 \text{ as } n \rightarrow \infty. \]  

(3.3)

Thus (3.1) holds if and only if (3.2) holds and \( \Psi(\cdot) \equiv G(\cdot) \). We remark that under the assumption (1.4) we did not use (2.1).

Assume now (1.3) holds. Then by (2.1), for every \( \varepsilon > 0 \), there exists a positive number \( K_\varepsilon \) such that for every \( n \geq 1 \)

\[ P(|S_n - L_n|/s_n \geq K_\varepsilon) \leq \varepsilon. \]

Furthermore, we may and do assume \( 0 < \varepsilon_1 < \varepsilon_2 \) implies \( K_{\varepsilon_1} \geq K_{\varepsilon_2} \) and that \( K_\varepsilon \rightarrow \infty \text{ as } \varepsilon \rightarrow 0. \)

Let us put

\[ \psi(n) = \max\{k : s_k \leq s_n^{1/2}\}, \]

\[ \varepsilon(n) = 2 \inf \{\varepsilon > 0 : K_\varepsilon < s_n^{1/4}, \varepsilon > s_n^{-1/4}\}, \]

\[ \varrho(n) = \min\{\psi(n), (\varepsilon(n))^{-1/2}\}. \]

We have \( s_n \rightarrow \infty \), hence \( \varepsilon(n) \rightarrow 0, \psi(n) \rightarrow \infty \) and \( K_{\varepsilon(n)} \rightarrow \infty \) as \( n \rightarrow \infty. \)

Furthermore, for every \( 1 \leq i \leq \varrho(n) \)

\[ P(|S_i - L_i|/s_n > s_n^{-1/4}) \leq P(|S_i - L_i|/s_i s_n^{1/2} > s_n^{-1/4}) \leq P(|S_i - L_i|/s_i > s_n^{1/4}) \leq \]

\[ \leq P(|S_i - L_i|/s_i > K_{\varepsilon(n)}) \leq \varepsilon(n) \]

and, in consequence,

\[ P(|S_i - L_i|/s_n > s_n^{-1/4}, N_n = i) \leq \varepsilon(n). \]

Thus, for every \( t \) such that \( |t| < s_n^{1/8} \), we get

\[ |E(\exp\{it(S_{N_n} - L_n)/s_n\} - \exp\{it(L_{N_n} - L_n)/s_n\}) I[N_n \leq \varrho(n)]| \leq \]

\[ \leq E|\exp\{it(S_{N_n} - L_{N_n})/s_n\} - 1| I[N_n \leq \varrho(n), \max_{1 \leq i \leq \varrho(n)} |S_i - L_i|/s_n \leq s_n^{-1/4}] + \]

\[ + \sum_{i \leq \varrho(n)} 2P(|S_i - L_i|/s_n > s_n^{-1/4}) \leq 2|t| s_n^{-1/4} + 2\varrho(n)\varepsilon(n) \leq 4(\varepsilon(n))^{1/2}. \]
Similarly, replacing \( S_i \) by \( Z_i \), we get
\[
|E(\exp\{it(Z_{N_n} - L_n)/s_n\} - \exp\{it(L_{N_n} - L_n)/s_n\}) I[N_n \leq \varrho(n)]| \leq 4(\varepsilon(n))^{1/2},
\]
so that
\[
|E(\exp\{it(S_{N_n} - L_n)/s_n\} - \exp\{it(Z_{N_n} - L_n)/s_n\}) I[N_n \leq \varrho(n)]| \leq 8(\varepsilon(n))^{1/2}.
\]
On the other hand, step by step as in above, for \(|t| < s_n^{1/8}\) we also get
\[
E|\exp\{it(S_{\varrho(n)} - L_{\varrho(n)})/s_n\} - 1| \leq 4(\varepsilon(n))^{1/2},
\]
and
\[
E|\exp\{it(Z_{\varrho(n)} - L_{\varrho(n)})/s_n\} - 1| \leq 4(\varepsilon(n))^{1/2},
\]
where \([x]\) denotes the integral part of \( x \). Hence, taking into account the inequalities obtained above and using the triangle inequality, for \(|t| \leq s_n^{1/8}\) we have
\[
I_n(t) \leq |E(\exp\{it(S_{N_n} - L_n)/s_n\} - \\
- \exp\{it(Z_{N_n} - L_n)/s_n\}) I[N_n > \varrho(n)]| + 8(\varepsilon(n))^{1/2} \leq \\
\leq |E(\exp\{it(S_{N_n} - L_n - S_{\varrho(n)} + L_{\varrho(n)})/s_n\} - \\
- \exp\{it(Z_{N_n} - L_n - S_{\varrho(n)} + L_{\varrho(n)})/s_n\}) I[N_n > \varrho(n)]| + 8(\varepsilon(n))^{1/2} \leq \\
\leq |E(\exp\{it(S_{N_n} - S_{\varrho(n)} - L_n + L_{\varrho(n)})/s_n\} - \\
- \exp\{it(Z_{N_n} - Z_{\varrho(n)} - L_n + L_{\varrho(n)})/s_n\}) I[N_n > \varrho(n)]| + 16(\varepsilon(n))^{1/2} \leq \\
\leq \sum_{k > \varrho(n)} |E \exp\{it(S_k - S_{\varrho(n)} - L_n + L_{\varrho(n)})/s_n\} I[N_k = k] - \\
- E \exp\{it(S_k - S_{\varrho(n)} - L_n + L_{\varrho(n)})/s_n\} P[N_k = k]| + 16(\varepsilon(n))^{1/2} \leq \\
\leq r([\varrho(n)] + 1) + 16(\varepsilon(n))^{1/2} \to 0 \text{ as } n \to \infty.
\]
Thus the proof of Lemma 1 is finished. \( \square \)

**Lemma 2.** If \( \{X_n, n \geq 1\} \) and \( \{Y_n, n \geq 1\} \) are tight sequences of random variables, then the following sequences are also tight:

(a) \( \{X_n + Y_n, n \geq 1\} \),
(b) \( \{X_nY_n, n \geq 1\} \),
(c) \( \{(X_n, Y_n), n \geq 1\} \).

The proof is simple and therefore omitted.
Lemma 3. Let \( \{X_n, n \geq 1\} \) be a sequence of independent random variables and let \( \{N_n, n \geq 1\} \) be a sequence of positive integer-valued random variables satisfying (1.3) or (1.4). Let \( \{s_n, n \geq 1\} \) and \( \{L_n, n \geq 1\} \) be sequences of real numbers such that \( 0 < s_n, n \geq 1, \) and \( s_n \to \infty \) as \( n \to \infty. \) Assume (2.1) and (3.1) hold with nondegenerate distribution function \( F(\cdot), \) then the sequence \( \{s_{N_n}/s_n, n \geq 1\} \) is tight.

Proof: Let \( \{Y_n, n \geq 1\} \) and \( \{V_n, n \geq 1\} \) be independent sequences of independent random variables and independent of the sequences \( \{X_n, n \geq 1\} \) and \( \{N_n, n \geq 1\}, \) such that \( \mathcal{L}(X_n) = \mathcal{L}(Y_n) = \mathcal{L}(V_n), n \geq 1. \) Let us put

\[
Z_n = \sum_{k=1}^{n} Y_k, \quad U_n = \sum_{k=1}^{n} V_k, \quad n \geq 1.
\]

Then

\[
(Z_n - L_n)/s_n \xrightarrow{D} F(\cdot), \quad (U_n - L_n)/s_n \xrightarrow{D} F(\cdot), \quad \text{as } n \to \infty,
\]

and, by Lemma 1,

\[
(Z_{N_n} - L_n)/s_n \xrightarrow{D} \Psi(\cdot), \quad (U_{N_n} - L_n)/s_n \xrightarrow{D} \Psi(\cdot), \quad \text{as } n \to \infty.
\]

By Lemma 2 (a) the sequences \( \{(Z_{N_n} - U_{N_n})/s_n, n \geq 1\} \) and \( \{(Z_n - U_n)/s_n, n \geq 1\} \) are tight. Moreover,

\[
(Z_n - L_n)/s_n \xrightarrow{D} \int_{-\infty}^{\infty} F(x + \cdot) F(dx) \quad \text{as } n \to \infty.
\]

Because \( F(\cdot) \) is nondegenerate distribution function and \( \int F(x + \cdot) F(dx) \) is symmetric distribution function so that there exists \( c > 0 \) such that \( \int F(x + c) F(dx) > 0. \)

Assume that \( \{s_{N_n}/s_n, n \geq 1\} \) is not tight. Thus, for some \( \varepsilon > 0 \) there exist the sequences \( \{k_n, n \geq 1\} \) and \( \{l_n, n \geq 1\} \) such that \( k_n \in \{1, 2, \ldots\}, \) \( k_n \to \infty, \) \( l_n \to \infty \) as \( n \to \infty, \) and \( P(s_{N_{k_n}}/s_{k_n} > l_n) > \varepsilon, n \geq 1. \) Hence, for sufficiently large \( n, \)

\[
P(Z_{N_{k_n}} - U_{N_{k_n}} \geq c l_n s_{k_n}) \geq \sum_{m : s_m > l_n s_{k_n}} P(Z_m - U_m \geq c l_n s_{k_n}) P(N_{k_n} = m) \geq \sum_{m : s_m > l_n s_{k_n}} P(Z_m - U_m \geq c s_m) P(N_{k_n} = m) \geq (1 - \int_{-\infty}^{\infty} F(x + c) F(dx)) P(s_{N_{k_n}} \geq l_n s_{k_n})/2 \geq (1/4)(1 - \int_{-\infty}^{\infty} F(x + c) F(dx)) \varepsilon > 0.
\]

Thus we get a contradiction, and this ends the proof. \( \Box \)
**Lemma 4.** Let \( \{Y_n, n \geq 1\} \) be a sequence of independent random variables and let \( \{N_n, n \geq 1\} \) be a sequence of positive integer-valued random variables independent of \( \{Y_n, n \geq 1\} \). If \( \{L_n, n \geq 1\} \) and \( \{s_n, n \geq 1\} \) are sequences of real numbers such that \( 0 < s_n, n \geq 1, s_n \to \infty \) as \( n \to \infty \) and the sequences \( \{(Z_n - L_n)/s_n, n \geq 1\} \) and \( \{s_{N_n}/s_n, n \geq 1\} \) are tight, then the sequence \( \{(Z_{N_n} - L_{N_n})/s_n, n \geq 1\} \) is tight, too.

**Proof:** We have

\[
P(|Z_{N_n} - L_{N_n}|/s_{N_n} > K) = \sum_{m=1}^{\infty} P(|Z_m - L_m|/s_m > K) P(N_n = m) \leq \varepsilon
\]

provided, for every \( m \geq 1 \), \( P(|Z_m - L_m|/s_m > K) \leq \varepsilon \). Thus the sequence \( \{(Z_{N_n} - L_{N_n})/s_{N_n}, n \geq 1\} \) is tight, so that the sequence \( \{(Z_{N_n} - L_{N_n})/s_n = (Z_{N_n} - L_{N_n})/s_{N_n})(s_{N_n}/s_n), n \geq 1\} \) is tight by Lemma 2(b).

**Lemma 5.** Let \( \{L_n, n \geq 1\} \) and \( \{s_n, n \geq 1\} \) be sequences of real numbers such that \( 0 < s_n, n \geq 1, \) and \( s_n \to \infty \) as \( n \to \infty \). Let \( \{N_n, n \geq 1\} \) be a sequence of positive integer-valued random variables such that the sequence \( \{(L_{N_n} - L_n)/s_n, n \geq 1\} \) is tight. Then (2.3) or (2.4) or (2.5) implies (2.10).

**Proof:** Assume (2.4) holds. Then

\[
P(|L_{N_n} - L_n|/s_n > K) \geq P(|N_n - n|/s_n > K/C) \geq \geq P((N_n - n)/s_n < -K/C) = P(N_n < s_n(n/s_n - K/C)).
\]

Thus, taking into account the tightness of \( \{(L_{N_n} - L_n)/s_n, n \geq 1\} \) and the second part of (2.4), we get

\[
P(N_n < s_n(n/s_n - n/2s_n)) = P(N_n < n/2) \to 0, \text{ as } n \to \infty,
\]

so that (2.10) holds with \( \alpha(n) = n/2, n \geq 1 \). Let us suppose (2.5). If \( L_n/s_n \to \infty \) as \( n \to \infty \), then

\[
P(|L_{N_n} - L_n|/s_n > K) \geq P((L_{N_n} - L_n)/s_n < -K) = P(L_{N_n} < s_n(L_n/s_n - K)).
\]

Now the tightness and \( L_n/s_n \to \infty \) as \( n \to \infty \) imply

\[
P(L_{N_n} < s_n(L_n/s_n - L_n/2s_n)) = P(L_{N_n} < L_n/2) \to 0, \text{ as } n \to \infty,
\]

so that (2.10) holds with \( \alpha(n) = \inf\{k \in \mathbb{N} : L_k \geq L_n/2\}, n \geq 1 \). Of course, since \( L_n \to \infty \) as \( n \to \infty \), we get \( \alpha(n) \to \infty \) as \( n \to \infty \).

If \( L_n/s_n \to \infty \) as \( n \to \infty \), the proof of (2.10) is the same. The equivalence of (2.3) and (2.10) has been explained after Theorem 1.

\[\square\]
Lemma 6. Let $A(\cdot, \cdot)$ and $A'(\cdot, \cdot)$ be two distribution functions. If for every $t \in \mathbb{R}$
\[
\int \int \exp(-t^2x/2 + ity) A'(dx, dy) = \int \int \exp(-t^2x/2 + ity) A(dx, dy),
\]
then $A = A'$

The proof is easy and therefore omitted.

Lemma 7. Let \(\{X_n, n \geq 1\}\) be a sequence of independent random variables and let \(\{N_n, n \geq 1\}\) be a sequence of positive integer-valued random variables independent of \(\{X_n, n \geq 1\}\) and satisfying (2.10). Assume for arbitrary \(\tau > 0\), some sequence of real numbers \(\{a_k, k \geq 1\}\) and nondecreasing sequence of positive real numbers \(\{s_n, n \geq 1\}\),
\[
\sum_{j=1}^{n} (b_j + \int_{-\infty}^{\infty} x/(1 + x^2) dF_j(x + b_j) - a_j)/s_n \to \gamma, \text{ as } n \to \infty,
\]
and uniformly on compact sets with respect to \(t\)
\[
\int_{-\infty}^{\infty} (e^{itx/s_n} - 1 - itx/(1 + x^2)) (1 + x^2)/x G(dx), \text{ as } n \to \infty,
\]
where
\[F_j(x) = P[X_j < x], \quad b_j = \int_{|x|<\tau} x dF_j(x), \quad j \geq 1,
\]
and \(G(\cdot)\) is nondecreasing bounded function. Then uniformly on compact sets
\[
J_n(t) = |E \exp\{it \sum_{j=1}^{N_n} (b_j + \int_{-\infty}^{\infty} x/(1 + x^2) dF_j(x + b_j) - a_j)/s_n (s_{N_n}/s_n) + it(L_{N_n} - L_n)/s_n +
\]
\[
+ \int_{-\infty}^{\infty} (e^{itx/s_n} - 1 - itx(s_{N_n}/s_n)/(1 + x^2)s_{N_n})) d \sum_{j=1}^{N_n} F_j(x + b_j)\} -
\]
\[
- E \exp\{it\gamma(s_{N_n}/s_n) + it(L_{N_n} - L_n)/s_n +
\]
\[
+ \int_{-\infty}^{\infty} (e^{itx(s_{N_n}/s_n) - 1 - itx(s_{N_n}/s_n)/(1 + x^2)(1 + x^2)/x dG(x)}\} \to 0
\]
as \(n \to \infty\),

where
\[L_n = \sum_{j=1}^{n} a_j, \quad L_{N_n} = \sum_{j=1}^{N_n} a_j, \quad n \geq 1.\]
PROOF: Let us remark that for every $\varepsilon > 0$

$$P[\left| \sum_{j=1}^{N_n} (b_j + \int_{-\infty}^{\infty} \frac{x}{1 + x^2} dF_j(x + b_j) - a_j)/s_{N_n} - \gamma \right| > \varepsilon] \leq$$

$$\leq P[N_n \leq \alpha(n)] + \sup_{k_n \geq \alpha(n)} \left| \sum_{j=1}^{k_n} (b_j + \int_{-\infty}^{\infty} \frac{x}{1 + x^2} dF_j(x + b_j) - a_j)/s_{k_n} - \gamma \right| / \varepsilon \to 0 \text{ as } n \to \infty,$$

where $\{\alpha(n), n \geq 1\}$ is defined in (2.10). Similarly

$$P[\sup_{|t| < K_1} \left| \int_{-\infty}^{\infty} (e^{itx/s_{N_n}} - 1 - itx/((1 + x^2)s_{N_n})) d \sum_{j=1}^{N_n} F_j(x + b_j) - \right.\left. - \int_{-\infty}^{\infty} (e^{itx} - 1 - itx/(1 + x^2))(1 + x^2)/x dG(x) \right| > \varepsilon] \to 0 \text{ as } n \to \infty.$$

On the other hand, for each positive number $K_i, \varepsilon_i, i = 1, 2$, we have

$$\sup_{|t| < K_1} J_n(t) \leq P[|s_{N_n}/s_n| > K_2] +$$

$$+ 2P[\left| \sum_{j=1}^{N_n} (b_j + \int_{-\infty}^{\infty} \frac{x}{1 + x^2} dF_j(x + b_j) - a_j)/s_{N_n} - \gamma \right| > \varepsilon_1/K_1] +$$

$$+ 2P[\sup_{|y| < K_1 K_2} \left| \int_{-\infty}^{\infty} (e^{iyx/s_{N_n}} - 1 - iyx/((1 + x^2)s_{N_n})) d \sum_{j=1}^{N_n} F_j(x + b_j) - \right.\left. - \int_{-\infty}^{\infty} (e^{iyx} - 1 - iyx/(1 + x^2))(1 + x^2)/x dG(x) \right| > \varepsilon_2] + 2\varepsilon_1 + 2\varepsilon_2, \quad n \geq 1.$$

Let now $K_1 > 1$ and $\varepsilon$ be arbitrary positive numbers and let $n_1$ be such that for every $n \geq n_1$

$$P[\left| \sum_{j=1}^{N_n} (b_j + \int_{-\infty}^{\infty} \frac{x}{1 + x^2} dF_j(x + b_j) - a_j)/s_{N_n} - \gamma \right| > \varepsilon/(9K_1)] \leq \varepsilon/9.$$

Now we put $K_2$ such that for every $n \geq n_1$

$$P[|s_{N_n}/s_n| > K_2] \leq \varepsilon/9,$$

and $n_2$ such that for every $n \geq n_2$

$$P[\sup_{|y| < K_1 K_2} \left| \int_{-\infty}^{\infty} (e^{iyx/s_{N_n}} - 1 - iyx/((1 + x^2)s_{N_n})) d \sum_{j=1}^{N_n} F_j(x + b_j) - \right.\left. - \int_{-\infty}^{\infty} (e^{iyx} - 1 - iyx/(1 + x^2))(1 + x^2)/x dG(x) \right| > \varepsilon/9] \leq \varepsilon/9.$$
Thus for every $n \geq \max(n_1, n_2)$

$$\sup_{|t| < K_1} J_n(t) \leq \varepsilon/9 + 2\varepsilon/9 + 2\varepsilon/9 + 2\varepsilon/9 + 2\varepsilon/9 = \varepsilon,$$

which ends the proof. \hfill \Box

4. Proofs.

Proof of Theorem 1: At first we prove that (2.6) ⇒ (2.7). Let $\{U_n, n \geq 1\}$ be a sequence of independent random variables and independent of $\{N_n, n \geq 1\}$ and such that

$$\int e^{itx} \mathcal{L}(U_n)(dx) = \exp\{it(b_n + \int_{-\infty}^{\infty} x/(1 + x^2) \, F_n(dx + b_n)) + \int (e^{itx} - 1 - itx/(1 + x^2)) \, F_n(dx + b_n)\},$$

where

$$b_n = \int_{|x| < 1} x \, dF_n(x), \quad F_n(x) = P[X_n < x], \quad n \geq 1.$$  

By Lemma 1 we may and do assume that $\{X_n, n \geq 1\}$ and $\{N_n, n \geq 1\}$ are independent. Note that by Theorem 4 [7, Chapter IV, §2, p. 115] and Lemma 5, the assumptions of Lemma 7 hold. By Lemma 7 it is enough to prove that

$$I_n(t) = |E \exp\{it(V_{N_n} - L_n)/s_n\} - E \exp\{it(S_{N_n} - L_n)/s_n\}| \rightarrow 0, \quad \text{as} \quad n \rightarrow \infty,$$

uniformly on compact sets with respect to $t$, where

$$V_n = \sum_{j=1}^{n} U_j.$$  

Let $C$ and $\varepsilon$ be arbitrary positive numbers. Let $n_1 \in \mathbb{N}$ be such that

$$P[N_n < \alpha(n)] < \varepsilon/3,$$

for every $n \geq n_1$. Here, and in what follows, $\{\alpha(n), n \geq 1\}$ is defined in Lemma 5. By (2.6) we may put $C_\varepsilon$ such that

$$P[|s_{N_n}/s_n| > C_\varepsilon] \leq \varepsilon/3,$$

for every $n \geq n_1$. By (3.5) and (3.6) it is possible to choose $n_2 \in \mathbb{N}$ such that

$$\sup_{|u| < C_\varepsilon} \sup_{j : \alpha(n)} |E \exp\{iu(S_j - L_j)/s_j\} - E \exp\{iu(V_j - L_j)/s_j\}| < \varepsilon/3,$$
for every $n \geq n_2$. Thus

$$\sup_{|yt|<C} I_n(t) \leq \int_{0<x<C_\varepsilon} \sup_{|yt|<C_\varepsilon} \sup_{|t|<C_\varepsilon} \sup_{j:j>\alpha(n)} |E \exp\{itx(S_j - L_j)/s_j\} - E \exp\{itx(V_j - L_j)/s_j\}| + P[N_n < \alpha(n)] + P[s_{N_n}/s_n > C_\varepsilon] < \varepsilon, \quad \text{for } n > \max(n_1, n_2).$$

Since the left hand side of the above inequality is independent of $\varepsilon$, we have

$$\lim_{n \to \infty} \sup_{|t|<C} I_n(t) = 0.$$ 

Thus the proof that (2.6) $\Rightarrow$ (2.7) is ended.

Assume now that (2.7) holds. Then, by Lemma 3, the sequence $\{s_{N_n}/s_n, n \geq 1\}$ is tight. Moreover, by Lemma 1 and Lemma 4, the sequence $\{(Z_{N_n} - L_n)/s_n, n \geq 1\}$ and $\{(Z_{N_n} - L_{N_n})/s_n, n \geq 1\}$ are tight, too, where $\{Z_n, n \geq 1\}$ is the sequence defined in Lemma 1. Thus by Lemma 2 (a) the sequence $\{(L_{N_n} - L_n)/s_n, n \geq 1\}$ is also tight, so that Lemma 2 (c) implies the tightness of the sequence $\{(s_{N_n}/s_n, (L_{N_n} - L_n)/s_n), n \geq 1\}$. \hfill $\Box$

**Proof of Theorem 2:** The implication (2.14) $\Rightarrow$ (2.15) can be proved similarly as the implication (2.5), (2.6) $\Rightarrow$ (2.7). In this case, let $\{U_n, n \geq 1\}$ be a sequence of independent random variables and independent of $\{N_n, n \geq 1\}$ and such that $\mathcal{L}(U_n) = G_{\alpha_1}(-c_1, -c_2)$, $e_0, (c_1 + c_2, e_1) \in I$, $n \geq 1$, then

$$E \exp\{it\sum_{j=1}^{N_n} U_j - L_n)/s_n\} = E \exp\{-|t|^\alpha(s_{N_n}/s_n + i \text{sgn}(t)\omega(\alpha, \beta_N/s_n, t)) + it(L_{N_n} - L_n)/s_n\} = E \exp\{-|t|^\alpha(s_{N_n}/s_n + i \text{sgn}(t)(\beta_n/s_n)\omega(\alpha, 1, t)) - |t|^\alpha i \text{sgn}(t)(\beta_n/s_n)\omega(\alpha, 1, t) + it(L_{N_n} - L_n)/s_n\} \to$$

$$\int \int \int \exp\{-|t|^\alpha(x + i \text{sgn}(t)\omega(\alpha, \beta, t)) - |t|^\alpha i \text{sgn}(t)\omega(\alpha, 1, t)y + itz\} A(dx, dy, dz),$$

as $n \to \infty$. We omit further details.

The second part of Theorem 2 can also be obtained similarly as the second part of Theorem 1. Namely, as in Theorem 1, we prove that the sequence $\{(s_{N_n}/s_n, (L_{N_n} - L_n)/s_n), n \geq 1\}$ is tight. Thus the sequence $\{(s_{N_n}/s_n, (L_{N_n} - L_n)/s_n), n \geq 1\}$ is tight, too. Now (2.15) follows, if we show that the sequence $\{(\beta_n - \beta_n)/s_n, n \geq 1\}$ is tight. But this fact follows from the tightness of the sequence $\{s_{N_n}/s_n, n \geq 1\}$. Namely, we have

$$|\beta_n/s_n| \leq 1, \quad |\beta_n/s_{N_n}| \leq 1 \quad \text{a.s.}$$

and

$$|\beta_n - \beta_n|/s_n \leq s_{N_n}/s_n + 1 \quad \text{a.s.}$$
Hence the proof of Theorem 2 is completed. □

**Proof of Theorem 3:** The implication (2.21) $\Rightarrow$ (2.22) follows from the first part of Theorem 1 as the Gaussian law is the special case of Levy laws. The tightness of sequence defined on the left hand side of (2.21) follows from Theorem 1, too. Assume that

$$\left(\frac{s_{N_{n'}}}{s_{n'}}, \frac{L_{N_{n'}} - L_{n'}}{s_{n'}}\right) \xrightarrow{D} A'(\cdot, \cdot) \quad \text{as} \quad n' \to \infty$$

and

$$\left(\frac{s_{N_{n''}}}{s_{n''}}, \frac{L_{N_{n''}} - L_{n''}}{s_{n''}}\right) \xrightarrow{D} A''(\cdot, \cdot) \quad \text{as} \quad n'' \to \infty.$$

Then applying two times the implication (2.21) $\Rightarrow$ (2.22), which is already proved, we get

$$\hat{\Psi}(t) = \iint_{\mathbb{R}^2} \exp(-t^2 x/2 + ity) A'(dx, dy) = \iint_{\mathbb{R}^2} \exp(-t^2 x/2 + ity) A''(dx, dy).$$

By Lemma 6, $A' = A''$, which ends the proof of Theorem 3. □

Corollaries 1, 2 and 3 easily follow from Theorems 2 and 3, respectively. We note only that if

$$\left(\frac{N_n - n}{n^{1/\alpha}}\right) \xrightarrow{D} \text{(some)} \ A(\cdot), \quad \text{as} \quad n \to \infty, \quad 0 < \alpha < 2,$$

then

$$\frac{N_n}{n} \xrightarrow{P} 1, \quad \text{as} \quad n \to \infty.$$

**Acknowledgement.** The authors would like to thank the referee for his remarks and helpful comments.

**References**


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*(Received August 5, 1992, revised March 11, 1993)*