On the Novak number of a hyperspace

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Abstract. An estimate for the Novak number of a hyperspace with the Vietoris topology is given. As a consequence it is shown that this cardinal function can decrease passing from a space to its hyperspace.

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The motivation for this paper comes from a question posed in [3]. There it was proved (relatively to the locally finite topology, but the same reasoning applies to the Vietoris topology) that for any dense in itself $T_1$ space $X$ the Novak number of $exp(X)$ is not greater than that of $X$. The point left open was whether the two cardinal numbers can actually be different.

Here we present an estimate for the Novak number of a hyperspace in terms of the netweight of the base space. Using this theorem we give a couple of examples showing that the Novak number can actually decrease passing to the hyperspace. This fact is rather surprising considering the behaviour of practically all other cardinal functions.

Given a topological space $X$, the hyperspace $exp(X)$ is the set of all non empty closed subsets of $X$.

If $S$ is a family of subsets of $X$ then the symbol $\langle S \rangle$ denotes the set of all $A \in exp(X)$ satisfying $A \subset \cup S$ and $A \cap S \neq \emptyset$ for every $S \in S$.

The Vietoris topology on $exp(X)$ is defined by taking as a base all the sets of the form $\langle U_1, \ldots, U_n \rangle$, where $U_1, \ldots, U_n$ are open subsets of $X$. For more information on the Vietoris topology the reader is referred to [5].

Notice that if $C_1, \ldots, C_n$ are closed subsets of $X$ then the set $\langle C_1, \ldots, C_n \rangle = exp(X) \setminus (\langle X \setminus C_1 \rangle \cup \cdots \cup \langle X \setminus C_n \rangle \cup \langle X, X \setminus (C_1 \cup \cdots \cup C_n) \rangle)$ is closed in $exp(X)$.

A network $B$ for the topological space $X$ is a family of subsets having the property that for any open set $U \subset X$ and any $x \in U$ there exists a member $B \in B$ such that $x \in B \subset U$.

The netweight of the space $X$, denoted by $nw(X)$, is the smallest cardinality of a network for $X$.

Given a dense in itself $T_1$ space $X$, the Novak number of $X$, denoted by $n(X)$, is the smallest cardinality of a cover of $X$ consisting of nowhere dense sets.

More details on the Novak number can be found in [1] and [2] and the bibliography listed there.
Theorem 1. If $X$ is a dense in itself regular $T_1$ space then $n(exp(X)) \leq nw(X)^{\aleph_0}$.

Proof: Let $\mathcal{B}$ be a network of $X$ satisfying $|\mathcal{B}| = nw(X)$. As the space $X$ is regular, we can assume that $\mathcal{B}$ consists of closed sets. Denote by $\mathcal{B}_1$ the collection of all countable infinite subsets of $\mathcal{B}$ consisting of pairwise disjoint elements. For any $\mathcal{C} = \{C_1, \ldots, C_n, \ldots\} \in \mathcal{B}_1$ let

$$F_{\mathcal{C}} = \bigcap_{n \in N^+} \langle X, C_n \rangle.$$  

Since every $C_n$ is closed, it follows that also $F_{\mathcal{C}}$ is closed. Moreover, it is clear that every point in $F_{\mathcal{C}}$ is an infinite subset of $X$. On the other hand, every basic open set in $exp(X)$ contains finite subsets of $X$ and therefore it follows that each $F_{\mathcal{C}}$ is nowhere dense. We claim that every infinite closed subset $A$ of $X$ is contained in some $F_{\mathcal{C}}$. To see this, let us begin by taking two disjoint elements $C', C'' \in \mathcal{B}$ such that $C' \cap A \neq \emptyset \neq C'' \cap A$. At least one of these two sets, say $C'$, satisfies $|A \setminus C'| \geq \aleph_0$. Let $C_1 = C'$ and suppose we have already chosen pairwise disjoint sets $C_1, \ldots, C_n \in \mathcal{B}$ in such a way that $(*) C_i \cap A \neq \emptyset$ for $i \leq n$ and $|A \setminus (C_1 \cup \cdots \cup C_n)| \geq \aleph_0$. Then we proceed by induction selecting $C_{n+1} \in \mathcal{B}$ disjoint from $C_1, \ldots, C_n$ and satisfying $(*)$. Letting $\mathcal{C} = \{C_1, \ldots, C_n, \ldots\}$ it is clear that $A \subseteq F_{\mathcal{C}}$. Now let $F_n$ be the subset of $exp(X)$ consisting of all subsets of $X$ having cardinality not bigger than $n$. Since $X$ is dense in itself and $T_2$, it is not difficult to see that $F_n$ is closed and nowhere dense in $exp(X)$. To finish, observe that $\{F_n : n \in N^+\} \cup \{F_{\mathcal{C}} : \mathcal{C} \in \mathcal{B}_1\}$ is a nowhere dense cover of $exp(X)$ of cardinality not exceeding $nw(X)^{\aleph_0}$. \hfill \Box

In order to obtain our first example, we recall the construction of a certain linearly ordered topological group.

For any ordinal $\nu$ denote by $\mathcal{R}^{\nu}$ the set of all functions $f : \nu \to \mathcal{R}$ ordered lexicographically, that is $f < g$ if and only if $f \neq g$ and $f(\alpha) < g(\alpha)$, where $\alpha = \min\{\beta : f(\beta) \neq g(\beta)\}$. The order so defined is actually a linear order and $\mathcal{R}^{\nu}$ can be equipped with the standard interval topology. If, moreover, we define $f + g$ by the rule $(f + g)(\alpha) = f(\alpha) + g(\alpha)$ then $\mathcal{R}^{\nu}$ becomes a linearly ordered topological abelian group.

For any $\alpha \in \nu$ denote by $\varepsilon_\alpha$ the element of $\mathcal{R}^{\nu}$ defined by $\varepsilon_\alpha(\alpha) = 1$ and $\varepsilon_\alpha(\beta) = 0$ if $\beta \neq \alpha$.

Let $\mathcal{R}^{<\nu} = \cup_{\alpha \in \nu} \mathcal{R}^{\alpha}$ and for any $\varphi \in \mathcal{R}^{<\nu}$ denote by $||\varphi||$ the ordinal $\alpha$ such that $\varphi \in \mathcal{R}^{\alpha}$. Let $[\varphi] = \{f \in \mathcal{R}^{\nu} : f \upharpoonright \alpha = \varphi\}$. Observe that $[\psi] \subset [\varphi]$ if and only if $\varphi \subseteq \psi$.

Each $[\varphi]$ is open in $\mathcal{R}^{\nu}$, in fact if $\varphi \in \mathcal{R}^{\alpha}$ and $f \in [\varphi]$ then the interval $(f - \varepsilon_{\alpha+1}, f + \varepsilon_{\alpha+1})$ is contained in $[\varphi]$. Furthermore, the collection of all sets of the form $[\varphi]$ is a base for the topology of $\mathcal{R}^{\nu}$. To see this, fix an interval $(f, g)$ and an element $h \in (f, g)$ and let $\alpha_1 = \min\{\beta : f(\beta) \neq h(\beta)\}$ and $\alpha_2 = \min\{\beta : h(\beta) \neq g(\beta)\}$. If $\alpha = \max\{\alpha_1, \alpha_2\} + 1$ then it is easily seen that $h \in [h \upharpoonright \alpha] \subset (f, g)$.

The next proposition is somewhat related to a result of Sikorski ([6, 4.15]).

Proposition 1. If $\nu$ is a regular cardinal then $n(\mathcal{R}^{\nu}) > \nu$.

Proof: It is enough to show that any family $\{A_\alpha : \alpha \in \nu\}$ of dense open subsets of $\mathcal{R}^{\nu}$ has a non empty intersection. We construct by induction the sequence
\{\varphi_{\alpha} : \alpha \in \nu\} \subset \mathbb{R}^{<\nu} \text{ satisfying the condition }

[\varphi_{\alpha}] \subset A_{\alpha} \cap (\cap_{\beta \in \alpha} [\varphi_{\beta}]).

Assume that the family \{\varphi_{\beta} : \beta \in \alpha\} has already been constructed. Since \nu is regular, the set \{||\varphi_{\beta}|| : \beta \in \alpha\} is bounded in \nu. Thus the function \psi = \cup\{\varphi_{\beta} : \beta \in \alpha\} is a member of \mathbb{R}^{<\nu}. To finish the induction, select \varphi_{\alpha} \in \mathbb{R}^{<\nu} in such a way that [\varphi_{\alpha}] \subset A_{\alpha} \cap [\psi]. Now let \varphi = \cup\{\varphi_{\alpha} : \alpha \in \nu\}. If \varphi is defined on all \nu then \varphi \in \mathbb{R}^{\nu} and \varphi \in \cap_{\alpha \in \nu} A_{\alpha}. If, on the other hand, ||\varphi|| < \nu then any \ f \in [\varphi] \ belongs to \cap_{\alpha \in \nu} A_{\alpha}\. \square

Recall that assuming Martin’s axiom (MA) the cardinality of the continuum \c is regular and, for every \kappa < \c, 2^{\kappa} \leq \c (see [4, Section 2.2]). Taking this into account, we see that the space \mathbb{R}^{c} has a base (and a fortiori a network) of cardinality not exceeding \,|\mathbb{R}^{<\c}| = \sum_{\alpha \in c} 2^{\aleph_0}|\alpha| = c\,. Thus from Theorem 1 and Proposition 1 we get:

**Corollary 1** (MA). There exists a linearly ordered topological group \(X\), namely \(\mathbb{R}^{c}\), for which \(n(exp(X)) < n(X)\).

Under more complicate set-theoretic assumptions, it is possible to find a compact space for which the Novak number decreases passing to the hyperspace. Indeed Theorem 5.2 in [1] describes two models of ZFC in which the Novak number of \(N^*\), the Čech-Stone remainder of \(N\), is greater than \(c\). Taking into account that \(N^*\) is a compact space of netweight \(c\), another application of Theorem 1 gives:

**Corollary 2.** It is consistent with ZFC the existence of a compact space \(X\), namely \(N^*\), for which \(n(exp(X)) < n(X)\).

Observe that, since for any compact space the locally finite topology coincides with the Vietoris topology, Corollary 2 also furnishes a direct answer to the question posed in [3].

We do not know whether the inequality \(n(X) \leq 2^{n(exp(X))}\) holds for every dense in itself \(T_1\) space \(X\). Certainly, however, it cannot be improved. In fact, in one of the models described in [1], it is true that \(n(N^*) = 2^{\aleph_0} = 2^{n(exp(N^*))}\).

**References**


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