\textbf{$\in$-representation and set-prolongations}

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Abstract. By an $\in$-representation of a relation we mean its isomorphic embedding to $E = \{ \langle x, y \rangle; x \in y \}$. Some theorems on such a representation are presented. Especially, we prove a version of the well-known theorem on isomorphic representation of extensional and well-founded relations in $E$, which holds in Zermelo-Fraenkel set theory. This our version is in Zermelo-Fraenkel set theory false. A general theorem on a set-prolongation is proved; it enables us to solve the task of the representation in question.

Keywords: isomorphic representation, extensional relation, well-founded relation, set-prolongation

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We prove that, in the alternative set theory, each weakly extensional and well-founded set-relation is strongly $\in$-representable. It means that there exists a set-mapping which is an isomorphism of the relation in question and a subrelation of the relation $E = \{ \langle x, y \rangle; x \in y \}$. We present a general theorem on a set-prolongation. This theorem guarantees, to a given weakly extensional and well-founded relation, its set-superrelation with the same two properties. Thus the relation in question has an $\in$-representation. Consequently, each model with absolute equality of Zermelo-Fraenkel set theory is $\in$-representable. For countable models, this result was firstly proved by Vopěnka (unpublished).

Convention. We use the usual notation of the alternative set theory. We put, having a relation $R$, $\text{fld}(R) = \text{dom}(R) \cup \text{rng}(R)$. We denote the class of all finite subsets of a class $X$ as $P_f(X)$.

$\in$-representations of set-relations.

Let $R$ be a binary relation. We shall write $R(x)$ instead of $R''\{x\}$ and $R[y]$ instead of $R^{-1''}\{y\}$.

Convention. In this paper, let $R$ be a binary nonempty relation and let $0_R$ be an element from $\text{dom}(R) - \text{rng}(R)$.

We have, consequently, $R[0_R] = \emptyset$.

A mapping $H$ is said to be an $\in$-representation of $\langle R, 0_R \rangle$ if we have

1) $H : \text{fld}(R) \to V$ is a one-one mapping,
2) $x, y \in \text{fld}(R) \Rightarrow (\langle x, y \rangle \in R \iff H(x) \in H(y) \& H(0_R) = \emptyset)$.

An $\in$-representation $H$ is strong if we have, moreover,

3) $y \in \text{rng}(R) \Rightarrow H(y) = H''R[y]$,
4) $x \in \text{dom}(R) - \text{rng}(R) - \{0_R\} \Rightarrow H(x)$ is infinite.
We say that \( R \) is weakly extensional – formally \( \text{wex}(R) \) – if we have
\[
x, y \in \text{rng}(R) \; \& \; x \neq y \Rightarrow R[x] \neq R[y].
\]

\( R \) is said to be well-founded – formally \( \text{wf}(R) \) – if we have
\[
(\forall u \subseteq \text{fld}(R))(u \neq \emptyset \Rightarrow (\exists y \in u)(R[y] \cap u = \emptyset)).
\]

Note that having a nonempty well-founded set-relation \( r \), we can see that \( (\exists x \in \text{dom}(r) - \text{rng}(r))(r[x] = \emptyset) \). Especially, \( \text{dom}(r) - \text{rng}(r) \neq \emptyset \) holds.

**Theorem.** Let \( r \) be a set-relation, \( 0_r \in \text{dom}(r) - \text{rng}(r) \). Then \( r \) is weakly extensional and well-founded iff there exists a strong \( \in \)-representation of \( \langle r, 0_r \rangle \) which is a set.

**Proof:** The implication from the right to the left is easy. Suppose that \( r \) is weakly extensional and well-founded. Put \( v = \text{rng}(r) \) and \( w = \text{dom}(r) - \text{rng}(r) \). We have \( 0_R \in w \). We denote by \( \tau(x) \) the type of a set \( x \), i.e. \( \tau(x) = \min\{\alpha; x \in P_\alpha\} - 1 \), where \( P_0 = \emptyset \) and \( P_{\alpha+1} = P(\alpha) \). By an \( \in \)-chain of the length \( \delta \) we mean a set \( \{z_\alpha; 1 \leq \alpha \leq \delta\} \) such that we have \( z_\delta \in z_{\delta-1} \in \ldots \in z_1 \). We denote such a chain as \( z|\delta \). We say that \( z|\delta \) is under \( x \) if we have \( z_1 \in x \). We have for each \( \delta \geq 1 \): \( \tau(x) = \delta \) implies that there is an \( \in \)-chain of the length \( \delta \) which is under \( x \). Assume \( \gamma \geq 1 \). Suppose, moreover, that each \( \in \)-chain under \( x \) has the length less than \( \gamma \). Then \( \tau(x) < \gamma \).

Suppose that \( x \in N \) is such a number that we have

i) \( \theta > ||v|| \), where \( ||v|| \) is the set-cardinality of the set \( v \), i.e. \( ||v|| \in N \) and there exists a one-one set-mapping between \( v \) and \( ||v|| \),

ii) there exists a set \( \{e_x; \ x \in w - \{0_r\}\} \) such that each \( e_x \) is infinite, \( \tau(z) = \theta \) holds for each \( z \in e_x \) and we have, for each \( x, y \in w - \{0_r\} \), \( x \neq y \Rightarrow e_x \neq e_y \).

We define sets \( u_\alpha \) as follows: \( u_0 = w \), \( u_{\alpha+1} = \{x \in v; r[x] \subseteq u_\alpha\} \cup u \). We can see that \( u_\alpha \subseteq u_{\alpha+1} \) holds for each \( \alpha \). We have, moreover, a number \( \gamma \) such that \( \alpha \geq \gamma \Rightarrow u_\alpha = u_\gamma = v \cup w \).

We define, for each \( \alpha \), the mapping \( h_\alpha : u_\alpha \rightarrow V \) by the relations: \( h_0(0_r) = \emptyset \), \( h_0(x) = e_x \) for each \( x \in w - \{0_r\} \), \( h_{\alpha+1}(y) = h_\alpha''r[y] \) for each \( y \in u_{\alpha+1} - w (= u_{\alpha+1} \cap v) \), \( h_{\alpha+1}(y) = h_0(y) \) for each \( y \in w \). We can easily prove that, for each \( \alpha \), \( h_\alpha \subseteq h_{\alpha+1} \) holds.

Let us formulate two lemmas. We denote by \( \text{Univ}(x) \) the universe of the set \( x \).

**Lemma.** Assume that \( y \in \text{rng}(r) \cap u_\alpha \) and let \( \text{Univ}(h_\alpha(y)) \cap \{e_x; x \in w - \{0_r\}\} = \emptyset \). Then \( \tau(h_\alpha(y)) \leq ||\text{rng}(r)|| \) holds.

**Proof:** Let \( z|\delta \) be an \( \in \)-chain under \( h_\alpha(y) \). Let us prove that \( \delta \leq ||\text{rng}(r)|| \).

We shall write \( h \) instead of \( h_\alpha \). Thus we have \( z_\delta \in z_{\delta-1} \in \ldots \in z_1 \in h(y) \), where \( \{z_\alpha; 1 \leq \alpha \leq \delta\} = z|\delta \). We deduce from the fact \( h(y) = h''r[y] \) that there exists a set \( y_1 \) such that \( y_1 \in r[y] \) and \( z_1 = h(y) \). Suppose that \( r[y] \cap (w - \{0_r\}) \neq \emptyset \). Then \( e_x \in h(y) \) holds for some \( x \in w - \{0_r\} \). It follows from the formula \( x \in r[y] \cap (w - \{0_r\}) \).
\{0_r\} \Rightarrow h(x) = e_x = h(y). \text{ We deduce from this that } Univ(h(y)) \cap \{w - \{0_r\}\} \neq \emptyset, \text{ which is a contradiction.} \text{ Thus we have } r[y] \subseteq v \cup \{0_r\}. \text{ Assuming } y_1 = 0_r, \text{ we obtain that } z_1 = h(0_r) = \emptyset. \text{ Thus } \delta = 1. \text{ Suppose } \delta > 1. \text{ Then } y_1 \in v.

Assume that 1 \leq \beta \leq \delta \text{ and let } \{y_\alpha; 1 \leq \alpha \leq \beta\} \subseteq v \text{ be a set such that } y_\beta r y_{\beta-1} r \ldots r y_1 r y \text{ and let } h(y_\alpha) = z_\alpha \text{ for each } 1 \leq \alpha \leq \beta. We have } z_{\beta+1} \in h(y_{\beta}) = h''r[y_\beta]. \text{ Thus there exists a set } y_{\beta+1} \in r[y_\beta] \text{ such that } z_{\beta+1} = h(y_{\beta+1}). \text{ Assume that } y_{\beta+1} = 0_r. \text{ Then } z_{\beta+1} = h(0_r) = \emptyset \text{ and, consequently, } \beta + 1 = \delta \text{ holds. Assume } \beta + 1 < \delta. \text{ Then } y_{\beta+1} \in v. \text{ It follows from the fact that } y_{\beta+1} \in w - \{0_r\} \text{ implies } z_{\beta+1} \in \{e_x; x \in w - \{0_r\}\} \cap Univ(h(y)) \text{ which is a contradiction.}

Thus, there exists a set \{y_\alpha; 1 \leq \alpha < \delta\} \subseteq v \text{ such that } y_{\delta-1} r y_{\delta-2} r \ldots r y_1 r y \text{ holds. The relation } r \text{ is well-founded. We deduce from this that } \delta \leq ||v||. \text{ Thus each } \epsilon \text{-chain under } h(y) \text{ has the length less or equal to } ||v||. \text{ Consequently, } \tau(h(y)) \leq ||v|| \text{ holds.} \square

**Lemma.** Each mapping \(h_\alpha\) is a one-one mapping.

**Proof:** We shall prove it by induction on \(\alpha\). If \(\alpha = 0\) then the assertion holds. Assume that \(h_\alpha\) is a one-one mapping; we shall prove that \(h_{\alpha+1}\) has the same properties. Suppose that \(x, y \in u_{\alpha+1}\) are such that \(h_{\alpha+1}(x) = h_{\alpha+1}(y)\).

a) \(x, y \in w\). Then \(x = y\) follows directly from the definition of \(h_{\alpha+1}\).

b) \(x, y \in v\). Then \(h_{\alpha+1}''r[x] = h_{\alpha+1}(x) = h_{\alpha+1}(y) = h_{\alpha+1}''r[y]\). We deduce from the induction hypothesis that \(r[x] = r[y]\). The equality \(x = y\) follows from this by using the weak extensionality of \(r\).

c) \(x \in v, y \in w\). Assume, at first, that \(y = 0_r\). We have \(h_{\alpha+1}(y) = \emptyset, h_{\alpha+1}(x) = \emptyset\). But \(h_{\alpha+1}(x) = h_{\alpha+1}''r[x] \neq \emptyset\), which is a contradiction. Assume, secondly, that \(y \neq 0_r\). We have \(h_{\alpha+1}(x) = h_{\alpha+1}(y) = e_y\). Suppose that \(Univ(h_{\alpha+1}(x)) \cap \{e_z; z \in w - \{0_r\}\} \neq \emptyset\). Then \(\tau(h_{\alpha+1}(x)) > \tau(e_y)\), which is a contradiction. Suppose that \(Univ(h_{\alpha+1}(x)) \cap \{e_z; z \in w - \{0_r\}\} \neq \emptyset\). We deduce from this assumption and by using the previous lemma that \(\tau(h_{\alpha+1}(x)) \leq ||v|| < \tau(e_y)\), which is impossible. \square

Let us finish the proof of our theorem. Choose \(\delta\) such that \(u_\delta = v \cup w (= dom(r) \cup \text{rng}(r))\) and put \(u = u_\delta\) and \(h = h_\delta\). Now, we have the following: \(h\) is a one-one mapping such that \(x \in \text{rng}(r) \Rightarrow h(x) = h''r[x], x \in \text{dom}(r) - \text{rng}(r) - \{0_r\} \Rightarrow h(x)\) is infinite, \(h(0_r) = \emptyset\) and \(\langle x, y \rangle \in r \Rightarrow h(x) \in h(y)\). Thus, only the following must be proved:

\[ x, y \in \text{dom}(r) \cup \text{rng}(r) \Rightarrow (h(x) \in h(y) \Rightarrow \langle x, y \rangle \in r). \]

Suppose that \(x, y \in \text{dom}(r) \cup \text{rng}(r)\) and let \(h(x) \in h(y)\). We have \(y \neq 0_r\).

a) \(x, y \in w\). Then \(h(y) = e_y\) and, consequently, \(h(x) \in h(y)\) is false. (Indeed, we have \(h(x) = e_x\) or \(h(x) = 0_r\). But neither \(e_x \in e_y\) for some \(x, y \in w - \{0_r\}\) nor \(\emptyset \in e_y\) holds.)

\(\beta\) \(y \in v\). We have \(h(x) \in h''r[y] (= h(y))\). Thus \(h(x) = h(z)\) holds for some \(z \in r[y]\). The mapping \(h\) is a one-one. Consequently \(z = x\) is satisfied and we have \(\langle x, y \rangle \in r\).

\(\gamma\) \(x \in v, y \in w\). Suppose that

\(\ast\) \(Univ(h(x)) \cap \{e_z; z \in w - \{0_r\}\} \neq \emptyset\)
We deduce from this that \( \tau(h(x)) > \tau(h(y)) \). But it is a contradiction with our assumption that \( h(x) \in h(y) = e_y \). Suppose that \((*)\) is not true. We have \( \tau(h(x)) \leq ||v|| \). But the relation \( \tau(h(x)) = \theta \) follows from the assumption that \( h(x) \in e_y \). We have \( \theta > ||v|| \), which is a contradiction.

**Set-prolongation.**

Our aim is to present a method of a prolongation of a given class, say \( X \), to a set, say \( d \), such that \( X \subseteq d \) and the set \( d \) has some properties as \( X \). We see that this purpose is essentially limited by the fact that \( d \) is a formally finite set. Thus, only some properties of \( X \) can be transferred on \( d \).

We formulate a theorem on set-prolongation below. Before we give it, let us introduce one definition.

Let \( X \) be a class and let \( \Gamma \) be a class of set formulas of the language \( FL_V \) with exactly one free-variable \( x \). We say that \( \Gamma \) is an \( f \)-type over \( X \) if we have for each finite set \( \{ \varphi_1, \ldots, \varphi_k \} \subseteq \Gamma \) the following

\[
(\forall u \in P_f(X))(\exists v \in P_f(X))(u \subseteq v \land \varphi_1(v) \land \ldots \varphi_k(v)),
\]

where \( \varphi_i(v) \) denotes the formula which is obtained from \( \varphi \) by replacing all of the occurrence of the variable \( x \) by \( v \).

**Theorem (on set-prolongation).** Let \( \Gamma \) be an \( f \)-type over a class \( X \). Then there exists an endomorphism \( F \) and a set \( d \) such that we have:

1) \( F''X = F''V \cap d \).

2) If \( \varphi(x, p_1, p_2, \ldots, p_l) \in \Gamma \) and \( \varphi(x, x_1, x_2, \ldots, x_n) \) is a formula of the language \( FL \), then \( \varphi(d, F(p_1), F(p_2), \ldots, F(p_l)) \).

3) Let \( \varphi(x, p_1, p_2, \ldots, p_l) \) be a set-formula of the language \( FL_V \) with exactly one set-variable \( x \) and suppose that \( (\exists u \in P_f(X))(\forall v \in P_f(X))(u \subseteq v \Rightarrow \varphi(v, p_1, p_2, \ldots, p_l)) \). Then \( \varphi(d, F(p_1), F(p_2), \ldots, F(p_l)) \) holds.

**Proof:** We sketch a proof by using the notion of the coherency \([V]\) which states the following. Let \( M \) be an ultrafilter on the ring \( Sd_V \) of all set-theoretically definable classes. Then \( F, M, d \) are coherent if \( \{ x; \varphi(x, p_1, p_2, \ldots, p_l) \} \in M \iff \varphi(d, F(p_1), F(p_2), \ldots, F(p_l)) \) holds for each set-formula \( \varphi(x_0, p_1, p_2, \ldots, p_l) \) of \( FL_V \) with exactly one free-variable \( x_0 \) and such that \( p_1, p_2, \ldots, p_l \in \text{dom}(F) \).

Let

\[
M_0 = \{ x; \varphi(x, p_1, p_2, \ldots, p_l) \}; \varphi(x_0, p_1, p_2, \ldots, p_l) \in \Gamma \) or \( \varphi(x_0, p_1, p_2, \ldots, p_l) \) is a set-formula of \( FL_V \) with exactly one free-variable \( x_0 \) such that \( (\exists u \in P_f(X))(\forall v \in P_f(X))(u \subseteq v \Rightarrow \varphi(v, p_1, p_2, \ldots, p_l)) \}.
\]

Then \( M_0 \) is a centered system of set-theoretically definable classes. Let \( M \) be an ultrafilter on \( Sd_V \) such that \( M_0 \subseteq M \). There exists an endomorphism \( F \) and a set \( d \) such that \( F, M, d \) are coherent. It follows from the first theorem of Section 2, Chapter V in \([V]\). We can see that 2), 3) hold. Let us prove 1). We have
\[\{x; y \in x\} \in \mathcal{M} \iff y \in X \text{ and } \{x; y \in x\} \in \mathcal{M} \iff \mathcal{F}(y) \in d. \text{ Thus } \mathcal{F}(y) \in d \iff y \in X \text{ holds.} \]

\[\varepsilon\text{-representations.}\]

We say that a binary relation \(R\) is without cycles if there is no sequence \(\{x_1, x_2, \ldots, x_n\} \subseteq \text{fld}(R)\) such that \(x_1 R x_n R x_{n-1} \ldots R x_1\) holds.

**Theorem.** Let \(R\) be a weakly extensional relation without cycles and let \(0_R \in \text{dom}(R) - \text{rng}(R)\). Then we have:

1) There exist a relation \(S\) and \(0_S\) such that \(\langle R, 0_R \rangle\) is isomorphic to \(\langle S, 0_S \rangle\) and there exists a weakly extensional and well-founded set-relation \(r\) such that \(S \subseteq r\) and \(0_S \in \text{dom}(r) - \text{rng}(r)\).

2) There exists a class \(K\) such that \(\emptyset \in K\) and \(\langle \text{fld}(R), R, 0_R \rangle\) is isomorphic to \(\langle K, \varepsilon \cap K^2, \emptyset \rangle\).

**Proof:** Let us prove, at first, that \(\{\text{wex}(x), \text{wf}(x)\}\) is an \(f\)-type over \(R\). Assume that \(s \subseteq R\) is finite. It is easy to see that \(s\) is well-founded. We must find a finite weakly-extensional relation \(r\) such that \(s \subseteq r \subseteq R\). Put \(v = \text{rng}(s)\) and, for each \(\{x, y\} \in [v]^2\), let \(d_{xy} = \triangle(R[x], R[y])\), where \(\triangle\) is the symmetric difference. Put \(r = s \cup \{(d_{xy}, x) \in R; \{x, y\} \in [v]^2\}\). We have \(\text{rng}(r) = v\) and \(\{x, y\} \in [v]^2\) implies \(d_{xy} = \triangle(r[x], r[y])\). Thus \(r\) is weakly extensional.

We can easily see that \(\{x; (\exists y, z)(x = \{1\} \times y \cup \{2\} \times z \& \text{wex}(y) \& \text{wf}(y) \& z \in \text{dom}(y) - \text{rng}(y))\}\) is an \(f\)-type over \(\{1\} \times R \cup \{2\} \times \{0_R\}\).

Now, we deduce from the previous theorem that there exist an endomorphism \(F\), a set-relation \(r\) and a set \(e\) such that \(F''(\{1\} \times R \cup \{2\} \times \{0_R\}) = F'' \triangle \{\{1\} \times r \cup \{2\} \times \{e\}\}\). Put \(S = F''R\). We have \(\langle x, y \rangle \in R \iff \langle F(x), F(y) \rangle \in S\), i.e. \(F\) is an isomorphism of \(R\) and \(S\). Put \(0_S = F(0_R)\). We have \(0_S \in \text{dom}(F''R) - \text{rng}(F''R)\). Thus \(\langle S, 0_S \rangle\) has the required properties.

2) We know that there exists a strong \(\varepsilon\)-representation \(h\) of \(\langle r, 0_S \rangle\). Let us define a mapping \(H : \text{fld}(R) \to V\) by \(H(x) = h(F(x))\) and put \(K = H'' \text{fld}(R)\). Then \(H\) is an isomorphism of \(R\) and \(\varepsilon \cap K^2\). We have, moreover, \(H(0_R) = h(0_S) = \emptyset\).

**Corollary.** Let \(\langle A, R \rangle\) be a model of \(ZF\) with absolute equality and let \(0_R \in A\) be such that \(\langle A, R \rangle \models "0_R\text{ is the empty set}"\). Then there exists a class \(M\) such that the structures \(\langle A, R, 0_R \rangle\) and \(\langle M, \varepsilon \cap M^2, \emptyset \rangle\) are isomorphic.

**Proof:** It is clear that \(R\) is an extensional relation and, consequently, weakly extensional one. \(R\) is without cycles, too. We have \(\text{dom}(R) - \text{rng}(R) = \{0_R\}\). We deduce from the previous theorem that there exists a class \(M\) with the required properties.

**Note:** The just presented assertion can be strengthened. We can find the class \(M\) in question such that, in addition, some gödelian operations are absolute for the model \(\langle M, \varepsilon \cap M^2, \emptyset \rangle\). Naturally, the transitivity of \(M\) cannot be guaranteed.

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