Short proofs of two theorems in topology

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Abstract. We present short and elementary proofs of the following two known theorems in General Topology:

(i) [H. Wicke and J. Worrell] A $T_1$ weakly $\delta\theta$-refinable countably compact space is compact.

(ii) [A. Ostaszewski] A compact Hausdorff space which is a countable union of metrizable spaces is sequential.

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Throughout this note, $\kappa$ denotes an infinite cardinal number and all topological spaces are assumed to be $T_1$.

A space $X$ is called $\kappa$-refinable if every open cover $\gamma$ of $X$ has an open refinement $\lambda$ such that $\lambda = \bigcup_{\alpha < \kappa} \lambda_{\alpha}$ and for each $x \in X$, there exists $\alpha < \kappa$ such that $1 \leq |\{V \in \lambda_{\alpha} : x \in V\}| \leq \kappa$. An example of a (hereditary) $\kappa$-refinable space is any space that can be represented as a union of $\leq \kappa$ metrizable subspaces.

The $\omega_0$-refinable spaces are the same as weakly $\delta\theta$-refinable spaces, the spaces introduced by H. Wicke and J. Worrell. In 1976 they proved that countably compact weakly $\delta\theta$-refinable spaces are compact [WW]. A slightly different proof of this theorem appears in [B]. See also [A] for a generalization of weak $\delta\theta$-refinability and yet another proof of this theorem. Below, we present a proof which is shorter and much more elementary than these proofs. Moreover, the theorem is more general than that of Wicke and Worrell's.

Recall that a topological space is called initially $\kappa$-compact if every open cover of it of cardinality $\leq \kappa$ has a finite subcover. Note that ‘initially $\omega_0$-compact’ is the same as ‘countably compact’. The reader is referred to [S] for a survey of initially $\kappa$-compact spaces.

**Theorem 1.** An initially $\kappa$-compact $\kappa$-refinable space is compact.

**Proof:** Assume the contrary, and let $X$ be an initially $\kappa$-compact $\kappa$-refinable space which is not compact. Let $\gamma$ be a maximal open cover of $X$ without a finite subcover. Let $\lambda = \bigcup_{\alpha < \kappa} \lambda_{\alpha}$ be an open refinement of $\gamma$ which witnesses the $\kappa$-refinability of $X$. For each $\alpha < \kappa$, and for each $x \in X$, let $\lambda_{\alpha}(x) = \{V \in \lambda_{\alpha} : x \in V\}$ and $X_{\alpha} = \{x \in X : 1 \leq |\lambda_{\alpha}(x)| \leq \kappa\}$. Then $X = \bigcup_{\alpha < \kappa} X_{\alpha}$. Since $X$ is initially $\kappa$-compact, there exists $\beta$ such that $X_{\beta}$ cannot be covered by $\kappa$ or less members.
of $\gamma$. Let $W = \bigcup \lambda_\beta$. Since $X_\beta \subseteq W$, $W \notin \gamma$. By the maximality of $\gamma$, there exists $U \in \gamma$ such that $X = W \cup U$. Then $X_\beta \setminus U$ cannot be covered by $\kappa$ or less members of $\gamma$.

By induction, we choose a sequence $x_1, x_2, \ldots$ of points in $X_\beta \setminus U$ as follows: let $x_1 \in X_\beta \setminus U$ be arbitrary. If $x_1, \ldots, x_n$ have already been chosen, then, since $|\bigcup_{i=1}^n \lambda_\beta(x_i)| \leq \kappa$, $X_\beta \setminus U$ is not contained in $\bigcup \left(\bigcup_{i=1}^n \lambda_\beta(x_i)\right)$. Choose $x_{n+1} \in (X_\beta \setminus U) \cup (\bigcup_{i=1}^n \lambda_\beta(x_i))$.

Let $S = \{x_1, x_2, \ldots\}$. Then $S \subseteq X \setminus U$ and, since $X \setminus U \subseteq W$, no point of $X \setminus U$ is a limit point of $S$. This is a contradiction, since $X \setminus U$ is countably compact. $\square$

A topological space $X$ is called sequential if every nonclosed subset $A$ of $X$ contains a sequence converging to a point in $X \setminus A$.

In [O], A. Ostaszewski proved that a countably compact regular space which can be represented as a union of countably many metrizable spaces is sequential. The proof consists of about four printed pages. Below, we present a short proof based on the Wicke-Worrell Theorem.

**Theorem 2.** A countably compact regular space which can be represented as a countable union of metrizable spaces is sequential (and compact).

**Proof:** Let $X$ be a countably compact regular space, and let $X = \bigcup_{i=1}^\infty X_i$, where each $X_i$ is metrizable. Let $A$ be a non-closed subset of $X$. Since $X$ is hereditary $\omega_0$-refinable (i.e. hereditarily weakly $\delta\theta$-refinable), $A$ cannot be countably compact. Therefore, there exists a sequence $S = \{x_1, x_2, \ldots\}$ in $A$ which has no cluster point in $A$. Let $Y = \overline{S} \setminus S$. Since $Y$ is non-empty and compact, $Y \cap X_i$ is not nowhere dense in $Y$ for some $i$. Hence, $Y \cap X_i$ contains a point which has countable character in $Y$ and thus in $\overline{S}$ as well. Therefore, $S$ contains a subsequence converging to a point in $Y$. $\square$

The last part of the above proof shows that any countably compact regular space which can be represented as a union of countably many first countable spaces contains a point of countable character. This is essentially the same as Theorem 3 of [O] attributed to M.E. Rudin and K. Kunen there. We have a much stronger theorem of this type which we prove by a different method.

**Theorem 3.** Let $X$ be a regular initially $\omega_1$-compact space which can be represented as a union of $\leq \omega_1$ subspaces of countable pseudocharacter. Then every non-empty $G_\delta$ subset of $X$ contains a point of countable character in $X$.

**Proof:** Let $X = \bigcup \{X_\alpha : \alpha < \omega_1\}$, where each $X_\alpha$ has countable pseudocharacter. Let $U$ be a non-empty $G_\delta$ subset of $X$ and suppose that no point of $U$ has countable character in $X$. By induction, we choose a decreasing sequence $\{F_\alpha : \alpha < \omega_1\}$ of non-empty closed $G_\delta$ subsets of $X$ as follows:

If $U \cap X_0 = \emptyset$, let $F_0$ be an arbitrary non-empty closed $G_\delta$ subset of $X$ such that $F_0 \subseteq U$. If $U \cap X_0 \neq \emptyset$, let $x \in U \cap X_0$. Then there exists a $G_\delta$ subset $V$ of $X$ such that $V \cap X_0 = \{x\}$. Since $X$ is countably compact and $x$ does not have countable character in $X$, $\{x\}$ is not $G_\delta$ in $X$. Therefore, $\emptyset \neq U \cap V \neq \{x\}$. Let
$F_0$ be a non-empty closed $G_δ$ subset of $X$ such that $F_0 \subseteq (U \cap V) \setminus \{x\}$. Then $F_0 \cap X_0 = \emptyset$.

If $\beta < \omega_1$, and for each $\alpha < \beta$, we have chosen $F_\alpha$, then, since $\bigcap_{\alpha < \beta} F_\alpha$ is a $G_δ$ subset of $X$, by repeating the above argument with $\bigcap_{\alpha < \beta} F_\alpha$ in place of $U$ and $X_\beta$ in place of $X_0$ we can find a non-empty closed $G_δ$ subset $F_\beta$ of $X$ such that $F_\beta \subseteq \bigcap_{\alpha < \beta} F_\alpha$ and $F_\beta \cap X_\beta = \emptyset$.

Let $F = \bigcap \{F_\alpha : \alpha < \omega_1\}$. Since $X$ is initially $\omega_1$-compact, $F \neq \emptyset$. On the other hand, since $F \cap X_\alpha = \emptyset$, for each $\alpha < \omega_1$, $F = \emptyset$. This is a contradiction. □

References


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