The product of distributions on $\mathbb{R}^m$

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Abstract. The fixed infinitely differentiable function $\rho(x)$ is such that \{n$\rho(nx)$\} is a regular sequence converging to the Dirac delta function $\delta$. The function $\delta_n(x)$, with $x = (x_1,\ldots,x_m)$ is defined by

$$\delta_n(x) = n_1\rho(n_1 x_1) \cdots n_m\rho(n_m x_m).$$

The product $f \circ g$ of two distributions $f$ and $g$ in $\mathcal{D}'^m$ is the distribution $h$ defined by

$$N\lim_{n_1 \to \infty} \ldots N\lim_{n_m \to \infty} \langle f_n g_n, \phi \rangle = \langle h, \phi \rangle,$$

provided this neutrix limit exists for all $\phi(x) = \phi_1(x_1) \cdots \phi_m(x_m)$, where $f_n = f * \delta_n$ and $g_n = g * \delta_n$.

Keywords: distribution, neutrix limit, neutrix product

Classification: 46F10

A commutative product of two distributions in $\mathcal{D}'^m$, the space of distributions defined on $\mathcal{D}_m$, the space of infinitely differentiable functions in $m$ variables with compact support, was considered in [1] and a non-commutative product of two distributions in $\mathcal{D}'_m$ was considered in [6]. In the following we are going to consider a commutative product of two distributions in $\mathcal{D}'_m$ which is similar to that given in [1] but simpler to deal with.

First of all we let $\rho$ be a fixed infinitely differentiable function with the properties

(i) $\rho(x) = 0$, $|x| \geq 1$,
(ii) $\rho(x) \geq 0$,
(iii) $\rho(x) = \rho(-x)$,
(iv) $\int_{-1}^{1} \rho(x) \, dx = 1.$

The function $\delta_n$ is defined by $\delta_n(x) = n\rho(nx)$ for $n = 1,2,\ldots$. It is obvious that \{\delta_n\} is a sequence of functions in $\mathcal{D}$ converging to the Dirac $\delta$ function $\delta$.

For an arbitrary distribution $f$ in $\mathcal{D}'$ the function $f_n$ is defined by

$$f_n(x) = (f * \delta_n)(x) = \langle f(x-t), \delta_n(t) \rangle.$$

It follows that \{f_n\} is a sequence of infinitely differentiable functions converging to the distribution $f$.

The following definition for the product of two distributions in $\mathcal{D}'$ was given in [3]:
Definition 1. Let \( f \) and \( g \) be distributions in \( \mathcal{D}' \) and let \( f_n = f * \delta_n \) and \( g_n = g * \delta_n \). The product \( f \cdot g \) is said to exist and be equal to the distribution \( h \) on the open interval \( (a, b) \), where \(-\infty \leq a \leq b \leq \infty \), if and only if
\[
\langle f \cdot g, \phi \rangle = \lim_{n \to \infty} \langle f_n g_n, \phi \rangle = \langle h, \phi \rangle,
\]
for all \( \phi \) in \( \mathcal{D}(a, b) \).

This definition generalizes the usual definition of a product of a distribution and an infinitely differentiable function or of a product of a distribution and a sufficiently smooth function and is clearly commutative.

The next definition for the neutrix product \( f \circ g \) of two distributions \( f \) and \( g \) in \( \mathcal{D}' \) was given in [5].

Definition 2. Let \( f \) and \( g \) be distributions in \( \mathcal{D}' \) and let \( f_n = f * \delta_n \) and \( g_n = g * \delta_n \). The neutrix product \( f \circ g \) of \( f \) and \( g \) is said to exist and be equal to \( h \) on the open interval \( (a, b) \), if and only if
\[
\text{N–lim}_{n \to \infty} \langle f_n g_n, \phi \rangle = \langle h, \phi \rangle,
\]
for all \( \phi \) in \( \mathcal{D}(a, b) \), where \( \text{N} \) is the neutrix, see van der Corput [2], having domain \( \text{N}' = \{1, 2, \ldots, n, \ldots\} \) and the range \( \text{N}'' \) the real numbers, with negligible functions finite linear sums of the functions
\[
n^\lambda \ln^{r-1} n, \quad \ln^r n
\]
for \( \lambda > 0 \) and \( r = 1, 2, \ldots \) and all functions which converge to zero in the normal sense as \( n \) tends to infinity.

Note that if
\[
\lim_{n \to \infty} \langle f_n g_n, \phi \rangle = \langle h, \phi \rangle
\]
for all \( \phi \) in \( \mathcal{D}(a, b) \), the neutrix product \( f \circ g \) reduces to the product \( f \cdot g \) of Definition 1 and so Definition 2 is a generalization of Definition 1. It is clear that the neutrix product \( f \circ g \) is commutative.

The following theorem holds, see [1].

Theorem 1. Let \( f \) and \( g \) be distributions in \( \mathcal{D}' \) and suppose that the neutrix products \( f \circ g \) and \( f \circ g' \) (or \( f' \circ g \)) exist on the open interval \( (a, b) \). Then the neutrix product \( f' \circ g \) (or \( f \circ g' \)) exists and
\[
(f \circ g)' = f' \circ g + f \circ g'
\]
on this interval.

In order to define a neutrix product \( f \circ g \) of two distributions \( f \) and \( g \) in \( \mathcal{D}'_m \), a \( \delta \)-sequence in \( \mathcal{D}_m \) was defined in [1] by
\[
\delta_n(x) = \delta_n(x_1, \ldots, x_m) = n^m \rho(nx_1) \cdots \rho(nx_m)
\]
for \( n = 1, 2, \ldots \). It is obvious that \( \{ \delta_n \} \) is a sequence of infinitely differentiable functions converging to \( \delta \) in the sense that
\[
\lim_{n \to \infty} \langle \delta_n(x), \phi(x) \rangle = \langle \delta(x), \phi(x) \rangle = \phi(0)
\]
for all test functions \( \phi \) in \( D_m \).

In the following, we use an alternative definition of a \( \delta \)-sequence, which is easier to work with. From now on the function \( \delta_n(x) \) will be defined by
\[
\delta_n(x) = n_1 \rho(n_1 x_1) \ldots n_m \rho(n_m x_m)
\]
for \( n_1, \ldots, n_m = 1, 2, \ldots \), where \( n = (n_1, \ldots, n_m) \). It is obvious that \( \{ \delta_n \} \) is a sequence of infinitely differentiable functions converging to \( \delta \) in the sense that
\[
\lim_{n_1 \to \infty} \ldots \lim_{n_m \to \infty} \langle \delta_n(x), \phi(x) \rangle = \langle \delta(x), \phi(x) \rangle = \phi(0)
\]
for all test functions \( \phi \) in \( D_m \), the result being independent of the order in which the limits are taken.

For an arbitrary distribution \( f \) in \( D_m' \) the function \( f_n \) is defined by
\[
f_n(x) = (f * \delta_n)(x) = \langle f(x - t), \delta_n(t) \rangle
\]
where \( t \) is in \( R^m \). It follows that \( \{ f_n \} \) is a sequence of infinitely differentiable functions converging to \( f \), in the sense that
\[
\lim_{n_1 \to \infty} \ldots \lim_{n_m \to \infty} \langle f_n(x), \phi(x) \rangle = \langle f(x), \phi(x) \rangle
\]
for all \( \phi \) in \( D_m \), the result again being independent of the order in which the limits are taken.

For our next definition and our main results we need the following lemmas, see Schwartz [7].

**Lemma 1.** The vector space \( X_m \) generated by the functions \( \phi_1(x_1) \ldots \phi_m(x_m) \), with \( \phi_1, \ldots, \phi_m \) in \( D \), is dense in \( D_m \).

**Lemma 2.** The convolution product of two direct products \( f_1(x) \times g_1(y) \) and \( f_2(x) \times g_2(y) \) is equal to the direct product of the convolution products \( f_1 * f_2 \) and \( g_1 * g_2 \), if the convolution products \( f_1 * f_2 \) and \( g_1 * g_2 \) exist, where \( f_1, f_2 \in D_m' \) and \( g_1, g_2 \in D_r' \), i.e.
\[
(f_1 \times g_1) * (f_2 \times g_2) = (f_1 * f_2) \times (g_1 * g_2).
\]

We also need the following lemma, see [4].
Lemma 3.\[\int_t^{1/n} s^k \delta_n^{(q)}(s) \, ds = \sum_{i=0}^{k} \frac{(-1)^{k+i+1} k!}{i!} t^i \delta_n^{(q-k+i-1)}(t)\]

for $k = 0, 1, 2, \ldots, q - 1$ and $q = 1, 2, \ldots$ and

\[
\int_t^{1/n} s^q \delta_n^{(q)}(s) \, ds = \sum_{i=1}^{q} \frac{(-1)^{q+i+1} q!}{i!} t^i \delta_n^{(i-1)}(t) + (-1)^q q! [1 - H_n(t)],
\]

for $q = 1, 2, \ldots$, where

\[H_n(t) = \int_{-1/n}^{t} \delta_n(s) \, ds.\]

The next definition is a generalization of Definition 2.

Definition 3. Let $f$ and $g$ be distributions in $\mathcal{D}'_m$ and let $f_n = f \ast \delta_n$ and $g_n = g \ast \delta_n$.

If $h$ is a distribution in $\mathcal{D}'_m$ such that

\[
\lim_{n_1 \to \infty} \ldots \lim_{n_m \to \infty} \langle f_n g_n, \phi \rangle = \langle h, \phi \rangle,
\]

or more briefly

\[
\lim_{n \to \infty} \langle f_n g_n, \phi \rangle = \langle h, \phi \rangle
\]

for all functions $\phi$ in $\mathcal{X}_m$ with support contained in the interval $(a, b)$, where $a = (a_1, \ldots, a_m)$ and $b = (b_1, \ldots, b_m)$, and $h$ is independent of the order in which the limits are taken, we say that the neutrix product $f \circ g$ exists and is equal to $h$ on $(a, b)$.

Note that if

\[
\lim_{n \to \infty} \langle f_n g_n, \phi \rangle = \langle h, \phi \rangle
\]

for all $\phi$ in $\mathcal{X}_m$, we simply say that the product $f \circ g = f \cdot g$ exists and equals $h$.

Note further that since $\mathcal{X}_m$ is dense in $\mathcal{D}_m$, the distribution $h$ in this definition will be uniquely defined.

The proof of Theorem 1 can be modified to give the following theorem.

Theorem 2. Let $f$ and $g$ be distributions in $\mathcal{D}'_m$ and suppose that the neutrix products $f \circ g$ and $f \circ D_i g$ (or $D_i f \circ g$) exist on the open interval $(a, b)$. Then the neutrix product $D_i f \circ g$ (or $f \circ D_i g$) exists and

\[D_i(f \circ g) = D_i f \circ g + f \circ D_i g\]

on this interval, where $D_i$ denotes the partial derivative with respect to $x_i$.\]
**Theorem 3.** Let $f$ and $g$ be distributions in $\mathcal{D}'_m$ such that
\[
f(x) = f_1(x_1) \times \cdots \times f_m(x_m), \quad g(x) = g_1(x_1) \times \cdots \times g_m(x_m),
\]
with $f_1, \ldots, f_m, g_1, \ldots, g_m \in \mathcal{D}'$, and suppose that the neutrix products $f_1 \circ g_1, \ldots, f_m \circ g_m$ exist and equal $h_1, \ldots, h_m$ respectively. Then the neutrix product $f \circ g$ exists and
\[
f \circ g = h_1 \times \cdots \times h_m.
\]
In particular, if the products $f_1 \cdot g_1, \ldots, f_m \cdot g_m$ exist, then the product $f \cdot g$ exists and is equal to $h_1 \times \cdots \times h_m$.

**Proof:** Putting
\[
f_{in_i}(x_i) = f_i(x_i) * \delta_{n_i}(x_i), \quad g_{in_i}(x_i) = g_i(x_i) * \delta_{n_i}(x_i),
\]
for $i = 1, \ldots, m$ and
\[
f_n(x) = f_{1n_1}(x_1) \times \cdots \times f_{mn_m}(x_m) = f(x) * \delta_n(x),
\]
\[
g_n(x) = g_{1n_1}(x_1) \times \cdots \times g_{mn_m}(x_m) = g(x) * \delta_n(x),
\]
we have on applying Lemma 2
\[
\langle f_n(x)g_n(x), \phi_1(x_1) \cdots \phi_m(x_m) \rangle = \prod_{i=1}^{m} \langle f_{in_i}(x_i), g_{in_i}(x_i)\phi_i(x_i) \rangle
\]
for all $\phi_1, \ldots, \phi_m$. Now since the neutrix product $f_i \circ g_i$ exists and equals $h_i$, it follows that
\[
\lim_{n \to \infty} \langle f_n(x)g_n(x), \phi_i(x_i) \cdots \phi_m(x_m) \rangle = \prod_{i=1}^{m} \lim_{n \to \infty} \langle f_{in_i}(x_i), g_{in_i}(x_i)\phi_i(x_i) \rangle
\]
\[
= \prod_{i=1}^{m} \lim_{n_i \to \infty} \langle h_i, \phi_i \rangle
\]
\[
= \langle h_1 \times \cdots \times h_m, \phi_1 \cdots \phi_m \rangle.
\]
The result of the theorem follows. \qed

If now
\[
\lambda = (\lambda_1, \ldots, \lambda_m), \quad \lambda_1, \ldots, \lambda_m \neq 0, \pm 1, \pm 2, \ldots,
\]
\[
r = (r_1, \ldots, r_m), \quad r_1, \ldots, r_m = 0, 1, 2, \ldots,
\]
we define
\[
cosec(\pi \lambda) = cosec(\pi \lambda_1) \cdots cosec(\pi \lambda_m),
\]
\[
(-1)^r = (-1)^{r_1 + \cdots + r_m}, \quad r! = r_1! \cdots r_m!.
\]
\[
x_\lambda^+ = (x_1)^{\lambda_1 +} \times \cdots \times (x_m)^{\lambda_m +}, \quad x_\lambda^- = (-x_\lambda)^+,
\]
\[
x_\lambda^r = (x_1)^{r_1 +} \times \cdots \times (x_m)^{r_m +}, \quad x_\lambda^- = (-x_\lambda)^r,
\]
\[
\delta^r(x) = \delta(r_1)(x_1) \times \cdots \times \delta(r_m)(x_m).
\]
We then have
Theorem 4. The neutrix products \( x_+^\lambda \circ x_-^{\lambda-r} \) and \( x_-^{\lambda-r} \circ x_+^\lambda \) exist in \( D'_m \) and
\[
x_+^\lambda \circ x_-^{\lambda-r} = x_-^{\lambda-r} \circ x_+^\lambda = \frac{(-\pi)^m \csc(\pi \lambda)}{2^m (r-1)!} \delta(r-1)(x),
\]
for \( \lambda_1, \ldots, \lambda_m \neq \pm 1, \pm 2, \ldots \) and \( r_1, \ldots, r_m = 1, 2, \ldots, \) where
\[
r - 1 = (r_1 - 1, \ldots, r_m - 1).
\]

In particular, the products \( x_+^\lambda \cdot x_-^{\lambda-1} \) and \( x_-^{\lambda-1} \cdot x_+^\lambda \) exist in \( D'_m \) for \( \lambda_1, \ldots, \lambda_m \neq 0, \pm 1, \pm 2, \ldots. \)

Proof: In the one variable case, suppose first of all that \( \lambda > -1 \) and choose a non-negative integer \( q \) such that \( -\lambda - r + q > -1. \) Then
\[
(x_+^\lambda)_n = x_+^\lambda \ast \delta_n = \int_{-1/n}^x (x-t)^\lambda \delta_n(t) \, dt,
\]
\[
(x_-^{\lambda-r})_n = x_-^{\lambda-r} \ast \delta_n = \frac{\Gamma(\lambda + r - q)}{\Gamma(\lambda + r)} \int_x^{1/n} (s-x)^{-\lambda-r+q} \delta_n^q(s) \, ds,
\]
where \( \Gamma \) denotes the Gamma function. The support of \((x_+^\lambda)_n(x_-^{\lambda-r})_n\) is clearly contained in the interval \((-1/n, 1/n)\) and it follows that
\[
\frac{\Gamma(\lambda + r)}{\Gamma(\lambda + r - q)} \int_{-1/n}^{1/n} (x_+^\lambda)_n(x_-^{\lambda-r})_n x^k \, dx =
\]
\[
= \int_{-1/n}^{1/n} \delta_n(t) \int_t^1 \delta_n^q(s) \int_s^t x^k(x-t)^\lambda (s-x)^{-\lambda-r+q} \, ds \, dt =
\]
\[
= n^{r-k-1} \int_{-1}^1 \rho(u) \int_u^1 \rho^q(v) \int_v^u w^k(w-u)^\lambda (v-w)^{-\lambda-r+q} \, dw \, dv \, du,
\]
where the substitutions \( nt = u, ns = v \) and \( nx = w \) have been made. Thus
\[
(2) \quad \lim_{n \to \infty} \int_{-1/n}^{1/n} (x_+^\lambda)_n(x_-^{\lambda-r})_n x^k \, dx = 0
\]
for \( k = 0, 1, 2, \ldots, r-2 \) and
\[
(3) \quad \lim_{n \to \infty} \int_{-1/n}^{1/n} \left| (x_+^\lambda)_n(x_-^{\lambda-r})_n x^r \right| \, dx = 0
\]
In the particular case \( k = r-1 \), we have on making the substitution \( x = t(1-y) + sy \)
\[
(4) \quad \int_{-1/n}^{1/n} \delta_n(t) \int_t^1 \delta_n^q(s) \int_s^{t(t-1)} x_{r-1}(x-t)^\lambda (s-x)^{-\lambda-r+q} \, dx \, ds \, dt =
\]
\[
= \int_{-1/n}^{1/n} \delta_n(t) \int_t^1 \delta_n^q(s) \int_0^1 (s-t)^{q-r+1} [t(1-y) + sy]^{r-1} y^\lambda (1-y)^{-\lambda-r+q} \, dy \, ds \, dt.
\]
On expanding \((s - t)^{q-r+1}\) and \([t(1 - y) + sy]^{r-1}\) in powers of \(s\) and \(t\), it follows that this integral is a linear sum of integrals of the form

\[
\int_{-1/n}^{1/n} t^{q-k} \delta_n(t) \int_{t}^{1/n} s^k \delta_n(s) \, ds \, dt
\]

for \(k = 0, 1, \ldots q\).

On using Lemma 3, we see that when \(k < q\) each of these integrals is a linear sum of integrals of the form

\[
\int_{-1/n}^{1/n} t^{q-k+1} \delta_n(t) \delta_n(q-k+i-1)(t) \, dt = 0,
\]

since the integrands are all odd functions.

When \(k = q\) we have on using Lemma 3 again

\[
\int_{-1/n}^{1/n} \delta_n(t) \int_{t}^{1/n} s^q \delta_n(s) \, ds \, dt = \sum_{i=1}^{q} \frac{(-1)^{q+i+1}q!}{i!} \int_{-1/n}^{1/n} t^i \delta_n(t) \delta_n(i-1)(t) \, dt + \left(-1 \right)^{q!} \int_{-1/n}^{1/n} [1 - H_n(t)] \delta_n(t) \, dt
\]

\[
= 0 + \frac{(-1)^{q}q!}{2}.
\]

It now follows from the equations (1) and (4) that

\[
\frac{\Gamma(\lambda + r)}{\Gamma(\lambda + r - q)} \int_{-1/n}^{1/n} (x_+^\lambda)_n(x_-^{\lambda-r})_n x^{r-1} \, dx = \frac{(-1)^{q}q!}{2} \int_{0}^{1} y^{\lambda+r-1}(1 - y)^{-\lambda-r+q} \, dy
\]

\[
= \frac{(-1)^{q}q!}{2} B(\lambda + r, -\lambda - r + q + 1)
\]

\[
= (-1)^{q}q! \Gamma(\lambda + r) \Gamma(-\lambda - r + q + 1)/2
\]

for \(\lambda \neq 0, 1, 2, \ldots\), where \(B\) denotes the Beta function, and so

\[
\int_{-1/n}^{1/n} (x_+^\lambda)_n(x_-^{\lambda-r})_n x^{r-1} \, dx = (-1)^{q}q! \Gamma(\lambda + r) \Gamma(-\lambda - r + q + 1)/2
\]

\[
= (-1)^{q} \pi \csc(\pi(\lambda + r - q)/2
\]

\[
= (-1)^{r} \pi \csc(\pi\lambda)/2.
\]

Now let \(\phi\) be an arbitrary function in \(\mathcal{D}\). Then we can write

\[
\phi(x) = \sum_{k=0}^{r-1} \frac{\phi^{(k)}(0)}{k!} x^k + \frac{\phi^{(r)}(\xi x)}{r!} x^r,
\]
where $0 \leq \xi \leq 1$. Thus

$$
\langle (x^\lambda_+) n(x^{-\lambda-r})_n, \phi(x) \rangle = \sum_{k=0}^{r-1} \frac{\phi^{(k)}(0)}{k!} \int_{-1/n}^{1/n} (x^\lambda_+) n(x^{-\lambda-r})_n x^k \, dx + \frac{1}{r!} \phi^{(r)}(\xi x)(x^\lambda_+) n(x^{-\lambda-r})_n x^r \, dx
$$

and it follows from the equations (2), (3) and (5) that

$$
\lim_{n \to \infty} \langle (x^\lambda_+) n(x^{-\lambda-r})_n, \phi(x) \rangle = (-1)^r \csc(\pi \lambda) \frac{2(r-1)!}{\phi^{(r-1)}(0)},
$$

proving that

$$
x^\lambda_+ \circ x^{-\lambda-r} = \frac{\pi \csc(\pi \lambda)}{2(r-1)!} \delta^{(r-1)}(x)
$$

for $\lambda > -1, \lambda \neq 0, 1, 2, \ldots$ and $r = 1, 2, \ldots$. Note that in the case $r = 1$ the neutrix limit is not needed and so the product $x^\lambda_+ \cdot x^{-\lambda-r}$ exists in this case.

Also note that in the case $r = 0$, the above proof shows that the product $x^\lambda_+ \cdot x^{-\lambda}$ exists and

$$
x^\lambda_+ \cdot x^{-\lambda} = 0
$$

for $\lambda > -1$ and $\lambda \neq 0, 1, 2, \ldots$.

A routine induction proof using the equations (6) and (7) and Theorem 2 now shows that equation (6) holds for $\lambda \neq 0, \pm 1, \pm 2, \ldots$ and $r = 1, 2, \ldots$, the product existing in the case $r = 1$.

Replacing $x$ by $-x$ and $\lambda$ by $-\lambda - r$ in the equation (6) proves that

$$
x^{-\lambda-r} \circ x^\lambda_+ = \frac{\pi \csc(\pi \lambda)}{2(r-1)!} \delta^{(r-1)}(x)
$$

for $\lambda \neq 0, \pm 1, \pm 2, \ldots$ and $r = 1, 2, \ldots$, the product existing in the case $r = 1$.

The results of the theorem now follows immediately on using Theorem 3 and the equations (6) and (8).

**Theorem 5.** The neutrix product $x^r_+ \circ \delta^{(r+p)}(x)$ exists in $\mathcal{D}'_m$ and

$$
x^r_+ \circ \delta^{(r+p)}(x) = \frac{(-1)^r (r+p)!}{2^m p!} \delta^{(p)}(x),
$$

for $r_1, p_1, \ldots, r_m, p_m = 0, 1, 2, \ldots$. In particular, the product $x^r_+ \cdot \delta^{(r)}(x)$ exists in $\mathcal{D}'_m$ for $r_1, \ldots, r_m = 0, 1, 2, \ldots$.

**Proof:** In the one variable case we have

$$
(x^r_+)_n = \int_{-1/n}^{x} (x-t)^r \delta_n(t) \, dt.
$$
The support of \((x^r_+)_n \delta_n^{(r+p)}\) is clearly contained in the interval \((-1/n, 1/n)\) and it follows that
\[
\int_{-1/n}^{1/n} (x^r_+) \delta_n^{(r+p)}(x) x^k \, dx = \int_{-1/n}^{1/n} \delta_n(t) \int_t^{1/n} x^k(x-t)^r \delta_n^{(r+p)}(x) \, dx \, dt
\]
\[
= n^{p-k} \int_{-1}^1 \rho(u) \int_{u}^{1/n} v^k(v-u)^r \rho^{(r+p)}(v) \, dv \, du,
\]
where the substitutions \(nt = u\) and \(nx = v\) have been made. Thus
\[
\text{N-}\lim_{n \to \infty} \int_{-1/n}^{1/n} (x^r_+) \delta_n^{(r+p)}(x) x^k \, dx = 0
\]
for \(k = 0, 1, 2, \ldots, p-1\) and
\[
\lim_{n \to \infty} \int_{-1/n}^{1/n} |(x^r_+) \delta_n^{(r+p)}(x) x^{p+1}| \, dx = 0.
\]

In this particular case \(k = p\) we have from the equation (10)
\[
\int_{-1/n}^{1/n} (x^r_+) \delta_n^{(r+p)}(x) x^p \, dx = \int_{-1/n}^{1/n} \delta_n(t) \int_t^{1/n} x^p(x-t)^r \delta_n^{(r+p)}(x) \, dx \, dt
\]
\[
= \int_{-1/n}^{1/n} \delta_n(t) \int_t^{1/n} x^{r+p} \delta_n^{(r+p)}(x) \, dx \, dt,
\]
all other integrals in the sum, obtained by expanding \((x-t)^r\) by the binomial theorem, being zero by Lemma 3. On using Lemma 3 again, it now follows that
\[
\int_{-1/n}^{1/n} (x^r_+) \delta_n^{(r+p)}(x) x^p \, dx = (-1)^{r+p}(r+p)! \int_{-1/n}^{1/n} \delta_n(t)[1 - H_n(t)] \, dt
\]
\[
= (-1)^{r+p}(r+p)!/2,
\]

Now let \(\phi\) be an arbitrary function in \(D\). Then on using the equations (11), (12) and (13), it follows as in the proof of Theorem 4 that
\[
\text{N-}\lim_{n \to \infty} \langle (x^r_+) \delta_n^{(r+p)}(x) \phi(x) \rangle \, dx = \frac{(-1)^{r+p}(r+p)!}{2p!} \phi(p)(0),
\]
proving that
\[
x^r_+ \delta^{(r+p)}(x) = \frac{(-1)^{r+p}(r+p)!}{2p!} \delta^{(r+p)}(x)
\]
for \(r, p = 0, 1, 2, \ldots\). Note that in the case \(p = 0\) the neutrix limit is not needed and so the product \(x^r_+ \delta^{(r)}(x)\) exists in this case.

The result of the theorem now follows on using Theorem 3 and the equation (14).
Corollary. The neutrix product \( x^r \circ \delta^{(r+p)}(x) \) exists in \( \mathcal{D}'_m \) and

\[
x^r_\circ \delta^{(r+p)}(x) = \frac{(-1)^r(r+p)!}{2^m p!} \delta^{(p)}(x),
\]

for \( r_1, p_1, \ldots, r_m, p_m = 0, 1, 2, \ldots \). In particular, the product \( x^r_\circ \delta^{(r)}(x) \) exists in \( \mathcal{D}'_m \) for \( r_1, \ldots, r_m = 0, 1, 2, \ldots \).

Proof: The result follows immediately on replacing \( x \) by \( -x \) in the equation (9). \( \square \)

Theorem 6. The neutrix product \( \delta^{(r)}(x) \circ \delta^{(p)}(x) \) exists and

\[
\delta^{(r)}(x) \circ \delta^{(p)}(x) = 0
\]

for \( r_1, p_1, \ldots, r_m, p_m = 0, 1, 2, \ldots \).

Proof: It follows from the equation (14) with \( r = 0 \) that

\[
x^0_+ \circ \delta^{(p)}(x) = \frac{1}{2} \delta^{(p)}(x)
\]

for \( p = 0, 1, 2, \ldots \). Using Theorem 1, it follows that

\[
\delta(x) \circ \delta^{(p)}(x) = \frac{1}{2} \delta^{(p+1)}(x) - x^0_+ \delta^{(p+1)}(x) = 0
\]

for \( p = 0, 1, 2, \ldots \). It can now be proved easily by induction that

\[
\delta^{(r)}(x) \circ \delta^{(p)}(x) = 0
\]

for \( p = 0, 1, 2, \ldots \). The result of the theorem follows on using Theorem 3. \( \square \)

References

[4] Fisher B., The product of the distributions \( x^r_+ \delta^{(r-1/2)} \) and \( x^-_r \delta^{(r-1/2)} \), Proc. Camb. Phil. Soc. 71 (1972), 123–130.

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(Received April 13, 1992)